EDN: TDISWG
УДК 517.95

An Inverse Problem for Pseudoparabolic Equation with the Mixed Boundary Condition

Anna Sh. Lyubanova*
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 05.02.2023, received in revised form 10.06.2023, accepted 20.08.2023

Abstract. In this paper we study the inverse problem on identification of the leading coefficient in the pseudoparabolic equation. The problem involves the mixed boundary condition. The unknown coefficient is recovered by additional integral boundary data. The existence and uniqueness of the strong solution are proved. The result concerns with the identification of the hydraulic properties of fissured medium.

Keywords: filtration, inverse problem, pseudoparabolic equation, existence, uniqueness.

Citation: A.Sh. Lyubanova, An Inverse Problem for Pseudoparabolic Equation with the Mixed Boundary Condition, J. Sib. Fed. Univ. Math. Phys., 2023, 16(5), 661–672. EDN: TDISWG.

Introduction

This paper is devoted to the inverse problem of identification of an unknown coefficient in the pseudoparabolic equation

$$(u + L_1 u)_t + L_2 u = f \tag{0.1}$$

with the initial data

$$B_1 u|_{t=0} = U_0 \tag{0.2}$$

and the boundary condition

$$B_2 u|_{\partial\Omega} = \mu(t, x, k(t)) \tag{0.3}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$, $L_1 = \eta M$, $L_2 = k(t)M$, M and B_1 are linear differential operators of the second order in the spacial variables, B_2 is a linear operator. To find the unknown coeffcient $k(t)$, the additional data is used in the form of the condition of overdetermination

$$\int_{\partial\Omega} B_3(t)(\eta u_t + k(t)u)\,\omega(t,x)\,ds + k(t)\varphi_1(t) = \varphi_2(t) \tag{0.4}$$

where $B_3(t)$ is a linear operator in the spacial variables for every $t \in (0,T)$, $\omega(t,x)$ and $\varphi_i(t)$, $i = 1, 2$, are known functions. The conditions (0.3) and (0.4) must be independent in that (0.4) may not be evident from (0.3).

*lubanova@mail.ru

A main goal of this article is to investigate the correctness of the problem (0.1)–(0.4) with $B_1u = u$, $B_2u = \dfrac{\partial}{\partial \overline{N}}(\eta u_t + k(t)u) + \sigma(x)(\eta u_t + k(t)u)$, $\mu(t, x, k(t)) = \mu_2(t, x) - \mu_1(t, x)k(t)$ and $B_3 = I$ where I is the identity operator, $\sigma(x)$ and $\mu_i(t, x)$, $i = 1, 2$, are known functions, $\dfrac{\partial}{\partial \overline{N}}$ is the conormal derivative associated with the operator M (see Problem 1 below).

In [9, 11, 12], the problem (0.1)–(0.4) was considered in the case where $L_1 = \eta M$, $L_2 = k(t)M + g(t, x)I$, $B_1u = u + L_1u$, $B_2u = u$, $B_3 = \dfrac{\partial}{\partial \overline{N}}$. In [11], the existence and uniqueness of the strong solution are proved. The regularity of the solution is also investigated. The work [12] discusses the stabilization and the asymptotic behavior of the solution as $t \to +\infty$. It is shown in [9] that under certain conditions this solution tends to the solution of the appropriate parabolic inverse problem when $\eta \to 0$.

Applications of the inverse problems for (0.1) with various operators L_1 and L_2 involve the recovery of the unknown parameters indicating physical properties of a medium (the heat conductivity, the permeability of a porous medium, the elasticity, etc.). In particular, the equation (0.1) considered in [9, 11, 12] describes the filtration of a liquid in a fissured medium [1]. The coefficient $k(t)$ is in inverse proportion to the total effect of compressibility of the liquid and the fissured medium. Since the natural stratum is involved, the parameters in (0.1) should be determined on the basis of the investigation of its behaviour under the natural non-steady-state conditions. This leads to the interest in studying the inverse problems for (0.1) and its analogue.

The study of inverse problems for pseudoparabolic equations (0.1) goes back to 1980s. The first result [16] refers to the inverse problems of determining a source function f in (0.1) with $L_1 = L_2$. Most of the results on inverse problems are concerned with the identification of an unknown source f and coefficient in the lowest order term u as in [4–6, 15]. The work [8] is devoted to the inverse problem on reconstruction of the kernels in the integral term of the integro-differential operator L_2. We should mention also the results [5, 14, 15] concerning with coefficient inverse problems for (0.1). In [14], the uniqueness theorem is obtained and an algorithm of determining a constant a in the second order term is constructed. In [5], the solvability is established for two inverse problems of recovering the unknown coefficients in terms u (the lowest term of L_2u) and u_t of (0.1). In [15], an inverse problem of recovering time-depending right-hand side and coefficients of (0.1) is considered. The values of the solution at separate points are employed as overdetermination conditions. The existence and uniqueness theorems are proven for this problem.

The paper is organized as follows. Section 1 presents the formulation of the inverse problem and certain preliminary results concerning the direct initial boundary value problem for (0.1). In Section 2, the existence and uniqueness of the strong solution of the inverse problem are proved.

1. The statement of the problem and preliminaries

Let $\partial\Omega \in C^2$, $\overline{\Omega}$ be the closure of Ω. $Q_T = \Omega \times (0, T)$ is a cylinder with the lateral surface $S_T = (0, T) \times \partial\Omega$, $\overline{Q}_T$ is the closure of Q_T and the pair (t, x) is a point of Q_T.

