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Local Asymptotic Normality of Statistical Experiments in an Inhomogeneous Competing Risks Model under Random Censoring on the Right

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Abstract. The local asymptotic normality property of the likelihood ratio statistic in the competing risk model that corresponds to inhomogeneous and randomly right-censored observations is proved in the paper.

Keywords: local asymptotic normality, likelihood ratio statistic, competing risk model, random censoring, asymptotic representation.

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1. Introduction and preliminaries

There are many works on the study of asymptotic properties of the likelihood ratio statistics (LRS) for full samples. It was shown that local asymptotic normality (LAN) allows one to develop an asymptotic theory of maximum likelihood estimates and Bayesian estimates, as well as the contiguity of families of probability distributions [1–4]. The study of similar properties in the case of incomplete — censored observations is of considerable interest. Effective estimates for the unknown parameter were obtained from censored observations when the distribution of the censoring random variable also depends on the unknown parameter [5]. The properties of local asymptotic normality for LRS were established in some models of censoring observations in the presence of competing risks [6–8]. The properties of local asymptotic normality of the likelihood ratio statistic in the competing risks model under random censoring on the right are studied in this paper.

Let us consider a inhomogeneous competing risks model (CRM). Let $\{X_m, m \geqslant 1\}$ be a sequence of random variables (r.v.) defined on a probability space (Ω, A, P) with distribution functions (d.f.) $H(x;\theta)$, $\theta \in \Theta \subseteq R^1$ with values in a measurable space $(X_m,\, B_m)$. The joint properties of pairs $\left(X_m, A_m^{(i)}\right)$, $i=\overline{1,k}$; are of interest, where $A_m^{(1)}, \ldots, A_m^{(k)}$ — pairwise disjoint events $P\left(\bigcup_{i=1}^{k} A_m^{(i)}\right) = 1$. Let $\delta_m^{(i)} = I\left(A_m^{(i)}\right)$ be an indicator of the event $A_m^{(i)}$, $i=\overline{1,k}$; $m \geqslant 1$. Suppose that set $\left(X_m,\, A_m^{(1)}, \ldots, A_m^{(k)}\right)$ is randomly censored from the right by an r.v. Y with

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continuous d.f. K. Observation is an available set $\left(Z_m;\ B_m^{(0)}, B_m^{(1)}, \dots, B_m^{(k)}\right)$, where $Z_m = \min(X_m, Y)$, events $B_m^{(0)} = \{\omega : Y(\omega) \leqslant X_m(\omega)\}$ and $B_m^{(i)} = A_m^{(i)} \cap \{\omega : X_m(\omega) \leqslant Y\}$, $i = \overline{1,k};\ m \geqslant 1$. Let $\left\{X_m,\, Y_m;\, B_m^{(0)}, B_m^{(1)}, \dots, B_m^{(k)}\right\}_{m=1}^{\infty}$ be a sequence of independent copies of population $\left\{X_m,\, Y;\, B_m^{(0)}, B_m^{(1)}, \dots, B_m^{(k)}\right\}_{m=1}^{\infty}$ and there is a sample $\widetilde{Z}^{(n)} = \left(\widetilde{Z}_1, \dots, \widetilde{Z}_n\right)$ in the n-step of the experiment, where $\widehat{Z}_m = \left\{Z_m;\, \Delta_m^{(0)}, \Delta_m^{(1)}, \dots, \Delta_m^{(k)}\right\}$, $Z_m = \min(X_m, Y_m)$, $\Delta_m^{(i)} = I\left(B_m^{(i)}\right)$, $i = 0, 1, \dots, k$. Let us note that considering sample $\widehat{Z}^{(n)}$ the pairs $\left(X_m, A_m^{(i)}\right)$ are observable only in the case of $\Delta_m^{(i)} = 1$, $i = \overline{1,k}$; $m = \overline{1,n}$. It is easy to see that r.v. have d.f. where d.f. is interfering.

