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Some New Fixed Point Results in b-metric Space with Rational Generalized Contractive Condition

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Abstract. In this paper, we improve and generalize several results in fixed point theory to the b-metric space. Where we confirm the existence of the fixed point for self mapping T satisfying some rational contractive conditions. Over-more, we establish the uniqueness of the fixed point in some cases and give dynamic information linking the fixed points between them in the other cases. Some illustrative examples are furnished, which demonstrate the validity of the hypotheses.

Keywords: metric space, b-metric space, Picard sequence, Fixed point, Rational contraction mapping.

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Introduction

The fixed point theory is an exceptional combination of analysis (pure and applied), topology and geometry. This theory stems from purely mathematical thought, and herein lies the difficulty of developing and expanding this field. On the other hand, we find that the application of these theorems as tool to study of non-linear natural phenomena gave amazing results that match reality in various fields that include biology, chemistry, economics, engineering, game theory and physics, which increased its aesthetic and importance, for more detait we refer reader to [10]. Despite the difficulty of purely mathematical study, the fixed point theory developed rapidly because of its applications in diverse fields, especially after the emergence of Banach's contraction [6], which is a basic result on fixed points for contraction type mappings, it was introduced by great Polish mathematician Stefan Banach in 1922. It has been generalized in various directions. These generalizations are made either by using contractive conditions or

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by imposing some additional conditions on the ambient spaces for more detail see references [3–5, 13, 19, 20, 22, 27].

There exist various generalizations of usual metric spaces. One of them is b-metric space or metric-type space. This concept was first introduced by Bakhtin [5].

The b-metric space has been studied topologically in many works, including: such as S. Czerwik [9], N. Bourbaki [8] which confirmed the fundamental difference between it and the metric space, for example the b-metric is not necessarily continuous unlike the metric distance.

In 1993, Czerwik [9] extended the results of metric spaces that generalized the famous Banach contraction principle for b-metric space. Later, several authors extended the fixed point theorem in b-metric space. For fixed point results and more examples in b-metric spaces, the readers may refer to [1, 2, 7, 9, 11–18, 21–26]. The aim of this paper is to present some fixed point results for mappings satisfying generalized contractive condition in a b-metric space.

1. Preliminary

In this section, we look back on some famous notions and definition of the b-metric spaces which will be used in the sequel.

Definition 1 ([9]). *Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold:*
(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.
The triple (X, d, s) is called a b-metric space with constant $s \geqslant 1$.

Remark 1. *It is obvious from the above definition that the class of b-metric spaces is larger than that of metric spaces, since a b-metric space is a metric space when $s = 1$ but the converse is not true.*

Remark 2. *In general, the b-metric is not usually continuous (see example 4 in [19]).*

Definition 2 ([21]). *Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called*
(a) convergent if and only if there exists $x \in X$ such that $\lim\limits_{n\to+\infty} d(x_n, x) = 0$ and in this case we write $\lim\limits_{n\to+\infty} x_n = x$;
(b) Cauchy if and only if $\lim\limits_{n,m\to+\infty} d(x_n, x_m) = 0$.

Before starting, we present the following simple lemma proven by A. Aghajani, M. Abbas and J. R. Roshan [3] which has a fundamental role in proving our results.

Lemma 1 ([3]). *Let (X, d, s) be a b-metric space such that $s \geqslant 1$ and $\{x_n\}$ be a convergent sequence in X to x. Then for each $y \in X$, we have*

$$\frac{1}{s} d(x, y) \leqslant \liminf_{n\to+\infty} d(x_n, y) \leqslant \limsup_{n\to+\infty} d(x_n, y) \leqslant s d(x, y). \tag{1}$$

2. Results

Firstly, we state and prove our first theorem that generalize and improve the result of Khojasteh et al [20].

Theorem 1. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $s^2 a \leqslant \min\{c, f\}$ or $s^2 b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} d(x, y). \tag{2}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2 a \leqslant \min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (2), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} d(x_{n-1}, x_n). \end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.

