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Series of Hypergeometric Type and Discriminants

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Abstract. The monomial of solutions of a reduced system of algebraic equations are series of hypergeometric type. The Horn–Karpranov result for hypergeometric series is extended to the case of series of hypergeometric type.

Keywords: series of hypergeometric type, logarithmic Gauss map, discriminant locus, reduced system, conjugative radii of convergence.

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1. Introduction and preliminaries

Hypergeometric functions were studied in the 19th century by many famous mathematicians such as L. Euler, C. F. Gauss, E. Kummer, B. Riemann. Most of the researches were on one variable series. At the end of 19th and the first half of 20th century the hypergeometric functions were widespread considered, including several variables cases. Among them are the functions studied by G. Lauricella [11], J. Horn [8], P. Appell [3] (see also the books [4, 5]). The hypergeometric functions are still attractive recently (see [2, 6, 13, 14]). According to Horn [8] the series

$$H(x_1, \ldots, x_N) = \sum_{\alpha \in \mathbb{N}^N} c_\alpha x^\alpha \tag{1}$$

is called *hypergeometric* if the relations of neighboring coefficients

$$\mathfrak{h}_i(\alpha) = \frac{c_{\alpha+e_i}}{c_\alpha}, \quad i = 1, \ldots, N, \tag{2}$$

(where the set of e_i composes the standard basis in $\mathbb{Z}^N$), are rational functions in variables $\alpha = (\alpha_1, \ldots, \alpha_N)$. Limit values of functions $\mathfrak{h}_i$ along fixed directions $s = (s_1, \ldots, s_N) \in \mathbb{R}^N \setminus \{0\}$

$$\mathfrak{P}_i(s) := \lim_{k\to\infty} \mathfrak{h}_i(ks)$$

play an important role. We call the vector limit

$$\frac{1}{\mathfrak{P}(s)} = \left(\frac{1}{\mathfrak{P}_1(s)}, \ldots, \frac{1}{\mathfrak{P}_N(s)}\right)$$

the *Horn parameterization* or *Horn uniformization* for the hpereometric series (1). These vectors define the conjugative radii of convergence for the series (1) (about the conception of these radii see [16, Sec. 7, ch. 1]).

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In this paper we study the *hypergeometric type series*. Roughly speaking, these series satisfy the following conditions: there is a sublattice $L \subset \mathbb{Z}^N$ of rank N such that the restriction of H on the shifts of L are hypergeometric. The details about the hypergeometric type series refer to the Section 3.

We are interested in the hypergeometric type series in order to investigate the solutions to universal systems of polynomial equations. In particular, we intend to apply the discriminant apparatus considered here to the calculation of the convergence domain of these series.

Consider a general system of n polynomial equations with n unknowns $y_1, \dots, y_n$:

$$P_i := \sum_{\lambda \in A^{(i)}} a_\lambda^{(i)} y^\lambda = 0, \quad i = 1, \dots, n, \tag{3}$$

where $A^{(i)}$ are the finite subsets of $\mathbb{Z}^n$ and $y^\lambda = y_1^{\lambda_1} \dots y_n^{\lambda_n}$. We assume that all coefficients $a_\lambda^{(i)}$ are independent, and call (3) an *universal algebraic system.* Applying the Stepanenko's formula (see [10]) we get the hypergeometric type series presenting the monomials with positive integer exponents of the *principal solution* to the system (4).

We will explicit the relation between the Horn parameterization $\dfrac{1}{\mathfrak{P}(s)}$ for these series and the parameterization Ψ of the discriminant locus ∇ of the system (4) (see more about Ψ and ∇ in Section 2.). According to result in [1], the parameterization Ψ is the inverse of the *logarithmic Gauss map* for ∇. (The logarithmic Gauss map $\gamma : \nabla \subset \mathbb{C}^N \to \mathbb{CP}^{N-1}$ for a hypersurface ∇, defined by polynomial P, can be defined by the formula

$$(z_1, \dots, z_n) \longmapsto (z_1 \partial_1 P(z) : \dots : z_n \partial_n P(z)),$$

where ∂_j is the derivative $\partial/\partial z_i$ (see [9,12])).

