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Almost Inner Derivations of Some Leibniz Algebras

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Abstract. The present paper is devoted to almost inner derivations of thin and solvable Leibniz algebras. Namely, we consider a thin Lie algebra, solvable Lie algebra with nilradical natural graded filifform Lie algebra, natural graded thin Leibniz algebra, thin non-Lie Leibniz algebra and solvable Leibniz algebra with nilradical nul-filiform algebra. We prove that any almost inner derivations of all these algebras are inner derivations.

Keywords: Lie algebra, Leibniz algebra, solvable algebra, nilradical, thin Lie algebra, thin Leibniz algebra, derivation, inner derivation, almost inner derivation

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Introduction

Almost inner derivations of Lie algebras were introduced by C.S.Gordon and E.N.Wilson [13] in the study of isospectral deformations of compact manifolds. Gordon and Wilson wanted to construct not only finite families of isospectral nonisometric manifolds, but rather continuous families. They constructed isospectral but nonisometric compact Riemannian manifolds of the form G/Γ, with a simply connected exponential solvable Lie group G, and a discrete cocompact subgroup Γ of G. For this construction, almost inner automorphisms and almost inner derivations were crucial.

Gordon and Wilson considered not only almost inner derivations, but they studied almost inner automorphisms of Lie groups. The concepts of "almost inner" automorphisms and derivations, almost homomorphisms or almost conjugate subgroups arise in many contexts in algebra, number theory and geometry. There are several other studies of related concepts, for example, local derivations, which are a generalization of almost inner derivations and automorphisms [3,4].

In [7] authors study almost inner derivations of some nilpotent Lie algebras. The authors of this work proved the basic properties of almost inner derivations, calculated all almost inner derivations of Lie algebras for small dimensions.They also introduced the concept of fixed basis vectors for nilpotent Lie algebras defined by graphs and studied free nilpotent Lie algebras of the nilindex 2 and 3. In [8], almost inner derivations of Lie algebras over a field of characteristic zero has been studied and these derivations has been determined for free nilpotent Lie algebras, almost abelian Lie algebras, Lie algebras whose solvable radical is abelian and for several classes of filiform nilpotent Lie algebras. A family of n-dimensional characteristically nilpotent filiform Lie algebras f_n has been found for all $n \geqslant 13$, all derivations of which are almost inner. The almost inner derivations of Lie algebras considered over two different fields $K \supseteq k$ for a finite-dimensional field extension were compared.

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Motivated by the work [7], we studied almost inner derivations of some nilpotent Leibniz algebras [2] and in this work the almost inner derivations for Leibniz algebras were introduced and it was proved that on a filiform non-Lie Leibniz algebra there exists an almost inner derivation that is not an inner derivation.

In work [1] it is proved that any derivation complex maximal solvable extension of Lie algebras is inner [Theorem 4.1]. Moreover, it is proved that any non-maximal solvable extension of a nilpotent Lie algebra admits an outer derivation [Proposition 4.3]. Therefore, in this paper almost inner derivations of solvable Lie algebras with the nilradical naturally graded filiform Lie algebra and almost inner derivations of thin Lie algebras will be considered. In addition, almost inner derivations of natural graded thin Leibniz algebras, non-Lie thin Leibniz algebras and solvable Leibniz algebras with nilradical nul-filiform algebra will be studied.

1. Preliminaries

Definition 1.1. *An algebra $\mathfrak{g}$ over field $\mathbb{F}$ is called a Lie algebra if its multiplication satisfies:*
1) $[x,x]=0$,
2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$,
for all $x,y,z\in\mathfrak{g}$.

The product $[x,y]$ is called the bracket of x and y. Identity 2) is called the Jacobi identity.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For Lie algebra $\mathfrak{g}$ we consider the following central and derived series:

$$\mathfrak{g}^1=\mathfrak{g},\quad \mathfrak{g}^i=[\mathfrak{g}^{i-1},\mathfrak{g}],\quad i\geqslant 1,$$
$$\mathfrak{g}^{[1]}=\mathfrak{g},\quad \mathfrak{g}^{[k]}=[\mathfrak{g}^{[k-1]},\mathfrak{g}^{[k-1]}],\quad k\geqslant 1.$$

A Lie algebra $\mathfrak{g}$ is *nilpotent* (*solvable*) if there exists $m\geqslant 1$ such that $\mathfrak{g}^m=0$ ($\mathfrak{g}^{[m]}=0$).