Let us introduce sub-distributions

$$M_m^{(i)}(x;\theta) = P_\theta\left(Z_m < x,\, M_m^{(i)}\right), \quad i = 0, 1, \dots, k,$$

where

$$M_m^{(0)}(x;\theta) = P_\theta(Y_m \leqslant x \wedge X_m) = M_\theta\left[I(Y_j \leqslant x,\, X_m > Y_m)\right] =$$
$$= M_\theta\left\{M_\theta\left[I(X_m > Y_m/Y_m)\right] \cdot I(Y_m < x)\right\} = M_\theta\left[I(Y_m < x)(1 - H_m(Y_m;\theta))\right] =$$
$$= \int_{-\infty}^{x}(1 - H_m(u;\theta))dK(u),$$

and for $i = 1, \dots, k$

$$M^{(i)}(x;\theta) = P_\theta\left(X_n < x \wedge Y_m;\, A_m^{(i)}\right) = M_\theta\left[I\left(X_m < x;\, A_m^{(i)},\, Y_m > X_m\right)\right] =$$
$$= M_\theta\left\{M_\theta\left[I(Y_m > X_m/X_m)\right] \cdot I\left(Y_m < x;\, A_m^{(i)}\right)\right\} = M_\theta\left[I\left(X_m < x;\, A_m^{(i)}\right)(1 - K(X_m))\right] =$$
$$= \int_{-\infty}^{x}(1 - K(u))dH_m(u;i).$$

Then it is easy to see that integral intensity functions $\Lambda_m^{(i)}$ can be represented as

$$\Lambda_m^{(i)}(x;\theta) = \int_{-\infty}^{x}\frac{dM_m^{(i)}(u;\theta)}{1 - N_m(u;\theta)}, \quad m = \overline{1,n},\ i = \overline{1,k}.$$

Let $\left(Y^{(n)},\, U^{(n)},\, \widetilde{Q}_\theta^{(n)}\right)$ be a sequence of statistical experiments generated by observations $\widetilde{Z}^{(n)}$. Moreover, if the set of possible values of the r.v. Z is denoted by $\widetilde{Z}$ then we have

$$Y^{(n)} = \left\{\widetilde{Z} \otimes \{0,1\}^{(k+1)}\right\}^{(n)} = \left\{\overbrace{\widetilde{Z} \otimes \{0,1\}^{(k+1)} \otimes \cdots \otimes \widetilde{Z} \otimes \{0,1\}^{(k+1)}}^{n}\right\},$$

where $\{0,1\}^{(k+1)} = \underbrace{\{0,1\} \otimes \cdots \otimes \{0,1\}}_{k+1}$, $U^{(n)}$ is σ-algebra of Borel sets in $Y^{(n)}$, $Q_\theta^{(n)}$ distribution on $\left(Y^{(n)},\, U^{(n)}\right)$ is the n-fold product of "one-dimensional" distributions

$$\widetilde{Q}_{\theta m}\left(x, y^{(0)}, y^{(1)}, \dots, y^{(k)}\right) = P_\theta\left(Z_m < x,\, \Delta_m^{(0)} = y^{(0)}, \Delta_m^{(1)} = y^{(1)}, \dots, \Delta_m^{(k)} = y^{(k)}\right),$$

$x \in \overline{R}^1$, $y^{(i)} \in \{0,1\}$, $1 = \overline{1,k}$; $m = \overline{1;n}$, Θ is open set in R^1.

Let $h_m^{(i)}(x;\theta) = f_m^{(i)}(x;\theta) \prod\limits_{j\neq i} \left(1 - F_m^{(j)}(x;\theta)\right)$, $i = \overline{1,k}$; $m = \overline{1,n}$. Let us introduce the likelihood ratio statistics (LRS)

$$\frac{d\widetilde{Q}_{\theta_2}^{(n)}\left(\widetilde{Z}^{(n)}\right)}{d\widetilde{Q}_{\theta_1}^{(n)}\left(\widetilde{Z}^{(n)}\right)} = \prod_{m=1}^{n} \left\{ \prod_{i=1}^{k} \left[\frac{h_m^{(i)}(Z_m;\theta_2)}{h_m^{(i)}(Z_m;\theta_1)} \right] \right\}^{y_m^{(i)}} \cdot \left\{ \frac{1 - H_m(Z_m;\theta_2)}{1 - H_m(Z_m;\theta_1)} \right\}^{y_m^{(0)}},$$