Throughout this paper we use the following notation: $\|\cdot\|_R$ and $(\cdot, \cdot)_R$ are the norm and the inner product of $\mathbf{R}^n$; $\|\cdot\|$ and $(\cdot, \cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively; $\|\cdot\|_j$ and $\langle \cdot, \cdot \rangle_j$ are the norm of $W_2^j(\Omega)$ and the duality relation between $\mathring{W}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$, respectively ($j = 1, 2$).

We introduce the operator M by the following rule: for every $u \in W_2^1(\Omega)$ the element Mu

gives the functional

$$J(v) = \langle Mu, v\rangle_M + \int_{\partial\Omega} \sigma(x)uvdx$$

defined for all $v \in W_2^1(\Omega)$ where

$$\langle Mu, v\rangle_M \equiv \int_\Omega \{(\mathcal{M}(x)\nabla u, \nabla v)_R + m(x)uv\}dx.$$

Here $\mathcal{M}(x) \equiv ((m_{ij}(x)))$ is a matrix of functions $m_{ij}(x),\ i,j = 1,2,\dots,n,$ $m(x)$ and $\sigma(x)$ are scalar functions. We assume that the following conditions are fulfilled.

I. $m_{ij}(x),\ \partial m_{ij}/\partial x_l,\ i,j,l = 1,2,\dots,n,$ and $m(x)$ are bounded in Ω. M is an elliptic operator, that is, there exist positive constants m_0 and m_1 such that for any $v \in W_2^1(\Omega)$ and almost all $x \in \Omega$

$$m_0\|v\|_1^2 \leqslant \langle Mv, v\rangle_M + \int_{\partial\Omega} \sigma(x)uvdx \leqslant m_1\|v\|_1^2. \tag{1.1}$$

II. $m_{ij}(x) = m_{ji}(x),\ i,j = 1,2,\dots,n$ for $x \in \Omega$.

We are studying the following inverse problem.

Problem 1. For a given constant η and functions $f(t,x),\ g(t,x),\ u_0(x),\ \mu_1(t,x),\ \mu_2(t,x),$ $\sigma(x),\ \omega(t,x),\ \varphi_1(t),\ \varphi_2(t)$ find the pair of functions $(u(t,x), k(t))$ satisfying the equation

$$u_t + \eta M u_t + k(t)\, Mu = f(t,x), \quad (t,x) \in Q_T, \tag{1.2}$$

and the conditions

$$u\big|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.3}$$

$$\Big\{\eta\frac{\partial u_t}{\partial \overline{N}} + k(t)\frac{\partial u}{\partial \overline{N}} + \sigma(x)(\eta u_t + k(t)u)\Big\}\Big|_{S_T} + k(t)\mu_1(t,x) = \mu_2(t,x), \tag{1.4}$$

$$\int_{\partial\Omega} (\eta u_t + k(t)u)\omega(t,x)\, dS + \varphi_1(t)k(t) = \varphi_2(t), \quad t \in (0,T). \tag{1.5}$$

Here $\dfrac{\partial}{\partial \overline{N}} = (\mathbf{n}, \mathcal{M}(x)\nabla)$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

The conditions (1.4) and (1.5) may seem peculiar. However, such formulation of this conditions are rather natural for pseudoparabolic equations. For a deeper discussion of the conditions (1.4), (1.5) we refer the reader to [10].

We introduce functions $b(t,x)$ and $h^\eta(t,x)$ as the solutions of the boundary value problems

$$Mb = 0 \quad \text{in } \Omega, \quad \Big\{\frac{\partial b}{\partial \overline{N}} + \sigma(x)b\Big\}\Big|_{\partial\Omega} = \omega(t,x),$$

$$h^\eta + \eta M h^\eta = 0 \quad \text{in } \Omega, \quad \Big\{\frac{\partial h^\eta}{\partial \overline{N}} + \sigma(x)h^\eta\Big\}\Big|_{\partial\Omega} = \omega(t,x).$$

The existence and uniqueness results for Problem 1 rely upon two propositions for the direct problem (1.2)–(1.4) with the known function $k(t)$.

The first proposition concerns with the existence and uniqueness of the solution of the direct problem (1.2)–(1.4), which follows from the the results of [19] in the case of the constant coefficient $k(t) \equiv k$.

Lemma 1.1. 1) *Let the assumptions* I-II *be fulfilled,* $\eta > 0,\ k \in C([0,T]),\ f \in C([0,T]; L^2(\Omega)),$ $u_0 \in W_2^2(\Omega)$ *and* $\mu_1, \mu_2 \in C^1([0,T]; W_2^{1/2}(\partial\Omega))$. *Then there exists a unique solution* $u(t,x)$ *of problem (1.2)–(1.4) such that* $u \in C^1([0,T]; W_2^2(\Omega))$.