Definition 1.2. *A derivation of Lie algebra $\mathfrak{g}$ is a linear map $D:\mathfrak{g}\to\mathfrak{g}$ which satisfies the Leibniz law, that is,*

$$D([x,y])=[D(x),y]+[x,D(y)]$$

for all $x,y\in\mathfrak{g}$.

The set of all derivations of $\mathfrak{g}$ with respect to the commutation operation is a Lie algebra and it is denoted by $Der(\mathfrak{g})$. For all $a\in\mathfrak{g}$, the map ad_a on $\mathfrak{g}$ defined as $ad_a(x)=[a,x]$, $x\in\mathfrak{g}$ is a derivation and derivations of this form are called *inner derivation*. The set of all inner derivations of $\mathfrak{g}$, denoted $InDer(\mathfrak{g})$.

Definition 1.3. *A derivation $D\in Der(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is said to be almost inner, if $D(x)\in[\mathfrak{g},x]$ for all $x\in\mathfrak{g}$. The space of all almost inner derivations of $\mathfrak{g}$ is denoted by $AID(\mathfrak{g})$.*

We now give the definition and necessary facts of the Leibniz algebra.

Definition 1.4. *An algebra $\mathfrak{L}$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x,y,z\in\mathfrak{L}$, the Leibniz identity*

$$[x,[y,z]]=[[x,y],z]-[[x,z],y]$$

is satisfied, where $[-,-]$ is the multiplication in $\mathfrak{L}$.

The definitions of nilpotency, solvability and derivation for Leibniz algebras are introduced in a similar way as the definition of nilpotency, solvability and derivation of Lie algebras.

Let $\mathfrak{L}$ be a Leibniz algebras. For each $a\in\mathfrak{L}$, the operator $R_x:\mathfrak{L}\to\mathfrak{L}$ which is called the *right multiplication*, such that $R_x(y)=[y,x]$, $y\in\mathfrak{L}$, is a derivation. This derivation is called an *inner derivation* of $\mathfrak{L}$, and we denote the space of all inner derivations by $InDer(\mathfrak{L})$.

Now let us give the definitions of the almost inner derivations for the Leibniz algebras.

Definition 1.5 ([2]). *The derivation $D \in Der(\mathfrak{L})$ of the Leibniz algebra $\mathfrak{L}$ is called almost inner derivation, if $D(x) \in [x, \mathfrak{L}]$ holds for all $x \in \mathfrak{L}$; in other words, there exists $a_x \in \mathfrak{L}$ such that $D(x) = [x, a_x]$. The space of all almost inner derivations of $\mathfrak{L}$ is denoted by $AID(\mathfrak{L})$.*

2. Almost inner derivations of thin Lie algebras

In this section, we will consider almost inner derivations of thin Lie algebras. Let's consider the following so-called thin Lie algebra $\mathfrak{g}$ with a basis $\{e_i : i \in \mathbb{N}\}$, which is defined by the following table of multiplications of the basic elements:

$$M_1 : \quad [e_1, e_i] = e_{i+1}, \ i \geqslant 2, \tag{1}$$

$$M_2 : \begin{cases} [e_1, e_j] = e_{j+1}, & j \geqslant 2, \\ [e_2, e_i] = e_{i+2}, & i \geqslant 3, \end{cases} \tag{2}$$

and other products of the basic elements being zero [12].

Note that the algebras M_1 and M_2 are an infinite-dimensional analog of the filiform Lie algebras L_n and Q_n which are given in [10]. In papers [7] and [8] it was proved that every almost inner derivation of the algebras L_n and Q_n is inner.

The derivations of thin Lie algebras M_1 has the following form [6]:

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad D(e_2) = \sum_{i=1}^{n} \beta_i e_i, \, D(e_j) = ((j-2)\alpha_1 + \beta_2)e_j + \sum_{i=1}^{n} \beta_{i+2} e_{i+j}, \, j \geqslant 3,$$

where $\alpha_i, \beta_i \in \mathbb{C}, \ i = 1, \ldots, n$, and $n \in \mathbb{N}$.

The following theorem is one of the main results in this section.