Since $s^2 a \leqslant \min\{c, f\}$, then $0 \leqslant \theta_n < \dfrac{1}{s}$ for all $n \in \mathbb{N}$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$,

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0. \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \theta_n d(x_{n-1}, x_n) \leqslant \\ &\leqslant \theta_n \theta_{n-1} d(x_{n-2}, x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \theta_n \theta_{n-1} \cdots \theta_1 d(x_0, x_1) \leqslant \\ &\leqslant \theta_1^n d(x_0, x_1). \end{aligned}$$

Now for all n, $m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} \theta_1^i d(x_0, x_1) \leqslant \\ &\leqslant \frac{(s\theta_1)^n - (s\theta_1)^m}{1 - s\theta_1} \times \frac{1}{s^{n-1}} d(x_0, x_1). \end{aligned}$$

By passing to the limits $n, m \to +\infty$ on a both side of previous inequality we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \tag{3}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = x_n$ and $y = x_{n-1}$ in inequality (2), we find

$$d(x_n, x_{n+1}) \leqslant \frac{bd(x_{n-1}, x_{n+1})}{cd(x_n, x_{n+1}) + fd(x_{n-1}, x_n) + e} d(x_{n-1}, x_n).$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X, d, s) is complete, there exit $\dot{x} \in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
By putting $x = \dot{x}$, $y = x_n$ in inequality (2), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(\dot{x}, x_{n+1}) + bd(x_n, T\dot{x})}{cd(\dot{x}, T\dot{x}) + fd(x_n, x_{n+1}) + e} d(\dot{x}, x_n). \tag{4}$$

By taking limit on both sides of (4), we have $\lim_{n\to+\infty} x_n = T\dot{x}$. Because of the uniqueness of the limit, we find $T\dot{x} = \dot{x}$.
Step 3: Suppose that T have two distinct fixed points $\dot{x}, \dot{y}$ in X and we find the distance between them.
By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (2), we find,

$$d(\dot{x}, \dot{y}) \quad \leqslant \quad \frac{ad(\dot{x}, \dot{y}) + bd(\dot{y}, \dot{x})}{cd(\dot{x}, \dot{x}) + fd(\dot{y}, \dot{y}) + e} d(\dot{x}, \dot{y}) \quad \leqslant \quad \frac{(a + b)d(\dot{x}, \dot{y})}{e} d(\dot{x}, \dot{y}).$$

Then $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a + b}$. This complete the proof of the theorem. □

Remark 3. *If we take $a = b = c = f = e = 1$ and $s = 1$ in Theorem 1, we returne to results of Khojasteh et al [20].*

The following example support our Theorem 1.

Example 1. *Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = (x-y)^2$ for all $x, y \in X$. $(X, d, 2)$ is a complete b-metric space. Let $T : X \to X$ be a self mapping given by $T(0) = 0$, $T(1) = 0$ and $T(2) = 2$.*
If $x = y$, the equation is obviously verified. Now, we treat the other cases.
If $x = 0$ and $y = 2$,

$$d(0, 2) \leqslant (d(0, 2) + 5d(0, 2))d(0, 2).$$

If $x = 1$ and $y = 2$,

$$d(0, 2) \leqslant \frac{d(1, 2) + 5d(0, 2)}{4d(1, 0) + 1} d(1, 0).$$

If $x = 2$ and $y = 0$,

$$d(0, 2) \leqslant (d(0, 2) + 5d(0, 2))d(0, 2).$$

If $x = 2$ and $y = 1$,

$$d(0, 2) \leqslant \frac{d(0, 2) + 5d(1, 2)}{4d(1, 0) + 1} d(1, 0).$$

That mean that the equation (2) is verified with constants $a = 1$, $b = 5$, $c = 4$, $e = 1$ and $f = 4$. Over more, all conditions of Theorem 1 was satisfied, then T has at least one fixed point in X. We remark that T has exactly two fixed point $0, 2$, over more, $d(0, 2) \geqslant \dfrac{1}{6}$.

If we take $s = 1$ in Theorem 1, we get the following corollary.

Corollary 1. *Let (X, d) be a complete metric space and let T be a self mapping in X. If there exist five positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $a \leqslant \min\{c, f\}$ or $b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} d(x, y). \tag{5}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then, $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$.*

This example illustrates and supports Theorem 1 and Corollary 1.

Example 2. *Let $X = \{0, 1, 2\}$ associated with a metric d such that $d(0, 1) = 0.75$, $d(0, 2) = 1$ and $d(1, 2) = 0.25$. Also, $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, x) = 0$ for all $x \in X$.*

Let T be a self mapping in X such that $T(0) = 2$, $T(1) = 1$ and $T(2) = 2$.