Proof. Let us consider the problem

$$\begin{cases} v_t + G(v) = f, & (t,x) \in Q_T, \\ v|_{t=0} = (I + \eta M)u_0, & x \in \Omega, \end{cases} \tag{1.6}$$

where $v = (I + \eta M)u$ and the operator G acting from $C([0,T]; L^2(\Omega))$ into itself is defined as

$$G = k(t)M(I + \eta M)^{-1} \equiv \frac{k(t)}{\eta}\left(I - (I + \eta M)^{-1}\right).$$

The function v is the solution of the problem (1.6) if and only if the function $u = (I + \eta M)^{-1}v$ is a solution of problem (1.2)–(1.4). Therefore Theorem 1.1 will be proved once we prove the existence and uniqueness of the solution of the problem (1.6). From the hypotheses of the lemma it follows that G is a Lipschitz-continuous linear operator. Then, by Theorem 1.2 of [3, Chapter V], the problem (1.6) has a unique solution $v \in C^1([0,T]; L^2(\Omega))$. The lemma is proved. □

The next lemma is a maximum principle (a comparison theorem) for the direct problem (1.2)–(1.4). In general, the maximum principle does not hold for the pseudoparabolic equation [17]. However it has been possible to prove such assertions under certain additional assumptions on the input data of the initial boundary value problems. In [17, 18, 20], comparison theorems were proved for the first initial boundary value problem for (0.1) with the general linear elliptic operators $L_1 = L_2 = L$ of the second order and $k \equiv 1$. In [2], the comparison theorem is proved for (0.1) with the constant coefficient k, $L_1 = L_2 = -\Delta$ in the case of the mixed boundary conditions.

Lemma 1.2. *Under the assumptions of Lemma* 1.1*, let $u(t,x)$ be the solution of the problem (1.2)–(1.4) in $C^1([0,T]; W_2^2(\Omega))$. In addition, let $f(t,x) \geqslant 0$ for almost all $(t,x) \in \overline{Q}_T$, $u_0(x) \geqslant 0$ almost everywhere in $\overline{\Omega}$, $k(t) \geqslant 0$ for $t \in [0,T]$, $\mu_1 \leqslant 0$ and $\mu_2 \geqslant 0$ for almost all $(t,x) \in S_T$. Then*

$$u(t,x) \geqslant u_0(x)\, \exp\Big(-\frac{1}{\eta}\int_0^t k(\theta)d\theta\Big)$$

for almost all $(t,x) \in Q_T$.

Proof. It is sufficient to prove the assertion of Lemma 1.2 for the smooth solution of problem (1.2)–(1.4) since the following arguments can be justified by using the method of difference quotients or mollifiers.

The function $v = u - u_0 \exp\Big(-\frac{1}{\eta}\int\limits_0^t k(\theta)d\theta\Big)$ is the solution of the equation

$$v + \eta M v = \int_0^t \Big[f + \frac{k(\tau)}{\eta}v\Big] \exp\Big(-\int_\tau^t \frac{k(\theta)}{\eta}d\theta\Big)d\tau + u_0 \exp\Big(-\int_0^t \frac{k(\theta)}{\eta}d\theta\Big), \tag{1.7}$$

and obeys the boundary condition

$$\Big[\frac{\partial v}{\partial \overline{N}} + \sigma(x)v\Big]\Big|_{S_T} = \frac{1}{\eta}\int_0^t (\mu_2 - k(\tau)\mu_1) \exp\Big(-\frac{1}{\eta}\int_\tau^t k\, d\theta\Big)d\tau. \tag{1.8}$$

Let us define the functions $v_1 = \min_{(t,x)\in Q_T}\{v, 0\}$ and $v_2 = \max_{(t,x)\in Q_T}\{v, 0\}$. We multiply (1.7) by v_1 in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the second summand of the left side of the resulting equation. In view of (1.8) this gives

$$\|v_1\|^2 + \eta\langle Mv_1, v_1\rangle_M + \eta\int_{\partial\Omega} \sigma v_1^2 ds - \int_\Omega \int_0^t (f + \frac{k}{\eta}v_2) \exp\Big(-\int_\tau^t \frac{k}{\eta}d\theta\Big)d\tau v_1 dx -$$

$$-\int_{\partial\Omega}\int_0^t (\mu_2 - k\mu_1)\exp\Big(-\int_\tau^t \frac{k}{\eta}d\theta\Big)d\tau\, v_1 ds - \exp\Big(-\int_0^t \frac{k}{\eta}d\theta\Big)\int_\Omega u_0 v_1 dx =$$
$$= \int_\Omega\int_0^t \frac{k}{\eta}v_1 \exp\Big(-\int_\tau^t \frac{k}{\eta}d\theta\Big)d\tau v_1\, dx.$$

In the hypotheses of the lemma this relation implies the inequality

$$\|v_1\|^2 \leqslant C\int_0^t \|v_1\|^2 d\tau.$$

with the constant $C > 0$ depending on T, η, $\max_{t\in[0,T]} k(t)$, whence by Gronwall's lemma it follows that $v_1 = 0$, that is, $v \geqslant 0$ almost everywhere in Q_T. The lemma is proved. □

2. The existence and uniqueness

In this section we discuss the sufficient conditions for the solvability and the uniqueness of the solution of Problem 1. By a solution $\{u, k\}$ of Problem 1 we mean that

1) $u \in C^1([0,T]; W_2^2(\Omega))$, $k(t) \in C([0,T])$;

2) the pair $\{u, k\}$ obeys the equation (1.2) almost everywhere in Q_T and the conditions (1.3) for almost all $x \in \overline{\Omega}$, (1.4) almost everywhere in S_T and (1.5) for all $t \in [0,T]$.

The main result of this article is established by our next theorem.