Theorem 2.1. *Let $\mathfrak{g}$ be the thin Lie algebra. Then any almost inner derivation on thin Lie algebras is inner.*

Proof. First, consider the thin Lie algebra $\mathfrak{g} = M_1$ with multiplication table (1) and inner derivation of this algebra. Let $x = \sum\limits_{i=1}^{n} x_i e_i \in \mathfrak{g}$, $n \in \mathbb{N}$. For basis e_i define $ad_x(e_i)$:

$$ad_x(e_1) = [x, e_1] = \left[\sum_{i=1}^{n} x_i e_i, e_1\right] = -\sum_{i=2}^{n} x_i e_{i+1};$$
$$ad_x(e_j) = [x, e_j] = \left[\sum_{i=1}^{n} x_i e_i, e_j\right] = x_1 e_{j+1}, \ j \geqslant 2.$$

In the next step, we study an almost inner derivation of a thin Lie algebra $\mathfrak{g}$. Let $D \in AID(\mathfrak{g})$. For basis $e_i \in \mathfrak{g}$ exists $a_{e_i} \in \mathfrak{g}$ such that $D(e_i) = [a_{e_i}, e_i]$, for all $i \geqslant 1$. Then

$$D(e_1) = [a_{e_1}, e_1] = \left[\sum_{i=1}^{n} a_{1,i} e_i, e_1\right] = -\sum_{i=2}^{n} a_{1,i} e_{i+1},$$
$$D(e_j) = [a_{e_j}, e_j] = \left[\sum_{i=1}^{n} a_{j,i} e_i, e_j\right] = a_{j,1} e_{j+1}, \ j \geqslant 2.$$

Now we check the conditions of derivation:

$$D(e_3) = D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] = [e_1, a_{2,1} e_3] = a_{2,1} e_4.$$

On the other hand $D(e_3) = a_{3,1}e_4$. From, here we get $a_{2,1} = a_{3,1}$.
For $i \geqslant 3$ consider

$$D(e_i) = D([e_1, e_{i-1}]) = [e_1, D(e_{i-1})] = [e_1, a_{i-1,1}e_i] = a_{i-1,1}e_{i+1}.$$

On the other hand $D(e_i) = a_{i,1}e_{i+1},\ i \geqslant 3$. From here we have

$$a_{i,1} = a_{i-1,1},\ i \geqslant 3.$$

Hence

$$D(e_1) = -\sum_{i=2}^{n} a_{1,i}e_{i+1}, \quad D(e_j) = a_{2,1}e_{j+1},\ j \geqslant 2.$$

For arbitrary element $x \in \mathfrak{g}$ we take element $a = a_{2,1}e_1 - \sum\limits_{k=2}^{n} a_{1,k}e_{k+1} \in \mathfrak{g}$ such that $D(x) = ad_a(x)$, and this means that almost inner derivations D is inner.

Now, we investigate the case $\mathfrak{g} = M_2$.

Let $x = \sum\limits_{i=1}^{n} x_ie_i \in \mathfrak{g}$, $n \in \mathbb{N}$. For basis e_i define $ad_x(e_i)$:

$$\begin{aligned} ad_x(e_1) = [x, e_1] &= \left[\sum_{i=1}^{n} x_ie_i, e_1\right] = -\sum_{i=2}^{n} x_ie_{i+1}; \\ ad_x(e_2) = [x, e_2] &= \left[\sum_{i=1}^{n} x_ie_i, e_2\right] = x_1e_3 - \sum_{k=3}^{n} x_ke_{k+2},\ n \in \mathbb{N}; \\ ad_x(e_j) = [x, e_j] &= \left[\sum_{i=1}^{n} x_ie_i, e_j\right] = x_1e_{j+1} + x_2e_{j+2},\ j \geqslant 3. \end{aligned}$$

Let $D \in AID(\mathfrak{g})$. For basis e_i exists a_{e_i} such that $D(e_i) = [a_{e_i}, e_i]$, for all $i \geqslant 1$. Then

$$\begin{aligned} D(e_1) = [a_{e_1}, e_1] &= \left[\sum_{i=1}^{n} a_{1,i}e_i, e_1\right] = -\sum_{i=2}^{n} a_{1,i}e_{i+1}, \\ D(e_2) = [a_{e_2}, e_2] &= \left[\sum_{i=1}^{n} a_{2,i}e_i, e_2\right] = a_{2,1}e_3 - \sum_{k=3}^{n} a_{2,k}e_{k+2},\ n \in \mathbb{N}, \\ D(e_i) = [a_{e_i}, e_i] &= \left[\sum_{k=1}^{n} a_{i,k}e_k, e_i\right] = a_{i,1}e_{i+1} + a_{i,2}e_{i+2},\ i \geqslant 3. \end{aligned}$$