It is easy to conclude that (X, d) is a complete metric space and the inequality (5) was verified for all $x, y \in X$ with constant $a = b = \dfrac{1}{2}$, $c = f = 1$ and $e = \dfrac{1}{4}$. According to Corollary 1, we conclude that T has at least one fixed point. (exactly, it has two fixed point 1 and 2). Moreover, the distance between them is $d(1, 2) \geqslant \dfrac{1}{4}$.

Remark 4. *It should be noted that Khojasteh et al theorem [20] is not applicable in this example while the generalized Corollary 1 is applicable as shown in the example above, which proves the robustness of our results.*

Secondly, we state and prove our second theorem that generalize and improve the result of A. C. Aouine and A. Aliouche [4].

Theorem 2. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $s^2a \leqslant \min\{c, f\}$ or $s^2b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} \max\{d(x, Tx), d(y, Ty)\}. \tag{6}$$

Then T has a unique fixed point $\dot{x} \in X$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2a \leqslant \min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (6), we find

$$\begin{aligned}
d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\
&\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\
&\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\
&\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} d(x_{n-1}, x_n).
\end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.
Since $s^2 a \leqslant \min\{c, f\}$, then $0 \leqslant \theta_n < \dfrac{1}{s}$ for all $n \in \mathbb{N}$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$,

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \theta_n d(x_{n-1}, x_n) \leqslant \\ &\leqslant \theta_n \theta_{n-1} d(x_{n-2}, x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \theta_n \theta_{n-1} \cdots \theta_1 d(x_0, x_1) \leqslant \\ &\leqslant \theta_1^n d(x_0, x_1). \end{aligned}$$

Now, for all n, $m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} \theta_1^i d(x_0, x_1) \leqslant \\ &\leqslant \frac{(s\theta_1)^n - (s\theta_1)^m}{1 - s\theta_1} \times \frac{1}{s^{n-1}} d(x_0, x_1). \end{aligned}$$

By passing to the limits $n, m \to +\infty$ on a both side of previous inequality, we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \tag{7}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = x_n$ and $y = x_{n-1}$ in inequality (6), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{bd(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{cd(x_n, x_{n+1}) + fd(x_{n-1}, x_n) + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{\min\{c, f\}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + e} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \leqslant \\ &\leqslant \frac{bsd(x_{n-1}, x_n) + bsd(x_n, x_{n+1})}{\min\{c, f\}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + e} d(x_{n-1}, x_n). \end{aligned}$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X, d, s) is complete, there exit $\dot{x} \in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
Suppose that $d(\dot{x}, T\dot{x}) > 0$

Case 1. If $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_n$, $y = \dot{x}$ in inequality (6), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(x_n, T\dot{x}) + bd(x_{n+1}, \dot{x})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e}\max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \tag{8}$$

On the other hand, we have

$$\begin{aligned} d(\dot{x}, T\dot{x}) &\leqslant sd(\dot{x}, x_{n+1}) + sd(x_{n+1}, T\dot{x}) \leqslant \\ &\leqslant sd(\dot{x}, x_{n+1}) + s\frac{ad(x_n, T\dot{x}) + bd(x_{n+1}, \dot{x})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e}\max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \end{aligned} \tag{9}$$

By taking limit superior on both sides of (9), we have

$$d(\dot{x}, T\dot{x}) \leqslant \frac{sa\limsup_{n\to+\infty} d(x_n, T\dot{x})}{fd(\dot{x}, T\dot{x}) + e}d(\dot{x}, T\dot{x}). \tag{10}$$

According to Lemma 1, we get

$$d(\dot{x}, T\dot{x}) \leqslant \frac{s^2 a}{fd(\dot{x}, T\dot{x}) + e}d(\dot{x}, T\dot{x})^2, \tag{11}$$

then

$$1 \leqslant \frac{s^2 a}{fd(\dot{x}, T\dot{x}) + e}d(\dot{x}, T\dot{x}). \tag{12}$$

Since, $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, then $s^2 a \leqslant f$, then $s^2 ad(\dot{x}, T\dot{x}) < fd(\dot{x}, T\dot{x}) + e$ which contradict inequality (12). Then $d(\dot{x}, T\dot{x}) = 0$ that mean $T\dot{x} = \dot{x}$.