According to Kapranov's result in [9], the Horn parameterization $\dfrac{1}{\mathfrak{P}}$ for the hypergeometric series coincides with the parameterization $\Psi = \gamma^{-1}$ of the discriminant locus ∇. The following theorem gives an extension of the Kapranov's result in [9] for the series of hypergeometric type representing monomials of solutions to the reduced system (4).

Theorem 1. *The Horn parameterization* $\dfrac{1}{\mathfrak{P}(s)}$ *for the series (6) and the parameterization* $\Psi(s)$ *of discriminant set for the system (4) coincide:*

$$\Psi = \frac{1}{\mathfrak{P}}.$$

2. Reduced systems and their discriminants

Following the paper [1] we consider the reduced system of the system (3) in the forms

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)}} x_\lambda^{(j)} y^\lambda - 1 = 0, \quad j = 1, \dots, n, \tag{4}$$

where each m_j is a positive integer and $\Lambda^{(j)}$ does not contain $\lambda = 0$ and $\lambda = (0, \dots, m_j, \dots, 0)$.

Denote by ∇^0 the set of all the coefficients for which the system (3) has multiple zeros in the torus $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$, i.e. the Jacobian of P equals zero. The *discriminant locus* ∇ of the system (3) is the closure of the set ∇^0 in the space of coefficients of polynomials $P_1, \dots, P_n$.

Denote the matrix

$$\Lambda := (\Lambda^{(1)}, \dots, \Lambda^{(n)}) = (\lambda_1, \dots, \lambda_N),$$

where $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k)^T \in \Lambda^{(j)}$ are column-vector of exponents in monomials of equations (4). Also let ω_m denote the $n \times n$-diagonal matrix with values $\dfrac{1}{m_j}$ on the diagonal. Consider the matrices

$$\Phi := \omega\Lambda, \quad \tilde{\Phi} := \Phi - \chi,$$

where χ is the matrix, whose i-th row is assigned by the characteristic function of the subset $\Lambda^{(i)} \subset \Lambda$, i.e. elements of this row are 1 at the position $\lambda \in \Lambda^{(i)}$ and 0 at all positions $\lambda \in \Lambda \backslash \Lambda^{(i)}$. In addition, φ_k denotes the rows of Φ, and $\tilde{\varphi}_k$ denotes the rows of $\tilde{\Phi}$. Their elements are denoted by $\varphi_{k\lambda}$ and $\tilde{\varphi}_{k\lambda}$ correspondingly. We can interpret each row φ_k as a sequence of vectors $\varphi_k^{(1)}, \ldots, \varphi_k^{(n)}$.

We will follow two copies of $\mathbb{C}^N$. The first one is $\mathbb{C}_x^\Lambda$ with the coordinators $x = (x_\lambda)$, and the second one is $\mathbb{C}_s^\Lambda$ with the coordinators $s = (s_\lambda)$ constructed as a space with homogeneous coordinators for $\mathbb{CP}^{N-1}$. Following the result of Antipova and Tsikh (see [1]), the map

$$\Psi : \mathbb{CP}_s^{N-1} \to \mathbb{C}_x^N = \mathbb{C}_{x^{(1)}}^{\Lambda^{(1)}} \times \cdots \times \mathbb{C}_{x^{(n)}}^{\Lambda^{(n)}},$$

from a projective space to the space of coefficients $x = (x_\lambda)$ of the system (4), defined by

$$x_\lambda^{(j)} = -\frac{s_\lambda^{(j)}}{\langle \tilde{\varphi}_j, s \rangle} \prod_{k=1}^{n} \left(\frac{\langle \tilde{\varphi}_k, s \rangle}{\langle \varphi_k, s \rangle} \right)^{\varphi_{k\lambda}}, \quad \lambda \in \Lambda^{(j)}, \ j = 1, \ldots, n, \tag{5}$$

gives the parameterization for the discriminant locus ∇.