Theorem 2.1. *Let the assumptions* I–II *be fulfilled,* $\partial\Omega \in C^2$ *and* η *be a positive constant. Assume that*
(i) $f \in C([0,T]; L^2(\Omega))$, $\mu_1, \mu_2 \in C\big([0,T]; W_2^{1/2}(\partial\Omega)\big)$, $u_0 \in W_2^2(\Omega)$, $\sigma \in C(\partial\Omega)$, $\omega \in C^1\big([0,T]; W_2^{1/2}(\partial\Omega)\big)$, $\varphi_1, \varphi_2 \in C([0,T])$;
(ii) *the assumptions of Lemma 1.2 are fulfilled and there exist constants* $\alpha_0 > 0$ *and* $\Phi_0 > 0$ *such that*

$$\varphi_1 - \int_{\partial\Omega} \mu_1 \omega ds \geqslant \alpha_0, \tag{2.1}$$

$$\Phi(t) \equiv \varphi_2(t) - (f, h^\eta) - \int_{\partial\Omega} \mu_2 h^\eta ds \geqslant \Phi_0. \tag{2.2}$$

Then Problem 1 has a solution $\{u, k\}$ *and this solution is unique. Moreover,* $u \geqslant 0$ *almost everywhere in* Q_T *and the estimates*

$$0 < k_0 \leqslant k(t) \leqslant \overline{\Phi}\alpha^{-1}, \tag{2.3}$$

$$\|u\|_2 + \|u_t\|_2 \leqslant C_1 \tag{2.4}$$

hold with positive constants k_0 *and* C_1. *Here* $\overline{\Phi} = \max_{t\in[0,T]} \Phi$.

Proof. Following the idea in [11,13], we reduce Problem 1 to an equivalent inverse problem with a nonlinear operator equation for $k(t)$. To do this, we multiply (1.2) by h^η in terms of the inner product of $L^2(\Omega)$ and integrate by parts twice. This yeilds

$$(u_t, h^\eta) + (\eta M u_t + kMu, h^\eta) = (u_t, h^\eta) - \int_{\partial\Omega}\Big(\eta\frac{\partial u_t}{\partial \overline{N}} + k\frac{\partial u}{\partial \overline{N}}\Big)h^\eta ds+$$

$$+\langle \eta M u_t + kMu, h^\eta\rangle_M = (u_t, h^\eta) + k\int_{\partial\Omega}\mu_1 h^\eta ds - \int_{\partial\Omega}\mu_2 h^\eta ds +$$

$$+\int_{\partial\Omega}\sigma(x)(\eta u_t + ku)h^\eta ds + \int_{\partial\Omega}(\eta u_t + ku)\frac{\partial h^\eta}{\partial\overline{N}}ds + (\eta u_t + ku, Mh^\eta) =$$

$$= k\Big[-\varphi_1 + \int_{\partial\Omega}\mu_1 h^\eta ds - \frac{1}{\eta}(u, h^\eta)\Big] - \int_{\partial\Omega}\mu_2 h^\eta ds + \varphi_2 = (f, h^\eta),$$

whence we obtain that

$$k\Big[\varphi_1 - \int_{\partial\Omega}\mu_1 h^\eta ds + \frac{1}{\eta}(u, h^\eta)\Big] = \Phi(t)$$

We define the operator A which maps every element $y \in C_+([0,T]) = \{y \,|\, y \in C([0,T]), y > 0\}$ into the element $Ay \in C([0,T])$ by the rule

$$Ay = \Phi(t)\Big[\varphi_1 - \int_{\partial\Omega}\mu_1 h^\eta ds + \frac{1}{\eta}(u_y, h^\eta)\Big]^{-1} \tag{2.5}$$

where u_y is the solution of the problem (1.2)–(1.5) with $k(t) = y$. The element Ay is meaningful for every $y \in C_+([0,T])$. Indeed, the direct problem (1.2)–(1.4) with $k = y$ has a unique solution $u_y \in C^1([0,T]); W_2^2(\Omega))$ for every $y \in C_+([0,T])$ by Lemma 1.1. Moreover, by Lemma 1.2 and the maximum principle for elliptic equation $(u_y, h^\eta) \geqslant 0$, which implies in view of (2.1) that

$$\varphi_1 - \int_{\partial\Omega}\mu_1 h^\eta ds + \frac{1}{\eta}(u_y, h^\eta) \geqslant \alpha_0. \tag{2.6}$$

It can be shown that Problem 1 is solvable if and only if the operator equation

$$y = Ay \tag{2.7}$$

has a solution in $C_+([0,T])$. Really, the deduction of the equation (2.7) shows that if $\{u_y, y\}$ is a solution of Problem 1, then y is a fixed point of the operator A by (2.5). On the other hand, let y^* is a solution of equation (2.7) and u^* is a solution of (1.2)–(1.4) with $k(t) = y^*(t)$. Multiplying (1.2) by h^η in terms of the inner product of $L^2(\Omega)$ and integration by parts twice in the second and third summands implies in view of (1.4), (2.5), (2.7) that the pair $\{u^*(t,x), y^*(t)\}$ obeys the condition of overdetermination (1.5).

The relations (2.2), (2.5) and (2.6) imply the estimate

$$0 \leqslant Ay \leqslant \overline{\Phi}\alpha_0^{-1}.$$

Let us prove that there exists such $y_0 > 0$ that the operator A maps the set

$$Y = \{\, y \,|\, y \in C([0,T]), y_0 \leqslant y \leqslant \overline{\Phi}\alpha_0^{-1}\}$$

into itself. We multiply (1.2) with $k = y \in Y$ by the solution u_y of the problem (1.2)–(1.4) in terms of the inner product of $L_2(\Omega)$ and integrate by parts in the second and third summands. It gives

$$\frac{1}{2}\frac{d}{dt}\Big\{\|u_y\|^2 + \eta\langle Mu_y, u_y\rangle_M + \eta\int_{\partial\Omega}\sigma u_y^2 ds\Big\} + y\langle Mu_y, u_y\rangle_M + y\int_{\partial\Omega}\sigma u_y^2 ds =$$