According to the definition of derivation

$$\begin{aligned} D(e_3) &= D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] = \\ &= \left[-\sum_{i=2}^{n} a_{1,i}e_{i+1}, e_2\right] + \left[e_1, a_{2,1}e_3 - \sum_{k=3}^{n} a_{2,k}e_{k+2}\right] = \\ &= a_{2,1}e_4 + a_{1,2}e_5 + \sum_{k=3}^{n} (a_{1,k} - a_{2,k})e_{k+3}. \end{aligned}$$

On the other hand $D(e_3) = a_{3,1}e_4 + a_{3,2}e_5$. Comparing the coefficients at the basis elements, we obtain

$$\begin{cases} a_{2,1} = a_{3,1}, \\ a_{1,2} = a_{3,2}, \\ a_{1,k} = a_{2,k},\ 3 \leqslant k \leqslant n. \end{cases} \tag{3}$$

Hence

$$D(e_2) = a_{2,1}e_3 - \sum_{k=3}^{n} a_{1,k}e_{k+2},\ n \in \mathbb{N},$$

$$D(e_3) = a_{2,1}e_4 + a_{1,2}e_5.$$

For $i \geqslant 4$ consider the following:

$$D(e_i) = D([e_1, e_{i-1}]) = [e_1, D(e_{i-1})] = a_{i-1,1}e_{i+1} + a_{i-1,2}e_{i+2}.$$

On the other hand $D(e_i) = a_{i,1}e_{i+1} + a_{i,2}e_{i+2}$. Hence for $i \geqslant 4$ it follows that

$$\begin{cases} a_{i-1,1} = a_{i,1}, \\ a_{i-1,2} = a_{i,2}. \end{cases} \tag{4}$$

Combining (3) and (4) we get

$$D(e_1) = -\sum_{k=2}^{n} a_{1,k}e_{k+1},\ n \in \mathbb{N},$$
$$D(e_2) = a_{2,1}e_3 - \sum_{k=3}^{n} a_{1,k}e_{k+2},\ \ n \in \mathbb{N},$$
$$D(e_i) = a_{2,1}e_{i+1} + a_{1,2}e_{i+2},\ \ i \geqslant 3.$$

For every element $x \in \mathfrak{g}$ we take element $a = a_{2,1}e_1 + a_{1,2}e_3 + \sum\limits_{k=3}^{n} a_{1,k}e_k \in \mathfrak{g}$ such that $D(x) = ad_a(x)$, and this means that almost inner derivations D is inner. □

3. Almost inner derivation of naturally graded complex thin Leibniz algebras

In this section, we will consider almost inner derivation of naturally graded complex thin Leibniz algebras. In [14], the following theorem is given, which classifies the naturally graded complex thin Leibniz algebras.

Theorem 3.1 ([14])**.** *Up to isomorphism, there are three naturally graded complex thin Leibniz algebras, namely,*

$$\begin{array}{llll} L_1: & [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & i \geqslant 2, \\ L_2: & [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & i \geqslant 3, \\ L_3: & [e_i, e_1] = e_{i+1}, & [e_1, e_i] = -e_{i+1}, & i \geqslant 2, \end{array}$$

where $\{e_1, e_2, e_3, \dots\}$ are bases of the algebras L_1, L_2, L_3 and other products vanish.

The following lemma holds.

Lemma 3.1. *The derivations of naturally graded complex thin Leibniz algebras have the following forms:*

$$L_1: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_i) = ((i-1)\alpha_1 + \alpha_2)e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2}, \quad i \geqslant 2,\ n \in \mathbb{N};$$

$$L_2: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_2) = \sum_{k=2}^{n} \beta_k e_k,$$
$$D(e_i) = (i-1)\alpha_1 e_i + \alpha_3 e_{i+1} + \sum_{k=4}^{n} \alpha_k e_{k+i-2}, \quad i \geqslant 3, \quad n \in \mathbb{N};$$

$$L_3: \quad D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_2) = \sum_{k=1}^{n} \beta_k e_k,$$
$$D(e_i) = ((i-2))\alpha_1 + \beta_2)e_i + \sum_{k=3}^{n} \beta_k e_{k+i-2}, \quad i \geqslant 3,\ n \in \mathbb{N},$$

where $\alpha_i, \beta_i \in \mathbb{C}$, $1 \leqslant i \leqslant n,\ n \in \mathbb{N}$.