3. Solutions to reduced systems of algebraic equations

For the solution $y = (y_1, \ldots, y_n)$ to (4), we consider the series representing the monomial function $y^\mu = y_1^{\mu_1} \ldots y_1^{\mu_n}$

$$y^\mu = \sum_{\alpha \in \mathbb{N}^N} c_\alpha x^\alpha. \tag{6}$$

We focus on the so-called *principal solution* to system (4): they satisfy initial condition $y(0, \ldots, 0) = (1, \ldots, 1)$. When $\mu_j > 0$ the Stepanenko's result [10] claims that the coefficients c_α in (6) admit the following expression:

$$c_\alpha = (-1)^{\alpha_1 + \cdots + \alpha_N} \cdot \Gamma_\alpha \cdot \mathrm{R}_\alpha, \tag{7}$$

where

$$\begin{aligned} \Gamma_\alpha &= \frac{\prod\limits_{j=1}^{n} \Gamma\left(\frac{\mu_j + m_j}{m_j} + \langle \varphi_j, \alpha \rangle\right)}{\prod\limits_{i=1}^{N} \Gamma(\alpha_i + 1) \prod\limits_{j=1}^{n} \Gamma\left(\frac{\mu_j + m_j}{m_j} + \langle \varphi_j, \alpha \rangle - \sum\limits_{i \in \Lambda^{(j)}} \alpha_i\right)}, \\ \mathrm{R}_\alpha &= \det \left(\delta_i^{(j)} - \frac{\langle \varphi_i^{(j)}, \alpha^{(j)} \rangle}{\mu_j + \langle \varphi_j, \alpha \rangle} \right)_{(i,j) \in P_\alpha \times P_\alpha}, \end{aligned} \tag{8}$$

with $P_\alpha \subset \{1, \ldots, n\}$. We call Γ_α the *gamma-part* and R_α the *rational-part* of the coefficient c_α.

Remark that according to expressions (7) and (8) c_α admits the expression

$$c_\alpha = t^\alpha \, \mathrm{R}(\alpha) \prod_{j=1}^{M} \Gamma(\langle a_j, \alpha \rangle + b_j),$$

where $t^\alpha = t_1^{\alpha_1} \ldots t_N^{\alpha_N}$, $t_i, b_i \in \mathbb{C}$, $a_j \in \mathbb{Q}^N$, and $\mathrm{R}(\alpha)$ is a rational function. In the case when $a_j \in \mathbb{Z}^N$ this expression presents the general coefficient Ore-Sato for hypergeometric series (see [7, 15]).

4. Horn parameterization for hypergeometric type series

Here we give more details for the definition of the series of hypergeometric type and construct for them the analog of the Horn parameterization. Let $e_1, \dots, e_N$ denote the standard basis of $\mathbb{Z}^N$, i.e. $e_\lambda = (0, \dots, 1, \dots, 0)$ with 1 being on λ-th position. For a given $\nu = (\nu_1, \dots, \nu_N) \in (\mathbb{N}\backslash\{0\})^N$ we consider the sublattice $L_\nu \subset \mathbb{Z}^N$ generated by $\nu_1 e_1, \dots, \nu_N e_N$. For two vectors $\nu, s \in \mathbb{Z}^N$ we define their product $\nu s := (\nu_1 s_1, \dots, \nu_N s_N)$.