$$= (f, u_y) - \int_{\partial\Omega}(y\mu_1 - \mu_2)u_y ds.$$

We next integrate the last relation with respect to t from 0 to τ, $0 < \tau \leqslant T$, and rewrite as

$$\|u_y\|^2 + \eta\Big[\langle Mu_y, u_y\rangle_M + \int_{\partial\Omega} \sigma u_y^2 ds\Big] + 2\int_0^\tau y\Big[\langle Mu_y, u_y\rangle_M + \int_{\partial\Omega} \sigma u_y^2 ds\Big]dt =$$
$$= \|u_0\|^2 + \eta\Big[\langle Mu_0, u_0\rangle_M + \int_{\partial\Omega} \sigma u_0^2 ds\Big] + 2\int_0^\tau \Big\{(f, u_y) - \int_{\partial\Omega} (y\mu_1 - \mu_2)u_y ds\Big\}dt. \quad (2.8)$$

By (1.1), the left side of this equality is estimated from below as

$$\|u_y\|^2 + \eta m_0\|u_y\|_1^2 + \int_{\partial\Omega} \sigma u_y^2 ds.$$

One can estimate the right term of (2.8) with the help of the embedding theorem and the Cauchy inequality. This gives

$$\|u_0\|^2 + \eta\Big[\langle Mu_0, u_0\rangle_M + \int_{\partial\Omega} \sigma u_0^2 ds\Big] + 2\int_0^\tau \Big\{(f, u_y) - \int_{\partial\Omega} (y\mu_1 - \mu_2)u_y ds\Big\}dt \leqslant$$
$$\leqslant \frac{c^2}{\eta m_0}\Big(\overline{\Phi}\alpha_0^{-1}\|\mu_1\|_{L^2(S_T)} + \|\mu_2\|_{L^2(S_T)}\Big)^2 + \|f\|^2_{L^2(Q_T)} + \|u_0\|^2 +$$
$$+\eta\Big[\langle Mu_0, u_0\rangle_M + \bar\sigma\|u_0\|^2_{L^2(\partial\Omega)}\Big] + \int_0^\tau \big(\|u_y\|^2 + \eta m_0\|u_y\|_1^2\big)dt$$

where $\bar\sigma = \|\sigma\|_{C(\overline{\Omega})}$, $c > 0$ is the constant of the embedding $W_2^1(\Omega) \hookrightarrow L^2(\partial\Omega)$. Thus, we obtain from (2.8) that

$$\|u_y\|^2 + \eta m_0\|u_y\|_1^2 \leqslant \frac{c^2}{\eta m_0}\Big(\overline{\Phi}\alpha_0^{-1}\|\mu_1\|_{L^2(S_T)} + \|\mu_2\|_{L^2(S_T)}\Big)^2 + \|f\|^2_{L^2(Q_T)} +$$
$$+\|u_0\|^2 + \eta\Big[\langle Mu_0, u_0\rangle_M + \bar\sigma\|u_0\|^2_{L^2(\partial\Omega)}\Big] + \int_0^\tau \big(\|u_y\|^2 + \eta m_0\|u_y\|_1^2\big)dt,$$

whence by Gronwall's lemma we have

$$\|u_y\|^2 + \eta m_0\|u_y\|_1^2 \leqslant \Big[\frac{c^2}{\eta m_0}\Big(\overline{\Phi}\alpha_0^{-1}\|\mu_1\|_{L^2(S_T)} + \|\mu_2\|_{L^2(S_T)}\Big)^2 + \|f\|^2_{L^2(Q_T)} +$$
$$+ \|u_0\|^2 + \eta\Big(\langle Mu_0, u_0\rangle_M + \bar\sigma\|u_0\|^2_{L^2(\partial\Omega)}\Big)\Big]e^T \equiv C_2^2. \quad (2.9)$$

Coming back to (2.5) we can determine y_0. By (2.2), (2.9),

$$Ay \geqslant \Phi_0\Big[\bar\varphi_1 + \max_{t\in[0,T]}\big\{\|\mu_1\|_{L^2(\partial\Omega)}\|\omega\|_{L^2(\partial\Omega)} + \eta^{-1}C_2\|h^\eta\|\big\}\Big]^{-1} \equiv y_0, \quad (2.10)$$

where $\bar\varphi_1 = \|\varphi_1\|_{C([0,T])}$. Thus, the operator A maps the set Y with y_0 defined by (2.10) into itself.

Let $y_1, y_2 \in Y$ and u_{y_1}, u_{y_2} be the solutions of the problem (1.2)–(1.4) with $k = y_1$ and $k = y_2$, respectively. In view of (2.5)

$$|Ay_1 - Ay_2| = \eta^{-1}\Phi(t)\big|(u_{y_2} - u_{y_1}, h^\eta)\big|\Big\{\varphi_1 + \int_{\partial\Omega} \mu_1 h^\eta ds + \eta^{-1}(u_{y_1}, h^\eta)\Big\}^{-1} \times$$
$$\times\Big\{\varphi_1 + \int_{\partial\Omega} \mu_1 h^\eta ds + \eta^{-1}(u_{y_2}, h^\eta)\Big\}^{-1} \leqslant \frac{\overline{\Phi}\max_{t\in[0,T]}\|h^\eta\|}{\eta\alpha_0^2}\|u_{y_1} - u_{y_2}\|. \quad (2.11)$$