and its logarithm

$$L_n(u) = \log \left\{ \frac{d\widetilde{Q}_{\theta_2}^{(n)}\left(\widetilde{Z}^{(n)}\right)}{d\widetilde{Q}_{\theta_1}^{(n)}\left(\widetilde{Z}^{(n)}\right)} \right\} = \sum_{m=1}^{n} \sum_{i=1}^{k} \int_{-\infty}^{\infty} \log \left[\frac{h_m^{(i)}(x;\theta_2)}{h_m^{(i)}(x;\theta_1)} \right] dI\left(Z_m < x,\, \Delta_m^{(i)} = 1\right) +$$

$$+ \sum_{m=1}^{n} \int_{-\infty}^{\infty} \log \cdot \left[\frac{1 - H_m(x;\theta_2)}{1 - H_m(x;\theta_1)} \right] dI\left(Z_m < x,\, \Delta_m^{(0)} = 1\right).$$

2. LAN of a family of probability measures

Let us now formulate the regularity conditions. If these conditions are fulfilled then one can establish the local asymptotic normality (LAN) of the family of distributions $\left\{\widetilde{Q}_\theta^{(n)},\, \theta \in \Theta\right\}$. For simplicity, consider the case of homogeneous distributions θ.

(C1) Supports $N_{h_m^{(i)}} = \left\{x : h_m^{(i)}(x;\theta) > 0\right\}$, $i = \overline{1,k}$; $m = \overline{1,n}$, are independent of parameter θ and $\bigcap\limits_{m=1}^{n} \bigcap\limits_{n=1}^{k} N_{h_m^{(i)}}$ is not empty.

(C2) For any two points $\theta_1,\, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, $h_m^{(i)}(x;\theta_1) \neq h_m^{(i)}(x;\theta_2)$.

(C3) There exist derivatives $\left\{ \dfrac{\partial^l h_m^{(i)}(x;\theta)}{\partial \theta^l},\, l{=}1,2;\ i{=}\overline{1,k};\ m{=}\overline{1,n} \right\}$, and they are finite for all x, while

$$\int_{-\infty}^{\infty} \left| \frac{\partial^l h_m^{(i)}(x;\theta)}{\partial \theta^l} \right| \nu_m(dx), \quad l = 1,2; \quad i = \overline{1,k}; \quad m = \overline{1,n}.$$

(C4) Fisher information $J(\theta) = \sum\limits_{m=1}^{n} J_m(\theta)$ is finite and positive, where

$$J_m(\theta) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\partial \log h_m^{(i)}(x;\theta)}{\partial \theta} \right)^2 h_m^{(i)}(x;\theta)\, \nu_m(dx) +$$

$$+ \int_{-\infty}^{\infty} \left(\frac{\partial \log\left(1 - H_m(x;\theta)\right)}{\partial \theta} \right)^2 \left(1 - H_m(x;\theta)\right) dx.$$

Let us note that according to (C3)

$$\left| \frac{\partial^2 \left(1 - H_m(x;\theta)\right)}{\partial \theta^l} \right| \leqslant \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left| \frac{\partial^l h_m^{(i)}(x;\theta)}{\partial \theta^l} \right| \nu_m(dx) < \infty, \quad l = 1,2.$$

The lemma on the equality to zero of the mean of the contribution of the sample is valid.

Lemma 2.1. *Let regularity conditions (C1)–(C3) are fulfilled. Then*

$$\sum_{i=1}^{k} M_\theta \left[\Delta_m^{(i)} \frac{\partial h_m^{(i)}(Z_m;\theta)}{\partial\theta} \right] + M_\theta \left[\Delta_m^{(0)} \frac{\partial \log\left(1 - H_m(Z_m;\theta)\right)}{\partial\theta} \right] = 0. \qquad (1)$$

Proof. At $l = 1, 2$ for all $\theta \in \Theta$

$$\sum_{l=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\partial^l h_m^{(i)}(x;\theta)}{\partial\theta^l} \right)^2 \nu_m(dx) + \int_{-\infty}^{\infty} \frac{\partial^l \left(1 - H_m(x;\theta)\right)}{\partial\theta^l} \nu_m(dx) = 0. \qquad (2)$$

This equality is a differentiated version of the identity

$$M_m^{(0)}(+\infty;\theta) + \sum_{i=1}^{k} M_m^{(i)}(+\infty;\theta) = H_m(+\infty;\theta) = 1.$$

Now (1) is a consequence of (2).