Definition 1. We say that the power series

$$\sum_{\alpha\in\mathbb{N}^N} c_\alpha x^\alpha \tag{9}$$

is of *hypergeometric type* if there exists $\nu \in (\mathbb{N} \setminus \{0\})^N$ such that all subseries

$$H_l := \sum_{\alpha\in l+L_\nu\cap\mathbb{N}^N} c_\alpha x^\alpha = t^{\frac{l}{\nu}} \sum_{s\in\mathbb{N}^N} c'_s t^s, \quad l \in J,$$

are hypergeometric in variables $t_\lambda = x_\lambda^{\nu_\lambda}$, where $c'_s = c_{l+\nu s}$ and J is the sequence of all representatives for the factor $\mathbb{Z}^N/L_\nu$:

$$J = \{(l_1, \dots, l_N) \in \mathbb{Z}^n : 0 \leqslant l_i \leqslant \nu_i - 1, i = 1, \dots, N\}.$$

The subseries H_l is hypergeometric iff all the relations

$$\mathfrak{R}_\lambda(s) := \frac{c'_{s+e_\lambda}}{c'_s}, \quad \lambda = 1, \dots, N, \tag{10}$$

are rational functions of variables $s = (s_1, \dots, s_N)$.

Proposition 1. *The series (6) with the coefficient (7) is a hypergeometric type series.*

Proof. For a vector $\nu \in (\mathbb{N} \setminus \{0\})^N$ we take $\nu = (\tau, \dots, \tau)$ where τ is the least common multiple of $m_1, \dots, m_N$.

According to (8), the relations (10) become

$$\mathfrak{R}_\lambda(s) = \frac{c_{l+\nu(s+e_\lambda)}}{c_{l+\nu s}} = \frac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}} \frac{(-1)^\tau\, \mathrm{R}_{l+\tau s+\tau e_\lambda}}{\mathrm{R}_{l+\tau s}}, \quad \lambda = 1, \dots, N,\ l \in J, \tag{11}$$

where $J = \{l = (l_1, \dots, l_N)\ :\ 0 \leqslant l_1, \dots, l_N \leqslant \tau - 1\}$. The power of the exponent $(-1)^\tau$ comes from

$$\frac{(-1)^{|l+\tau(s+e_\lambda)|}}{(-1)^{|l+\nu s|}} = (-1)^{|\tau e_\lambda|} = (-1)^\tau,$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_N$.

Here $l + \tau s$ denotes the restriction of α on the shifted lattice $l + L_\nu$ (i.e. $\alpha =: l + \tau s$ for some $l \in J$). Thus $\Gamma_{l+\tau s}$ and $\mathrm{R}_{l+\tau s}$ are correspondingly the restrictions of the gamma-part Γ_α and the rational-part R_α of the series (6) on the such lattice. It is clear that the second ratio in (11), the ratio for R_α, is a rational function in s.

Introduce denotations

$$A_k := \varphi_k = (\varphi_{k1}, \dots, \varphi_{kN}), \quad A_{n+k} := \tilde{\varphi}_k = (\tilde{\varphi}_{k1}, \dots, \tilde{\varphi}_{kN}), \quad A_{2n+\lambda} := e_\lambda,$$

and rewrite (8) in such a way

$$\Gamma_\alpha = \frac{\prod\limits_{p=1}^{n} \Gamma(\langle A_p, \alpha\rangle + \eta_p)}{\prod\limits_{p=n+1}^{2n} \Gamma(\langle A_p, \alpha\rangle + \eta_p) \prod\limits_{p=2n+1}^{2n+N} \Gamma(\langle A_p, \alpha\rangle + 1)},$$

where η_p are some constants independing on α. To compute the ratio of gamma-parts in (11) we use the Pochhammer symbol

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)} = z(z+1)\dots(z+k-1), \quad k \in \mathbb{N} \setminus \{0\}$$

and the denotation $q_p^\lambda := \langle A_p, \tau e_\lambda \rangle$. Then it leads to

$$\frac{\Gamma_{\alpha+\tau e_\lambda}}{\Gamma_\alpha} = \frac{\prod\limits_{p=1}^{n} (\langle A_p, \alpha \rangle + \eta_p - 1 + q_p^\lambda)_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n} (\langle A_p, \alpha \rangle + \eta_p - 1 + q_p^\lambda)_{q_p^\lambda} \prod\limits_{p=2n+1}^{2n+N} (\langle A_p, \alpha \rangle + q_p^\lambda)_{q_p^\lambda}}.$$