Case 2. If $s^2 b \leqslant \min\{c, f\}$, by putting $x = \dot{x}$, $y = x_n$ in inequality (6), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(\dot{x}, x_{n+1}) + bd(x_n, T\dot{x})}{cd(\dot{x}, T\dot{x}) + fd(x_n, x_{n+1}) + e}\max\{d(\dot{x}, T\dot{x}), d(x_n, x_{n+1})\}. \tag{13}$$

Similarly, as Case 1, we can deduce that $T\dot{x} = \dot{x}$.

Step 3: Suppose that T have two fixed points $\dot{x}, \dot{y}$ in X. By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (6), we find

$$d(\dot{x}, \dot{y}) \leqslant \frac{ad(\dot{x}, \dot{y}) + bd(\dot{y}, \dot{x})}{cd(\dot{x}, \dot{x}) + fd(\dot{y}, \dot{y}) + e}\max\{d(\dot{x}, \dot{x}), d(\dot{y}, \dot{y})\}. \tag{14}$$

Then $d(\dot{x}, \dot{y}) = 0$, that mean, $\dot{x} = \dot{y}$, and this completes the proof of the theorem. □

Remark 5. *If we take $a = b = c = f = e = 1$ and $s = 1$ in Theorem 2, we returne to results of A. C. Aouine and A. Aliouche [4].*

The following example illustrates and supports our Theorem 2.

Example 3. *Let $X = [0, 4.5]$ and $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$. $(X, d, 2)$ is a complete b-metric space. Let $T : X \to X$ be a self mapping given by*

$$Tx = \begin{cases} 4.5 & \text{if } x \in [0, 2.5[\\ 4 & \text{if } x \in [2.5, 4.5] \end{cases}.$$

Let $x, y \in X$ and denote

$$m(x,y) = -d(Tx,Ty) + \frac{d(x,Ty)+d(y,Tx)}{4d(x,Tx)+4d(y,Ty)+1}\max\{d(x,Tx),d(y,Ty)\}.$$

if $x \in [0, 2.5[$ and $y \in [2.5, 4.5]$, we draw the curve of the function m over this domain (Fig. 1).

We remark that it is positive, which proves the validity of the inequality (6) for all $x \in [0, 2.5[$ and $y \in [2.5, 4.5]$. The other cases is trivial.
Therefore, by choosing $a = b = e = 1$ and $c = f = 4$ all conditions of Theorem 2 are satisfied. Hence T has a unique fixed point $\dot{x}$ in X (here $\dot{x} = 4$).

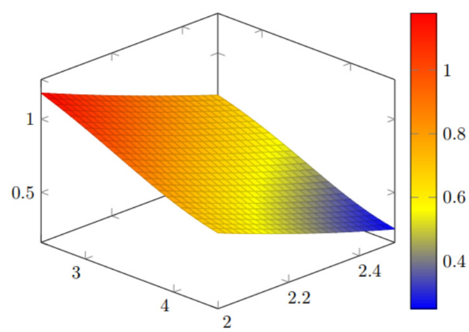

Fig. 1. Curve of the function m

If we take $s = 1$ in Theorem 2, we get the following corollary.

Corollary 2. *Let (X, d) be a complete metric space and let T be a self mapping in X. If there exist five positive real number $a, b, c, f,\ e \in \mathbb{R}^+$ such that $a \leqslant \min\{c, f\}$ or $b \leqslant \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx,Ty) \leqslant \frac{ad(x,Ty)+bd(y,Tx)}{cd(x,Tx)+fd(y,Ty)+e}\max\{d(x,Tx),d(y,Ty)\}. \tag{15}$$

Then T has a unique fixed point $\dot{x} \in X$.
Every Picard sequence converge to $\dot{x}$.

The following example illustrates and supports Corollary 2 and Theorem 2.

Example 4. *Let $X = \{0, 1, 2\}$ associated with a metric d such that $d(0, 1) = 0.6$, $d(0, 2) = 1$ and $d(1, 2) = 0.4$. Also $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, x) = 0$ for all $x \in X$.*

Let T be a self mapping in X such that $T(0) = 2$, $T(1) = 1$ and $T(2) = 1$.

It is easy to conclude that (X, d) is a complete metric space and the inequality (15) was verified for all $x, y \in X$ with constant $a = b = c = f = 3$ and $e = \frac{1}{4}$. According to Corollary 2, we conclude that T has a unique fixed point. In additional, every Picard sequence converge to $\dot{x}$.