Let us introduce $\psi^2(n;\theta) = \sum\limits_{m=1}^{n} I_m(\theta)$, $\varphi(n) = \varphi(n,t) = \psi^{-1}(n,t)$, and formulate a theorem on the LAN of a family of probability measures $\left\{ \widetilde{Q}_\theta^{(n)},\, \theta \in \Theta \right\}$.

Theorem 1. *Let regularity conditions (C1)–(C3) are fulfilled for any $T > 0$*

$$\lim_{n\to\infty} \sup_{|u|<T} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} \left(\frac{\partial}{\partial\theta} \sqrt{h_m^{(i)}\left(x;\ t + \frac{u}{\psi(n;t)}\right)} - \frac{\partial}{\partial t} \sqrt{h_m^{(i)}(x;t)} \right)^2 dx = 0, \qquad (3)$$

$$\lim_{n\to\infty} \sup_{|u|<T} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} \int \left(\frac{\partial}{\partial\theta} \sqrt{1 - H_m\left(x;\ t + \frac{u}{\psi(n;t)}\right)} - \frac{\partial}{\partial t} \sqrt{1 - H_m(x;t)} \right)^2 dx = 0 \qquad (4)$$

and the Lindberg condition holds

$$\lim_{n\to\infty} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} \sum_{i=1}^{k} M_\theta \left\{ \left| \frac{\partial}{\partial t} \log h_m^{(i)}(X_m;t) \right| \cdot I\left(\left| \frac{\partial \log h_m(X_m;t)}{\partial t} \right| > n\psi(n;t) \right) \right\} = 0, \qquad (5)$$

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} M_\theta \Bigg\{ & \left| \frac{\partial}{\partial t} \log\left(1 - H_m(X_m;t)\right) \right| \times \\ & \times I\left(\left| \frac{\partial \log\left(1 - H_m(X_m;t)\right)}{\partial t} \right| > n\psi(n;t) \right) \Bigg\} = 0. \end{aligned} \qquad (6)$$

Then the family of probability measures

$$\widetilde{Q}_\theta^{(n)}(A) = \int \underset{A}{\cdots} \int \prod_{m=1}^{n} \left\{ \prod_{i=1}^{k} \left[h_m^{(i)}(Z_m;\theta) \right] \right\}^{y_m^{(i)}} \cdot \left[\left(1 - H_m(Z_m;\theta)\right) \right]^{y_m^{(0)}} \cdot \nu_m(dZ_m)$$

satisfies the LAN property at the point $\theta = t$.

Let us introduce

$$\Delta_{n,t} = \varphi(n,t) \sum_{m=1}^{n} \left[\sum_{i=1}^{k} \Delta_m^{(i)} \cdot \frac{\partial \log h_m^{(i)}(Z_m;t)}{\partial t} + \Delta_m^{(0)} \cdot \frac{\partial \log\left(1 - H_m(Z_m;\theta)\right)}{\partial t} \right] =$$

$$= \varphi(n;t)\sum_{m=1}^{n}\sum_{i=1}^{k}\Delta_m^{(i)}\cdot\frac{\partial\log h_m^{(i)}(Z_m;t)}{\partial t}+\varphi(n;t)\sum_{m=1}^{n}\Delta_m^{0}\frac{\partial\log(1-H_m(Z_m;\theta))}{\partial t}=$$
$$=\Delta_{n,t}^{(1)}+\Delta_{n,t}^{(2)},$$

also $\xi_{n,m}^{(i)}=\left[\frac{h_m^{(i)}(Z_m;t+\varphi(n)u)}{h_m^{(i)}(Z_m;t)}\right]^{1/2}-1$ and $\eta_{n,m}=\left[\frac{1-H_m(Z_m;t+\varphi(n)u)}{1-H(Z_m;t)}\right]^{1/2}-1$. Further, the following assertion is also necessary.