With $\alpha = l + \tau s$, we get the ratio of gamma-parts restricted on the shifted lattice $l + L_\nu$:

$$\frac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}} = \frac{\prod\limits_{p=1}^{n} (\langle \tau A_p, s \rangle + \eta'_p + q_p^\lambda)_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n} (\langle \tau A_p, s \rangle + \eta'_p + q_p^\lambda)_{q_p^\lambda} \prod\limits_{p=2n+1}^{2n+N} (\tau \langle e_\lambda, s \rangle + l_\lambda + q_p^\lambda)_{q_p^\lambda}}, \tag{12}$$

where constants η'_p are independent on s.

Since m_j divide τ, the delation τA_p in (12) is a vector with integer coordinators. Then its turns out that the relation $\dfrac{\Gamma_{l+\tau s+\tau e_\lambda}}{\Gamma_{l+\tau s}}$ in (11) is a rational function of the variables $s_1, \dots, s_N$. Thus the series (6) is of hypergeometric type. $\square$

According to Horn (see [8]) the convergence radii of hypergeometric series are defined by the limits

$$\lim_{r\to\infty} \mathfrak{h}_i(rs), \quad i = 1, \dots, N,$$

where the rational functions $\mathfrak{h}_i$ are defined by (2). In the hypergeometric type case, the convergence radii of the series (9) are defined by the limits

$$\mathfrak{P}_\lambda(s_1, \dots, s_N) = \lim_{r\to\infty} \left(\mathfrak{R}_\lambda(rs)\right)^{\frac{1}{\tau}}, \quad \lambda = 1, \dots, N, \tag{13}$$

where $\mathfrak{R}_\lambda$ are rational relations (10) and τ is the least common multiple of $\nu_1, \dots, \nu_N$, $(s_1 : \dots : s_N) \in \mathbb{RP}^{N-1}$, $s_i > 0$. Indeed $(s_1, \dots, s_N)$ are homogeneous coordinates in $\mathbb{CP}^{N-1}$, and the limits $\mathfrak{P}_i$ are rational and homogeneous of degree zero. They depend only on the ratio $s = s_1 : \dots : s_N$. The vector limit

$$\frac{1}{\mathfrak{P}(s)} := \left(\frac{1}{\mathfrak{P}_1(s)}, \dots, \frac{1}{\mathfrak{P}_N(s)}\right)$$

are called by *Horn parameterization* (or *Horn uniformation*) for hypergeometric type series since Horn is the first person who considered such a limit for hypergeometric function (see [9]).

5. The proof of the Theorem 1

According to (12) we get the following formula for the limit values of relation (11) along direction $s := (s_1, \dots, s_N) \in \mathbb{R}^N \setminus \{0\}$.

Proposition 2.

$$\mathfrak{P}_\lambda(s_1,\dots,s_N) = -\frac{\langle\tilde{\varphi}_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle\varphi_p, s\rangle}{\langle\tilde{\varphi}_p, s\rangle}\right)^{\varphi_{p\lambda}}. \tag{14}$$

Proof. From the ratio (12) and the limits (13),

$$\mathfrak{P}_\lambda(s_1,\dots,s_N) := \lim_{r\to\infty}\left[\frac{c_{l+\tau rs+\tau e_\lambda}}{c_{l+\tau rs}}\right]^{\frac{1}{\tau}} = \lim_{k\to\infty}\left[\frac{\Gamma_{l+\tau rs+\tau e_\lambda}}{\Gamma_{l+\tau rs}}\frac{(-1)^\tau\,\mathrm{R}_{l+\tau rs+\tau e_\lambda}}{\mathrm{R}_{l+\tau rs}}\right]^{\frac{1}{\tau}} =$$