Proof. Let $D(e_1) = \sum\limits_{k=1}^{n} \alpha_k e_k$, $D(e_2) = \sum\limits_{k=1}^{n} \beta_k e_k$, $n \in \mathbb{N}$.

Using the definition of derivation of algebra L_1 from Theorem 3.1 we obtain the following:

$$\begin{aligned} D(e_3) &= D([e_1,e_1]) = [D(e_1),e_1] + [e_1, D(e_1)] = \sum_{k=1}^{n} \alpha_k [e_k,e_1] + \sum_{k=2}^{n} \alpha_k [e_2, e_k] = \\ &= (2\alpha_1 + \alpha_2)e_3 + \sum_{k=3}^{n} \alpha_k e_{k+2}. \end{aligned}$$

On the other hand

$$\begin{aligned} D(e_3) &= D([e_2,e_1]) = [D(e_2),e_1] + [e_2, D(e_1)] = \sum_{k=1}^{n} \beta_k [e_k, e_1] + \sum_{k=1}^{n} \alpha_k [e_2,e_k] = \\ &= (\beta_1 + \beta_2 + \alpha_1)e_3 + \sum_{k=3}^{n} \beta_k e_{k+1}. \end{aligned}$$

Comparing coefficients from basis elements we have

$$\begin{cases} \alpha_1 + \alpha_2 = \beta_1 + \beta_2, \\ \alpha_k = \beta_k,\ k \geqslant 3. \end{cases} \tag{5}$$

Consider the following:

$$0 = D([e_1, e_2]) = [D(e_1), e_2] + [e_1, D(e_2)] = \sum_{k=1}^{n} \alpha_k [e_k, e_2] + \sum_{k=1}^{n} \beta_k [e_1, e_k] = \beta_2 e_3.$$

From this, we get $\beta_1 = 0$. Then from equality (5) we obtain $\beta_2 = \alpha_1 + \alpha_2$. Hence,

$$D(e_2) = (\alpha_1 + \alpha_2)e_2 + \sum_{k=3}^{n} \alpha_k e_k,\ D(e_3) = (2\alpha_1 + \alpha_2)e_3 + \sum_{k=3}^{n} \alpha_k e_{k+1}.$$

Consider the following:

$$D(e_4) = D([e_3, e_1]) = [D(e_3), e_1] + [e_3, D(e_1)] = (3\alpha_1 + \alpha_2)e_4 + \sum_{k=3}^{n} \alpha_k e_{k+2}.$$

Continuing this process we have

$$D(e_i) = D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = ((i-1)\alpha_1 + \alpha_2)e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2}.$$

Thus, derivations of algebra L_1 has the following form:

$$D(e_1) = \sum_{k=1}^{n} \alpha_k e_k,\ D(e_i) = ((i-1)\alpha_1 + \alpha_2)e_i + \sum_{k=3}^{n} \alpha_k e_{k+i-2}, \quad i \geqslant 2,\ n \in \mathbb{N};$$

Derivations of algebras L_2 and L_3 are obtained in the same way. □

Note that in Theorem 3.1 the algebra L_3 is a thin Lie algebra, i.e., algebra with multiplication (3.1). Therefore, we will study almost inner derivations of thin Leibniz algebras L_1 and L_2.

The following theorem is one of the main results in this paper.

Theorem 3.2. *Let $\mathcal{L}$ be the naturally graded complex thin Leibniz algebra. Then any almost inner derivation on naturally graded complex thin Leibniz algebras is inner.*

Proof. Let $\mathcal{L} = L_1$ and $D \in AID(\mathcal{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathcal{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' = D - R_{a_{e_1}}$, then we have $D'(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

By definition of almost inner derivation for basis e_2 exists $a_{e_2} \in \mathcal{L}$ such that

$$D'(e_2) = [e_2, a_{e_2}] = [e_2, a_{2,1}e_1] = a_{2,1}e_3.$$

Then

$$0 = D'(e_3) = D'([e_2, e_1]) = [D'(e_2), e_1] = a_{2,1}e_4.$$

From this we get $D'(e_2) = 0$.