On the other hand, the difference $w_y = u_{y_1} - u_{y_2}$ satisfies the equation

$$w_{yt} + \eta M w_{yt} + y_1 M w_y = (y_2 - y_1) M u_{y2} \tag{2.12}$$

and the conditions

$$w_y(0,x) = 0,$$
$$\Big[\eta \frac{\partial w_{yt}}{\partial \overline{N}} + y_1 \frac{\partial w_y}{\partial \overline{N}} + \sigma(x)(\eta w_{yt} + y_1 w_y)\Big]\Big|_{S_T} = (y_2 - y_1)\Big[\Big(\frac{\partial u_{y2}}{\partial \overline{N}} + \sigma u_{y2}\Big)\Big|_{S_T} + \mu_1\Big]. \tag{2.13}$$

We multiply (2.12) by w_y in terms of the inner product of $L^2(\Omega)$ and integrate by parts in the second and third summands and the right side. In view of (2.13) this gives

$$\frac{1}{2}\frac{d}{dt}\Big\{\|w_y\|^2 + \eta\Big[\langle M w_y, w_y\rangle_M + \int_{\partial\Omega} \sigma w_y^2 ds\Big]\Big\} + y_1\Big\{\langle M w_y, w_y\rangle_M + \int_{\partial\Omega} \sigma w_y^2 ds\Big\} =$$
$$= (y_2 - y_1)\Big\{\langle M u_{2y}, w_y\rangle_M + \int_{\partial\Omega} (\sigma u_{2y} + \mu_1) w_y ds\Big\}. \tag{2.14}$$

By (1.1), (2.9) and the embedding theorem,

$$\Big|(y_2 - y_1)\Big\{\langle M u_{2y}, w_y\rangle_M + \int_{\partial\Omega} (\sigma u_{2y} + \mu_1) w_y ds\Big\}\Big| \leqslant$$
$$\leqslant c|y_2 - y_1| \left\{(m_1 + c\bar{\sigma})\|u_{y_2}\|_1 + \|\mu_1\|_{L^2(\partial\Omega)}\right\} \|w_y\|_1 \leqslant$$
$$\leqslant \frac{c^2}{2\eta m_0}|y_2 - y_1|^2 \left\{\frac{(m_1 + c\bar{\sigma})C_2}{(\eta m_0)^{1/2}} + \|\mu_1\|_{L^2(\partial\Omega)}\right\}^2 + \frac{\eta m_0}{2}\|w_y\|_1^2. \tag{2.15}$$

Integrating (2.14) with respect to t from 0 to τ, $0 < \tau < T$, and estimating the left side of the result with the use of (1.1) we obtain in view of (2.) that

$$\|w_y\|^2 + \eta m_0 \|w_y\|_1^2 \leqslant \frac{c^2}{\eta m_0}\int_0^\tau |y_2 - y_1|^2 \left\{\frac{(m_1 + c\bar{\sigma})C_2}{(\eta m_0)^{1/2}} + \|\mu_1\|_{L^2(\partial\Omega)}\right\}^2 dt +$$
$$+ \int_0^\tau \left(\|w_y\|^2 + \eta m_0 \|w_y\|_1^2\right) dt.$$

Accordig to Gronwall's lemma the last relation implies the estimate

$$\|w_y\|^2 \leqslant \frac{c^2 e^T}{\eta m_0}\left(\frac{(m_1 + c\bar{\sigma})C_2}{(\eta m_0)^{1/2}} + \|\mu_1\|_{C([0,T];L^2(\partial\Omega))}\right)^2 \int_0^\tau |y_2 - y_1|^2 dt. \tag{2.16}$$

Combining (2.11) and (2.16) we are led to the inequality

$$|Ay_1 - Ay_2| \leqslant K\Big(\int_0^\tau |y_1 - y_2|^2 dt\Big)^{1/2}$$

where

$$K = \frac{c\overline{\Phi}\max_{t\in[0,T]}\|h^\eta\|}{\eta^{3/2} m_0^{1/2} \alpha_0^2}\left(\frac{(m_1 + c\bar{\sigma})C_2}{(\eta m_0)^{1/2}} + \|\mu_1\|_{C([0,T];L^2(\partial\Omega))}\right) e^{T/2}.$$

Let us introduce an equivalent norm in $C([0,T])$ as

$$|\!|\!|\cdot|\!|\!|_\nu = \max_{t\in[0,T]}\{e^{-\nu t}|\cdot|\}$$

with a positive constant ν to be determined later. Then

$$| Ay_1 - Ay_2 |_\nu \leqslant \frac{K}{(2\nu)^{1/2}} | y_1 - y_2 |_\nu .$$

Choosing $\nu = 2K^{-2}$, we obtain the inequality

$$| Ay_1 - Ay_2 |_\nu \leqslant \frac{1}{2} | y_1 - y_2 |_\nu . \tag{2.17}$$

This means that the operator $A : Y \to Y$ is a contraction. Thus, in accordance with the principle of contracting mappings the operator A has a unique fixed point $k^* \in Y$. The pair $\{u^*, k^*\}$ gives the solution of the inverse problem (1.2)–(1.5) where u^* satisfies (1.2)–(1.4) with $k = k^*$ and $u^* \in C^1([0,T]; W_2^2(\Omega))$ by Lemma 1.1. Moreover, the estimates (2.3) and (2.9) are valid for k^* and u^*.