$$=: -\lim_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}},$$

where

$$A := \frac{\prod\limits_{p=1}^{n}(\langle rA_p, s\rangle + \eta'_p + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}(\langle rA_p, s\rangle + \eta'_p + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}\prod\limits_{p=2n+1}^{2n+N}(r\langle e_\lambda, s\rangle + \frac{l_\lambda}{\tau} + \frac{q_p^\lambda}{\tau})_{q_p^\lambda}},$$

$$B := \frac{\tau^{q_1^\lambda+\cdots+q_n^\lambda}}{\tau^{q_{n+1}^\lambda+\cdots+q_{2n}^\lambda}\cdot\tau^{q_{n+1}^\lambda+\cdots+q_{2n+N}^\lambda}},$$

$$C := \frac{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)}, l^{(j)}+\tau e_\lambda^{(j)}\rangle + \langle\varphi_i^{(j)}, \tau rs^{(j)}\rangle}{\mu_j+\langle\varphi_j, l+\tau e_\lambda\rangle+\langle\varphi_j,\tau rs\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)}, l^{(j)}\rangle + \langle\varphi_i^{(j)}, \tau rs^{(j)}\rangle}{\mu_j+\langle\varphi_j, l\rangle+\langle\varphi_j,\tau rs\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}.$$

Recall that

$$\begin{aligned} &A_k = (\varphi_{k1},\dots,\varphi_{kN}),\quad A_{n+k} = (\tilde{\varphi}_{k1},\dots,\tilde{\varphi}_{kN}),\quad A_{2n+\lambda} = e_\lambda,\\ &q_p^\lambda = \langle A_p, \tau e_\lambda\rangle,\quad p\in\{1,\dots,2n+N\}.\end{aligned} \tag{15}$$

Since $\tau e_\lambda = (0,\dots,\tau,\dots,0)$ with $\lambda\in\{1,\dots,N\}$,

$$q_p^\lambda = \begin{cases}\tau\varphi_{p\lambda} & \text{with } 1\leqslant p\leqslant n,\\ \tau\tilde{\varphi}_{(p-n)\lambda} & \text{with } n+1\leqslant p\leqslant 2n,\\ \tau & \text{with } p=2n+\lambda,\\ 0 & \text{with } p>2n \text{ and } p\neq 2n+\lambda.\end{cases} \tag{16}$$

Thus

$$q_1^\lambda+\cdots+q_n^\lambda = (\varphi_{1\lambda}+\cdots+\varphi_{n\lambda})\tau,\ q_{2n+1}^\lambda+\cdots+q_{2n+N}^\lambda = \tau, \tag{17}$$

and with the notice that $\lambda\in\Lambda^{(j)}$ for some j,

$$q_{n+1}^\lambda+\cdots+q_{2n}^\lambda = (\tilde{\varphi}_{1\lambda}+\cdots+\tilde{\varphi}_{n\lambda})\tau = (\varphi_{1\lambda}+\cdots+\varphi_{n\lambda}-1)\tau. \tag{18}$$

The sums in (17) and (18) lead to

$$B = \frac{\tau^{(\varphi_{1\lambda}+\cdots+\varphi_{n\lambda})\tau}}{\tau^{(\varphi_{1\lambda}+\cdots+\varphi_{n\lambda}-1)\tau}\cdot\tau^\tau} = 1,$$

Let r tend to the infinity we obtain the limits:

$$\lim_{r\to\infty}A = \frac{\prod\limits_{p=1}^{n}\langle A_p, s\rangle^{q_p^\lambda}}{\prod\limits_{p=n+1}^{2n}\langle A_p, s\rangle^{q_p^\lambda}\,\langle e_\lambda, s\rangle^\tau},$$

$$\lim_{r\to\infty} C = \frac{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)},\tau s^{(j)}\rangle}{\langle\varphi_j,\tau s\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}}{\det\left(\delta_i^{(j)} - \frac{\langle\varphi_i^{(j)},\tau s^{(j)}\rangle}{\langle\varphi_j,\tau s\rangle}\right)_{(i,j)\in P_\alpha\times P_\alpha}} = 1.$$