Lemma 2.2. *Suppose that conditions of Theorem 1 are hold then for any $u\in R^1$ we have*

$$\overline{\lim_{n\to\infty}}\sum_{m=1}^{n}M_t\left[\xi_{n,m}^{(i)}\right]^2\leqslant\frac{u^2}{4},\tag{7}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left[\eta_{n,m}^2\right]\leqslant\frac{u^2}{4},\tag{8}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left|\xi_{n,m}^{(i)}-\frac{1}{2}\varphi(n)u\cdot\frac{\partial\log h_m^{(i)}(Z_m;t)}{\partial t}\right|^2=0,\tag{9}$$

$$\lim_{n\to\infty}\sum_{m=1}^{n}M_t\left|\eta_{n,m}-\frac{1}{2}\varphi(n)u\cdot\frac{\partial\log(1-H_m(Z_m;t))}{\partial t}\right|^2=0.\tag{10}$$

Proof of Lemma 2.2. We have

$$\sum_{m=1}^{n}M_t\left[\xi_{n,m}^{(i)}\right]^2=\sum_{m=1}^{n}\int\limits_{\left\{x:\,h_m^{(i)}(x;\theta)\neq0\right\}}\left(\sqrt{h_m^{(i)}(x;\,t+\varphi(n)u)}-\sqrt{h_m^{(i)}(x;\,t)}\right)^2\cdot\nu_m(dx)\leqslant$$
$$\leqslant\sum_{m=1}^{n}\int\left(\int\limits_0^{\varphi(n)u}\frac{\frac{\partial}{\partial t}h_m^{(i)}(x;\,t+v)\,dv}{2\sqrt{h_m^{(i)}(x;\,t+\varphi(n)u)}}\right)^2\nu_m(dx)\leqslant\frac{u\varphi(n)}{4}\int\limits_0^{\varphi(n)u}\sum_{m=1}^{n}I_{m1}^{(i)}(t+v)\,dv,\tag{11}$$

where

$$I_{m1}^{(i)}(t)=\int_{-\infty}^{\infty}\left(\frac{\partial\log h_m^{(i)}(x;t)}{\partial t}\right)^2h_m^{(i)}(x;t)\,\nu_m(dx).$$

Also

$$\sum_{m=1}^{n}M_t\left[\eta_{n,m}\right]^2=\sum_{m=1}^{n}\int\limits_{\{x:\,H_m(x;t)=1\}}\left(\sqrt{1-H_m(x;t+\varphi(n)u)}-\sqrt{1-H_m(x;t)}\right)^2\nu_m(dx)\leqslant$$
$$\leqslant\sum_{m=1}^{n}\int\int\limits_0^{\varphi(n)u}\left(\frac{\frac{\partial}{\partial t}(1-H_m(x;\,t+v)\,dv)}{2\sqrt{(1-H_m(x;t+v))}}\right)^2\nu_m(dx)\leqslant\frac{u\varphi(n)}{4}\int\limits_0^{\varphi(n)u}\sum_{m=1}^{n}I_{m2}^{(i)}(t+v)\,dv,\tag{12}$$

where

$$I_{m2}^{(i)}(t)=\int_{-\infty}^{\infty}\left(\frac{\partial\log(1-H_m(x;t))}{\partial t}\right)^2(1-H_m(x;t))\,\nu_m(dx).\tag{13}$$

Next, using the inequality

$$|ab|<\alpha\cdot\frac{a^2}{2}+\frac{1}{2\alpha}b^2,$$

where $a_m = \dfrac{\frac{\partial}{\partial\theta} h_m^{(i)}(x;\theta)}{\sqrt{h_m^{(i)}(x;\theta)}}$ and $b_m = \dfrac{\frac{\partial}{\partial t} h_m^{(i)}(x;t)}{\sqrt{h_m^{(i)}(x;t)}}$ one can find

$$\left| \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} I_{m1}^{(i)}(\theta) - 1 \right| \leqslant \frac{1}{\psi^2(n;t)} \left| \int (a_m - b_m)(a_m + b_m)\,\nu_m(dx) \right| \leqslant$$

$$\leqslant \frac{1}{\psi^2(n;t)} \left[\alpha \sum_{m=1}^{n} \int (a_m - b_m)^2 \cdot \nu_m(dx) + \frac{1}{\alpha} \sum_{m=1}^{n} \left(\int a_m^2 \nu_m(dx) + \int b_m^2 \cdot \nu_m(dx) \right) \right].$$