We are coming now to the estimates for u and u_t in $W_2^2(\Omega)$. Multiplying (1.2) by u_t in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the second and third terms of the left side yields

$$\|u_t\|^2 + \eta\langle Mu_t, u_t\rangle_M + \eta\int_{\partial\Omega} \sigma(x)u_t^2 ds = -k\langle Mu, u_t\rangle_M - k\int_{\partial\Omega} \sigma(x)uu_t ds + \int_{\partial\Omega} \beta u_t ds + (f, u_t).$$

By (1.1), (2.3), (2.9) and the embedding theorem, the last equality implies the relation

$$\|u_t\|^2 + \eta m_0\|u_t\|_1^2 + \eta\int_{\partial\Omega} \sigma(x)u_t^2 ds \leqslant \Big\{\frac{\overline{\Phi}(m_1 + c^2\bar{\sigma})}{\alpha_0}\|u\|_1 + c\|\beta\|_{L^2(\partial\Omega)}\Big\}\|u_t\|_1 +$$
$$+\|f\|\|u_t\| \leqslant \frac{1}{2\eta m_0}\Big\{\frac{C_2\overline{\Phi}(m_1 + c^2\bar{\sigma})}{\alpha_0(\eta m_0)^{1/2}} + c\|\beta\|_{L^2(\partial\Omega)}\Big\}^2 + \frac{1}{2}\|f\|^2 +$$
$$+\frac{1}{2}\left(\|u_t\|^2 + \eta m_0\|u_t\|_1^2\right)$$

which implies the estimate

$$\|u_t\|^2 + \eta m_0\|u_t\|_1^2 \leqslant \frac{1}{\eta m_0}\Big\{\frac{C_2\overline{\Phi}(m_1 + c^2\bar{\sigma})}{\alpha_0(\eta m_0)^{1/2}} + c\|\beta\|_{L^2(\partial\Omega)}\Big\}^2 + \|f\|^2 \equiv C_3^2. \tag{2.18}$$

We are now in a position to get the estimates for u in $W_2^2(\Omega)$. To do this we rewrite the boundary condition (1.4) in the form

$$\Big[\frac{\partial u}{\partial \overline{N}} + \sigma u\Big]\Big|_{S_T} = \beta \tag{2.19}$$

where

$$\beta = \Big[\frac{\partial u_0}{\partial \overline{N}} + \sigma u_0\Big]\Big|_{S_T} \exp\Big(-\frac{1}{\eta}\int_0^t k\,d\theta\Big) + \frac{1}{\eta}\int_0^t (\mu_2 - k\mu_1)\exp\Big(-\frac{1}{\eta}\int_\tau^t k\,d\theta\Big)d\tau.$$

Multiplying (1.2) by Mu in terms of the scalar product of $L^2(\Omega)$ gives

$$\frac{\eta}{2}\frac{d}{dt}\|Mu\|^2 + k(t)\|Mu\|^2 = (f - u_t, Mu).$$

By (2.3) and (2.18),

$$\frac{\eta}{2}\frac{d}{dt}\|Mu\|^2 + y_0\|Mu\|^2 \leqslant \frac{1}{2y_0}(\|f\| + C_3)^2 + \frac{y_0}{2}\|Mu\|^2$$

whence

$$\|Mu\|^2 \leqslant \|Mu_0\|^2 e^{-\frac{y_0}{\eta}t} + \frac{1}{y_0}\int_0^t (\|f\| + C_3)^2 e^{-\frac{y_0}{\eta}(t-\tau)} d\tau. \tag{2.20}$$

The inequality [7, Chapter 2]

$$\|v\|_2 \leqslant K_1(\|Mv\| + \|\beta\|_{W_2^{1/2}(\partial\Omega)} + \|v\|_1) \tag{2.21}$$

is valid for all $v \in W_2^2(\Omega)$ satisfying the boundary condition (2.19) where the constant $K_1 > 0$ depends on m_0, m_1 and mesΩ. In view of (2.9), (2.18)–(2.21) we have

$$\|u\|_2 \leqslant K_1\Big[\|Mu_0\| + \Big(\frac{1}{y_0}\int_0^t (\|f\| + C_3)^2 d\tau\Big)^{1/2} + \|\beta\|_{W_2^{1/2}(\partial\Omega)} + C_2(\eta m_0)^{-1/2}\Big] \equiv C_4. \tag{2.22}$$

The estimates (2.18) and (2.20) enable one to conclude from (1.2) that

$$\eta\|Mu_t\| \leqslant \|f\| + C_3 + \frac{\overline{\Phi}}{\alpha_0}C_5. \tag{2.23}$$

The positive constant C_5 depends on C_2, C_3, y_0, T, η, $\|f\|_{L^2(Q_T)}$. By (1.4), (2.18) and (2.23),

$$\|\eta u_t + k(t)u\|_2 \leqslant K_1\Big(\|M(\eta u_t + k(t)u)\| + \|\beta - \sigma(\eta u_t + k(t)u)\|_{W_2^{1/2}(\partial\Omega)} +$$

$$+ \|\eta u_t + k(t)u\|_1\Big) \leqslant K_1\Big[\|f - u_t\| + \|\beta\|_{W_2^{1/2}(\partial\Omega)} + (c_0\bar{\sigma} + 1)\Big(\eta\|u_t\|_1 + \frac{\overline{\Phi}}{\alpha_0}\|u\|_1\Big)\Big] \leqslant$$

$$\leqslant K_1\Big[\|f\| + C_3 + \|\beta\|_{W_2^{1/2}(\partial\Omega)} + \frac{(c_0\bar{\sigma} + 1)}{(\eta m_0)^{1/2}}\Big(\eta C_3 + \frac{\overline{\Phi}}{\alpha_0}C_2\Big)\Big] \equiv C_6$$

whence

$$\eta\|u_t\|_2 \leqslant C_6 + \frac{\overline{\Phi}}{\alpha_0}\|u\|_2 \leqslant C_6 + \frac{\overline{\Phi}}{\alpha_0}C_4.$$

The last inequality and (2.22) implies the estimate (2.4).