Thus

$$\lim_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}} = \left[\frac{\prod\limits_{p=1}^{n}\langle A_p, s\rangle^{q_p^\lambda}}{\langle e_\lambda, s\rangle^{\tau}\prod\limits_{p=n+1}^{2n}\langle A_p, s\rangle^{q_p^\lambda}}\right]^{\frac{1}{\tau}}.$$

Substitute coordinators of the vectors A_p formulated in (15) and the value of q_p^α in (16), then the limit $\lim\limits_{r\to\infty}\Big(A\cdot B\cdot C\Big)^{\frac{1}{\tau}}$ equals

$$\left[\frac{\prod\limits_{p=1}^{n}\langle\varphi_p, s\rangle^{\tau\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle^{\tau}\prod\limits_{p=n+1}^{2n}\langle\tilde{\varphi}_{p-n}, s\rangle^{\tau\tilde{\varphi}_{(p-n)\lambda}}}\right]^{\frac{1}{\tau}}.$$

In the square brackets each factor is an exponentiation with the power τ. The radical $\dfrac{1}{\tau}$ applying on the square brackets leads to a simpler expression for the limit:

$$\frac{\prod\limits_{p=1}^{n}\langle\varphi_p, s\rangle^{\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle\prod\limits_{p=n+1}^{2n}\langle\tilde{\varphi}_{p-n}, s\rangle^{\tilde{\varphi}_{(p-n)\lambda}}}.$$

Rewrite the index for the production in the denominator of the last expression, it will become

$$\frac{\langle\tilde{\varphi}_j, s\rangle\prod\limits_{p=1}^{n}\langle\varphi_p, s\rangle^{\varphi_{p\lambda}}}{\langle e_\lambda, s\rangle\prod\limits_{p=1}^{n}\langle\tilde{\varphi}_p, s\rangle^{\varphi_{p\lambda}}}.$$

Combining the factors with the same index under the production signs in the numerator and in the denominator of the last expression we will get the result:

$$\frac{\langle\tilde{\varphi}_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle\varphi_p, s\rangle}{\langle\tilde{\varphi}_p, s\rangle}\right)^{\varphi_{p\lambda}}.$$

Consequently we get the formula for the limit $\mathfrak{P}_\lambda$:

$$\mathfrak{P}_\lambda(s_1,\ldots,s_N) = -\frac{\langle\tilde{\varphi}_j, s\rangle}{\langle e_\lambda, s\rangle}\prod_{p=1}^{n}\left(\frac{\langle\varphi_p, s\rangle}{\langle\tilde{\varphi}_p, s\rangle}\right)^{\varphi_{p\lambda}}.$$

The proposition holds. $\square$

Now we are ready to prove the Theorem 1.

Proof of theorem 1. From (5) and (14) it turns out that

$$x_\lambda^{(j)} = \frac{1}{\mathfrak{P}_\lambda}, \quad \lambda \in \Lambda^{(j)}, \ j = 1, \ldots, n.$$

Thus the parameterization $\Psi(s)$ for the discriminant locus ∇ of the system (4) composed by the coordinators $x_\lambda^{(j)}$ coincides with the limit vector of the hypergeometeric type series (6) composed by the coordinators $\dfrac{1}{\mathfrak{P}_\lambda}$. The theorem holds. □

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Ряды гипергеометрического типа и дискриминанты

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Аннотация. Одночлен решений редуцированной системы алгебраических уравнений представляет собой ряд гипергеометрического типа. Мы распространяем результат Хорна-Карпранова (для гипергеометрических рядов) на случай рядов гипергеометрического типа.

Ключевые слова: ряды гипергеометрического типа, логарифмическое отображение Гаусса, дискриминантное множество, редуцированная система, сопряженные радиусы сходимости.