Assuming $\alpha = 2$ in this inequality and taking into account (3) and the equality

$$\int a_m^2 \nu_m(dx) = I_{m1}^{(i)}(\theta),$$

we make sure that fraction $\dfrac{1}{\psi^2(n;t)} \displaystyle\sum_{m=1}^{n} I_{m1}^{(i)}(\theta)$ is bounded under the condition $|\theta - t| < \varphi(n)|u|$. Using this inequality for large enough α, we verify that

$$\lim_{n\to\infty} \sup_{|\theta-t|<\varphi(n)|u|} \left| \frac{1}{\psi^2(n;t)} \sum_{m=1}^{n} I_{m1}^{(i)}(\theta) - 1 \right|. \tag{14}$$

Now (7) follows from (11) and (13).

Similarly, setting in inequality (13) $a_m = \dfrac{\frac{\partial}{\partial\theta}(1 - H_m(x;\theta))}{\sqrt{(1 - H_m(x;\theta))}}$ and $b_m = \dfrac{\frac{\partial}{\partial t}(1 - H_m(x;t))}{\sqrt{(1 - H_m(x;t))}}$ and repeating all the inequalities, one can obtain

$$\lim_{n\to\infty} \sup_{|\theta-t|<\varphi(n)|u|} \left| \frac{1}{\psi^2(n;t)} \sum_{m=2}^{n} I_{m1}^{(i)}(\theta) - 1 \right| = 0 \tag{15}$$

which implies (8). It remains to prove (9) and (10). By virtue of (3) we have

$$\sum_{m=1}^{n} M_t \left(\xi_{n,m}^{(i)} - \frac{1}{2}\varphi(n)\,u \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right)^2 \leqslant$$

$$\leqslant \frac{1}{4} \sum_{m=1}^{n} \int \left[\int_0^{\varphi(n)u} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t+v)}{\sqrt{h_m^{(i)}(Z_m;t+v)}} - \frac{\frac{\partial}{\partial t} h_m^{(i)}(x_m;t)}{\sqrt{h_m^{(i)}(x_m;t)}} dv \right]^2 \nu_m(dx) \leqslant$$

$$\leqslant \frac{\varphi(n)\,u}{4} \int_0^{\varphi(n)u} dv \sum_{m=1}^{n} \int \left(\frac{\frac{\partial}{\partial t} h_m^{(i)}(x;t+v)}{\sqrt{h_m^{(i)}(x;t)}} - \frac{\frac{\partial}{\partial t} h_m^{(i)}(x;t)}{\sqrt{h_m^{(i)}(x;t)}} \right)^2 \nu_m(dx) \to 0, \quad n \to \infty.$$

Similarly, due to (4) we have

$$\sum_{m=1}^{n} M_t \left(\eta_{n,m} - \frac{1}{2}\varphi(n)\,u \cdot \frac{\frac{\partial}{\partial t}(1 - H_m(Z_m;t))}{1 - H_m(Z_m;t)} \right)^2 \leqslant$$

$$\leqslant \frac{1}{4} \sum_{m=1}^{n} \int \left[\int_0^{\varphi(n)u} \frac{\frac{\partial}{\partial t}(1 - H_m(x;t+v))}{\sqrt{(1 - H_m(x;t+v))}} - \frac{\frac{\partial}{\partial t}(1 - H_m(x;t))}{\sqrt{(1 - H_m(x;t))}} dv \right]^2 \nu_m(dx) \leqslant$$

$$\leqslant \frac{\varphi(n)u}{4}\int\limits_0^{\varphi(n)u} dv\sum_{m=1}^{n}\int\left(\frac{\frac{\partial}{\partial t}\left(1-H_m\left(x;t+v\right)\right)}{\sqrt{\left(1-H_m\left(x;t+v\right)\right)}}-\frac{\frac{\partial}{\partial t}\left(1-H_m\left(x;t\right)\right)}{\sqrt{\left(1-H_m\left(x;t\right)\right)}}\right)^2\nu_m\left(dx\right)\to 0,\, n\to\infty$$

which proves Lemma 2.2. □

Proof of Theorem 1. Under conditions

$$\max_{m=\overline{1,n}}\left|\xi_{n,m}^{(i)}\right|<\varepsilon \quad\text{and}\quad \max_{m=\overline{1,n}}\left|\eta_{n,m}^{(i)}\right|<\varepsilon$$