Remark 6. *It should be noted that A.C. Aouine and A. Aliouche. Theorem [2] is not applicable in this example while the generalized Corollary 2 is applicable as shown in the example above, which proves the robustness of our results.*

Third, we can generalize the previous theorems as follow:

Theorem 3. *Let (X, d, s) be a complete b-metric space and let T be a self mapping in X. If there exist five positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$ or $s^2 b \leqslant \frac{1}{2}\min\{c, f\}$ and for all $x, y \in X$,*

$$d(Tx, Ty) \leqslant \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} \max\{d(x, Ty), d(y, Tx)\}. \tag{16}$$

Then

1. *T has at least one fixed point $\dot{x} \in X$.*
2. *Every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point.*
3. *If T has two distinct fixed points $\dot{x}, \dot{y}$ in X then, $d(\dot{x}, \dot{y}) \geqslant \dfrac{e}{a+b}$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$) based on an arbitrary $x_0 \in X$. If there exist an $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_{n-1}$ and $y = x_n$ in inequality (16), we find

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_{n+1}) \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e}[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})] \leqslant \\ &\leqslant \frac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})]. \end{aligned}$$

We denote that $\theta_n = \dfrac{asd(x_{n-1}, x_n) + asd(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.
Since $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, then $0 \leqslant \theta_n < \dfrac{1}{2s}$ for all $n \in \mathbb{N}$.
On the other hand we have

$$d(x_n, x_{n+1}) \leqslant \theta_n[sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})],$$

then

$$d(x_n, x_{n+1}) \leqslant \frac{\theta_n s}{1 - \theta_n s} d(x_{n-1}, x_n).$$

We denote that $\lambda_n = \dfrac{\theta_n s}{1 - \theta_n s}$ for all $n \in \mathbb{N}$.
Since $0 \leqslant \theta_n < \dfrac{1}{2s}$ for all $n \in \mathbb{N}$, then $0 \leqslant \lambda_n < 1$.
Then $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, then $d(x_{n+1}, x_{n+2}) < d(x_{n-1}, x_n)$.
Furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ase\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \times \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} < 0. \end{aligned}$$

Then the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is decreasing, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \lambda_n d(x_{n-1}, x_n) \leqslant \\ &\leqslant \lambda_n \lambda_{n-1} d(x_{n-2}, x_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \lambda_n \lambda_{n-1} \cdots \lambda_1 d(x_0, x_1) \leqslant \\ &\leqslant \lambda_1^n d(x_0, x_1). \end{aligned}$$

Now for all n, $m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leqslant \\ &\leqslant \sum_{i=n}^{m-1} s^{i-n+1} \lambda_1^i d(x_0, x_1) \leqslant \\ &\leqslant \frac{(s\lambda_1)^n - (s\lambda_1)^m}{1 - s\lambda_1} \times \frac{1}{s^{n-1}} d(x_0, x_1). \end{aligned}$$

By passing to the limits $n, m \to +\infty$ on a both side of previous inequality, we get

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0. \tag{17}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Case 2. If $s^2 b \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_n$ and $y = x_{n-1}$ in inequality (16), we find

$$d(x_n, x_{n+1}) \leqslant \frac{bd(x_{n-1}, x_{n+1})}{cd(x_n, x_{n+1}) + fd(x_{n-1}, x_n) + e} d(x_{n-1}, x_{n+1}).$$

Similarly, as Case 1, we can deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the b-metric space (X, d, s) is complete, there exit $\dot{x} \in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: We check that $\dot{x}$ is a fixed point of T.
Suppose that $d(\dot{x}, T\dot{x}) > 0$.

Case 1. If $s^2 a \leqslant \frac{1}{2}\min\{c, f\}$, by putting $x = x_n$, $y = \dot{x}$ in inequality (16), we find

$$d(T\dot{x}, x_{n+1}) \leqslant \frac{ad(x_n, T\dot{x}) + bd(\dot{x}, x_{n+1})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e} \max\{d(\dot{x}, x_{n+1}), d(x_n, T\dot{x})\}. \tag{18}$$