The uniqueness of the solution $\{u, k\}$ follows from (2.17). Really, let $\{u_1, k_1\}$ and $\{u_2, k_2\}$ be two solutions of Problem 1. Hence, k_1 and k_2 satisfies the operator equation (2.7) and the inequality (2.17) is valid. Then

$$| k_1 - k_2 |_\nu = | Ak_1 - Ak_2 |_\nu \leqslant \frac{1}{2} | k_1 - k_2 |_\nu,$$

which proves that $k_1 = k_2$. From this in turn follows by (2.16) that $u_1 = u_2$. □

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement no. 075-02-2023-936).

References

[1] G.I.Barenblatt, Iu.P.Zheltov, I.N.Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured blocks [strata], *J. Appl. Math. Mech*, **24**(1960), 1286–1303.

[2] E.Di Bendetto, M.Pierre, On the Maximum Principle for Pseudoparabolic Equations, *Indiana University Mathematics Journal*, **30**(1981), 821–854.

[3] H.Gajewski, K.Gröger, K.Zacharias, *Nichtlinear Operatorgleichungen und Operatordifferentialgleichungen*, Mathematische Lehrbücher und Monographien. II, Abteilung, Mathematische Monographien, vol. 38, Akademie-Verlag, Berlin, 1974.

[4] V.E.Fedorov, N.D.Ivanova, Inverse problem for Oskolkov's system of equations, *Mathematical Methods in Applied Sciences*, **40**(2015), 6123–6126.

[5] A.I.Kozhanov, On the solvability of the coefficient inverse problems for equations of Sobolev type (O razreshimosti koeffitsientnikh obratnikh zadach dlya nekotorikh uravnenii sobolevskogo tipa), *Nauchniye vedomosti Belgorodskogo gosudarstvennogo universiteta. Seriya "Matematika. Phizika"*, **5**(2010), 88–98 (in Russian).

[6] T.-T.Li, L.W.White, Total Flux (Nonlocal) Boundary Value Problems for Pseudoparabolic Equation, *Appl. Anal.*, **16**(1983), 17–31.

[7] J.-L.Lions, E.Magenes, *Problemes aux Limites Non Homogenes et Applications*, Travaux et Recherches Mathematiques, No. 17, Vol. 1, Paris: Dunod, 1968.

[8] A.Lorenzi, E.Paparoni, Identification problems for pseudoparabolic integrodifferential operator equations, *J. Inver. Ill-Posed Probl.*, **5**(1997), 235–253.

[9] A.Sh.Lyubanova, On the Approximation of a Parabolic Inverse Problem by Pseudoparabolic One, *Journal of Siberian Federal University. Mathematics and Physics*, **5**(2012), 326–336.

[10] A.Sh.Lyubanova, A.Tani, On Inverse Problems for Pseudoparabolic and Parabolic Equations of Filtration, *Inverse Probl. Sci. En.*, **19**(2011), 1023–1042.

[11] A.Sh.Lyubanova, A.Tani, An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity, *Appl. Anal.*, **90**(2011), 1557–1571.

[12] A.Sh.Lyubanova, A.Tani, An inverse problem for pseudoparabolic equation of filtration. The stabilization, *Appl. Anal*, **92**(2013), 573–585.

[13] A.I.Prilepko, D.G.Orlovsky, I.A.Vasin, Methods for solving inverse problems in mathematical physics, "Marcel Dekker, Inc.", New York, 2000.

[14] M.Sh.Mamayusupov, The problem of determining coefficients of a pseudoparabolic equation (O zadache opredeleniya koeffitsiyentov psevdoparabolicheskogo uravneniya), *Studies in integro-differential equations*, Ilim, Frunze, 1983, no. 16, 290–297 (in Russian).

[15] C.G.Pyatkov, C.N.Shergin, On Some Inverse Coefficient Problems with the Pointwise Overdetermination for Mathematical Models of Filtration, *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software (Bulletin SUSU MMCS)*, **12**(2019), 82–95.

[16] W.Rundell, Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data, *Appl. An.*, **10**(1980), 231–242.

[17] W.Rundell, M.Stecher, The nonpositivity of solutions to pseudoparabolic equations, *Proc. Amer. Math. Soc.*, **75**(1979), 251–254.

[18] W.Rundell, M.Stecher, Maximum Principles for Pseudoparabolic Partial Differential Equations, *J. Math. Anal. Appl.*, **57**(1977), 110–118.

[19] R.E.Showalter, T.W.Ting, Pseudoparabolic partial differential equations, *SIAM J. Math. Ana*, **1**(1970), 1–26.

[20] T.W.Ting, Certain Non-Steady Flows of Second-order Fluids, *Arch. Ration Mech. An.*, **14**(1963), 1–26.

Обратная задача для псевдопараболического уравнения со смешанным граничным условием

Анна Ш. Любанова
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. В данной статье исследуется обратная задача идентификации старшего коэффициента в псевдопараболическом уравнении со смешанным граничным условием. Неизвестный коэффициент восстанавливается по дополнительным интегральным граничным данным. Доказано существование и единственность сильного обобщенного решения. Результат связан с идентификацией гидравлических свойств трещиноватой среды.

Ключевые слова: фильтрация, обратная задача, псевдопараболическое уравнение, существование, единственность.