we obtain

$$\sum_{m=1}^{n}\log\left[\frac{h_m^{(i)}\left(Z_m;t+\varphi\left(n\right)u\right)}{h_m\left(Z_m;t\right)}\right]=2\log\left(1+\xi_{n,m}^{(i)}\right)=2\sum_{m=1}^{n}\xi_{n,m}^{(i)}-\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2+\sum_{m=1}^{n}\gamma_{n,m}^{(i)}\left|\xi_{n,m}^{(i)}\right|^3,$$

and

$$\begin{aligned}\sum_{m=1}^{n}\log\left[\frac{1-H_m\left(Z_m;t+\varphi\left(n\right)u\right)}{1-H_m\left(Z_m;t\right)}\right]&=2\log\left(1+\eta_{n,m}\right)=\\ &=2\sum_{m=1}^{n}\eta_{n,m}-\sum_{m=1}^{n}\left[\eta_{n,m}\right]^2+\sum_{m=1}^{n}\beta_{n,m}\cdot\left|\eta_{n,m}\right|^3,\end{aligned}\tag{16}$$

where $\left|\gamma_{n,m}^{(i)}\right|<1$ and $\left|\beta_{n,m}\right|<1$, $m=\overline{1,n}$; $i=\overline{1,k}$ with probability 1. Let us prove the following relations for terms of expansions (16)

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\xi_{n,m}^{(i)}\right|>\varepsilon\right\}=0,\tag{17}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\eta_{n,m}\right|>\varepsilon\right\}=0,\tag{18}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2-\frac{u^2}{4}\right|>\varepsilon\right\}=0,\tag{19}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\eta_{n,m}^2-\frac{u^2}{4}\right|>\varepsilon\right\}=0,\tag{20}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|2\sum_{m=1}^{n}\xi_{n,m}^{(i)}-\varphi\left(n\right)u\cdot\sum_{m=1}^{n}\frac{\frac{\partial}{\partial t}\log h_m^{(i)}\left(Z_m;t\right)}{h_m^{(i)}\left(Z_m;t\right)}+\frac{u^2}{4}\right|>\varepsilon\right\}=0,\tag{21}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|2\sum_{m=1}^{n}\eta_{n,m}-\varphi\left(n\right)u\cdot\sum_{m=1}^{n}\frac{\frac{\partial}{\partial t}\log\left(1-H_m\left(Z_m;t\right)\right)}{\left(1-H_m\left(Z_m;t\right)\right)}+\frac{u^2}{4}\right|>\varepsilon\right\}=0\tag{22}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\sum_{m=1}^{n}\left|\xi_{n,m}^{(i)}\right|^3>\varepsilon\right\}=0\tag{23}$$

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\sum_{m=1}^{n}\left|\eta_{n,m}\right|^3>\varepsilon\right\}=0.\tag{24}$$

Using above relations, one needs to establish (17), (19), (21) and (23). The rest relations are proved quite similarly. Consider the inequality

$$\widetilde{Q}_t^{(n)}\left\{\max_{1\leqslant m\leqslant n}\left|\xi_{n,m}^{(i)}\right|>\varepsilon\right\}\leqslant\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\xi_{n,m}^{(i)}\right|>\varepsilon\right\}\leqslant$$

$$\leqslant\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\xi_{n,m}^{(i)}-\frac{\varphi(n)\,u}{2}\frac{\partial h_m^{(i)}(Z_m;t)}{\frac{\partial t}{h_m(Z_m;t)}}\right|>\varepsilon/2\right\}+\sum_{m=1}^{n}\widetilde{Q}_t^{(n)}\left\{\left|\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m(Z_m;t)}\right|>\frac{\varepsilon}{4\varphi(n)\,|u|}\right\},$$

where the Chebyshev inequality is used for the first component, and (9) is used for the second one. Now to prove (19) consider the following inequalities

$$\widetilde{Q}_t^{(n)}\left\{\left|\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2-\frac{1}{4}\varphi^2(n)\,u^2\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2\right|>\varepsilon\right\}\leqslant$$

$$\leqslant\frac{1}{\varepsilon}\sum_{m=1}^{n}M_t\left|\left[\xi_{n,m}^{(i)}\right]^2-\frac{1}{4}\varphi^2(n)\,u^2\cdot\left(\frac{\frac{\partial}{\partial t}h_m(Z_m;t)}{h_m(Z_m;t)}\right)^2\right|\leqslant$$

$$\leqslant\frac{\alpha}{2\varepsilon}\sum_{m=1}^{n}M_t\left|\xi_{n,m}-\frac{1}{2}\varphi(n)\,u\cdot\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m(Z_m;t)}\right|^2+\frac{1}{2\alpha\varepsilon}\left(1+\sum_{m=1}^{n}M_t\xi_{n,m}^{(i)}\right).$$