On the other hand, we have

$$d(\dot{x}, T\dot{x}) \leqslant sd(\dot{x}, x_{n+1}) + sd(x_{n+1}, T\dot{x}) \leqslant \tag{19}$$

$$\leqslant sd(\dot{x}, x_{n+1}) + \frac{sad(x_n, T\dot{x}) + sbd(\dot{x}, x_{n+1})}{cd(x_n, x_{n+1}) + fd(\dot{x}, T\dot{x}) + e} \max\{d(\dot{x}, x_{n+1}), d(x_n, T\dot{x})\}. \tag{20}$$

By taking limit superior on both sides of (20), we have

$$d(\dot{x}, T\dot{x}) \leqslant \frac{sa}{fd(\dot{x}, T\dot{x}) + e}\Big(\limsup_{n\to+\infty} d(x_n, T\dot{x})\Big)^2. \tag{21}$$

According to Lemma 1, we get

$$d(\dot{x},T\dot{x}) \leqslant \frac{sa}{fd(\dot{x},T\dot{x})+e}d(\dot{x},T\dot{x})^2, \tag{22}$$

then

$$1 \leqslant \frac{sa}{fd(\dot{x},T\dot{x})+e}d(\dot{x},T\dot{x}). \tag{23}$$

Since, $s^2a \leqslant \frac{1}{2}\min\{c,f\}$, then $s^2a \leqslant f$, then $s^2ad(\dot{x},T\dot{x}) < fd(\dot{x},T\dot{x})+e$ which contradict inequality (23). Then $d(\dot{x},T\dot{x})=0$ that mean $T\dot{x}=\dot{x}$.

Case 2. If $s^2b \leqslant \frac{1}{2}\min\{c,f\}$, by putting $x=\dot{x}$ and $y=x_n$ in inequality (16), we find

$$d(T\dot{x},x_{n+1}) \leqslant \frac{ad(\dot{x},x_{n+1})+bd(x_n,T\dot{x})}{cd(\dot{x},T\dot{x})+fd(x_n,x_{n+1})+e}\max\{d(\dot{x},x_{n+1}),d(x_n,T\dot{x})\}.$$

Similarly, as Case 1 we can deduce that $T\dot{x}=\dot{x}$.

Step 3: Suppose that T have two distinct fixed points $\dot{x},\dot{y}$ in X and we find the distance between them.
By putting $x=\dot{x}$, $y=\dot{y}$ in inequality (16), we find

$$d(\dot{x},\dot{y}) \leqslant \frac{ad(\dot{x},\dot{y})+bd(\dot{y},\dot{x})}{cd(\dot{x},\dot{x})+fd(\dot{y},\dot{y})+e}d(\dot{x},\dot{y}) \leqslant \frac{(a+b)d(\dot{x},\dot{y})}{e}d(\dot{x},\dot{y}).$$

Then, $d(\dot{x},\dot{y}) \geqslant \frac{e}{a+b}$. □

3. Discussion

- The Corollary 1 generalize the result of Khojasteh et al [1] and the Corollary 2 generalize the result of Aouine and Aliouche [2].
- We note that the choice of constants related to inequalities (2), (5), (6), (15) and (16) directly affects the dynamic result of Theorems 1, 2, and Corollaries 1, 2 and 3 respectively.
- Note that the ratio $\frac{ad(x,Ty)+bd(y,Tx)}{cd(x,Tx)+fd(y,Ty)+e}$ in the inequalities (2), (5), (6), (15) and (16) might be greater or less than 1, thus theorems is an special case of Banach contraction principle. Example 1 illustrates this point precisely.
- If *rangT* is a closed sub set of X, the inequalities (2), (5), (6), (15) and (16) can be restricted to *rangT*, and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.
- The above results can be generalized into several generalized metric spaces as q_1-q_2 b-metric space, partial metric space, ...

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Некоторые новые результаты с фиксированной точкой в b-метрическом пространстве с рациональным обобщенным условием сжатия

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Аннотация. В этой статье мы улучшаем и обобщаем некоторые результаты теории неподвижных точек на b-метрическое пространство. Где мы подтверждаем существование неподвижной точки для самоотображения T, удовлетворяющего некоторым рациональным сжимающим условиям. Более того, мы устанавливаем уникальность фиксированной точки в некоторых случаях и даем динамическую информацию, связывающую неподвижные точки между собой в других случаях. Приведены наглядные примеры, демонстрирующие справедливость гипотез.

Ключевые слова: метрическое пространство, b-метрическое пространство, последовательность Пикара, фиксированная точка, отображение рационального сжатия.