In this case, using the law of large numbers for sums

$$\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2$$

with the corresponding normalization, we have

$$\lim_{n\to\infty}\widetilde{Q}_t^{(n)}\left\{\left|\varphi^2(n)\sum_{m=1}^{n}\left(\frac{\frac{\partial}{\partial t}h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)}\right)^2-1\right|>\varepsilon\right\}=0.$$

Equality (19) is proved. Equality (23) is a consequence of (17) and (19). It remains to establish (21). It follows from (19) that $\sum\limits_{m=1}^{n}\left(\xi_{n,m}^{(i)}\right)^2$ converges in probability to $\dfrac{u^2}{4}$. Using (7), we obtain the equality

$$\lim_{n\to\infty}M_t\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2=\frac{u^2}{4}.$$

Using this equality and (11), the following relation is obtained

$$\lim_{n\to\infty}\sum_{m=1}^{n}\int\limits_{\left\{x:\,h_m^{(i)}(x;t)=0\right\}}h_m^{(i)}(x;t+\varphi(n)\,u)\,\nu_m(dx)=0.$$

Considering these two equalities and passing to the mathematical expectations in the identity

$$\sum_{m=1}^{n}\left[\xi_{n,m}^{(i)}\right]^2=\sum_{m=1}^{n}\left(\frac{h_m^{(i)}(Z_m;t+\varphi(n)\,u)}{h_m(Z_m;t)}-1\right)-2\sum_{m=1}^{n}\xi_{n,m}^{(i)},$$

we obtain

$$\lim_{n\to\infty} M_t \sum_{m=1}^{n} \xi_{n,m}^{(i)} = -\frac{u^2}{8}.$$

Next, for $n \geqslant n_0$ we find

$$\widetilde{Q}_t^{(n)} \left\{ \left| 2\sum_{m=1}^{n} \xi_{n,m}^{(i)} - \varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} + \frac{u^2}{4} \right| > \varepsilon \right\} \leqslant$$

$$\leqslant \widetilde{Q}_t^{(n)} \left\{ \left| 2\sum_{m=1}^{n} \left(\xi_{n,m}^{(i)} - M_t \xi_{\ n,m}^{\ (i)} \right) - \varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right| > \frac{\varepsilon}{2} \right\} \leqslant$$

$$\leqslant \frac{16}{\varepsilon^2} M_t \left[\sum_{m=1}^{n} \left(\xi_{n,m}^{(i)} - M_t \xi_{\ n,m}^{\ (i)} \right) - \frac{1}{2}\varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right]^2 \leqslant$$

$$\leqslant \frac{16}{\varepsilon^2} \sum_{m=1}^{n} M_t \left(\xi_{n,m}^{(i)} - \frac{1}{2}\varphi(n)\, u \sum_{m=1}^{n} \frac{\frac{\partial}{\partial t} h_m^{(i)}(Z_m;t)}{h_m^{(i)}(Z_m;t)} \right)^2$$

Now (21) follows from the last relation and (9). To prove the remaining relations (18), (20), (22) and (24) one should proceed in a similar way. Theorem is completely proved. □

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Локальная асимптотическая нормальность статистических экспериментов в неоднородной модели конкурирующих рисков при случайном цензурировании справа

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Аннотация. Статья посвящена доказательству свойства локальной асимптотической нормальности статистики отношения правдоподобия в модели конкурирующих рисков, отвечающих неоднородным и случайно цензурированных справа наблюдениям.

Ключевые слова: локальная асимптотическая нормальность, статистика отношения правдоподобия, модель конкурирующих рисков, случайное цензурирование, асимптотическое представление.