The next step consider the almost inner derivations of naturally graded thin Leibniz algebras $\mathcal{L} = L_2$. Let $D \in AID(\mathcal{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathcal{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' = D - R_{a_{e_1}}$, then we have $D'(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

By definition of almost inner derivation for basis e_2 exists $a_{e_2} \in \mathcal{L}$ such that

$$D'(e_2) = [e_2, a_{e_2}] = 0.$$

□

4. Almost inner derivation of complex non-Lie thin Leibniz algebras

In this section, we will consider almost inner derivation of complex non-Lie thin Leibniz algebras. We present the following theorem.

Theorem 4.1 ([14]). *Every complex non-Lie thin Leibniz algebra is isomorphic to one of the following two nonisomorphic non-Lie thin Leibniz algebras:*

$$\begin{aligned}
F_1^\infty: \quad & [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad i \geqslant 2, \\
& [e_1, e_2] = \sum_{k=1}^{n} \alpha_{p_k} e_{p_k}, \\
& [e_i, e_2] = \sum_{k=1}^{n} \alpha_{p_k} e_{p_k+i-2}, \qquad i \geqslant 2, \ n \in \mathbb{N}, \\
F_2^\infty: \quad & [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad i \geqslant 3, \\
& [e_1, e_2] = \sum_{s=1}^{m} \beta_{t_s} e_{t_s}, \\
& [e_i, e_2] = \sum_{s=1}^{m} \beta_{t_s} e_{t_s+i-2}, \qquad i \geqslant 3, \ m \in \mathbb{N},
\end{aligned}$$

where $4 \leqslant p_1 < p_2 < \cdots < p_n$ and $4 \leqslant t_1 < t_2 < \cdots < t_m$, and the other products vanish.

The following theorem is one of main the results of this section.

Theorem 4.2. *Let $\mathfrak{L}$ be the complex non-Lie thin Leibniz algebra. Then any almost inner derivation on complex non-Lie thin Leibniz algebras is inner.*

Proof. Let $\mathfrak{L} = F_1^\infty$ is a complex non-Lie thin Leibniz algebra and $D \in AID(\mathfrak{L})$. Then by definition of almost inner derivation, for basis e_1 there exists element $a_{e_1} \in \mathfrak{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' \in AID(\mathfrak{L})$ and $D' = D - R_{a_{e_1}}$, then we get $D'(e_1) = (D - R_{a_{e_1}})(e_1) = 0$. Since $D'(e_1) = 0$, then we obtain the following:

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0,$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

Let $D'(e_2) = \sum\limits_{k=1}^{n} b_k e_k, \ n \in \mathbb{N}$. By derivation conditions we have the following:

$$0 = D'(e_3) = D'([e_2, e_1]) = [D'(e_2), e_1] = \left[\sum_{k=1}^{n} b_k e_k, \, e_1\right] = (b_1 + b_2)e_3 + \sum_{k=4}^{n} b_k e_{k+1}.$$

It follows from the latter that

$$b_1 = -b_2, \ b_i = 0, \ 3 \leqslant i \leqslant n.$$

Then $D'(e_2) = b_1 e_1 - b_1 e_2$.

Since $4 \leqslant p_1 < p_2 < \cdots < p_n$, then

$$0 = D'([e_1, e_2]) = [e_1, D'(e_2)] = [e_1, b_1 e_1 - b_1 e_2] = b_1 e_3 - b_1 \sum_{k=1}^{n} \alpha_{p_k} e_{p_k}.$$

From this we get $b_1 = 0$. Hence, $D'(e_2) = 0$.

Let $\mathfrak{L} = F_2^\infty$. Then by definition AID for e_1 there exists $a_{e_1} \in \mathfrak{L}$ such that $D(e_1) = R_{a_{e_1}}$. Let $D' \in AID(\mathfrak{L})$ and $D' = D - R_{a_{e_1}}$, then we get $D'(e_1) = (D - R_{a_{e_1}})(e_1) = 0$. Then

$$D'(e_3) = D'([e_1, e_1]) = [D'(e_1), e_1] + [e_1, D'(e_1)] = 0;$$
$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] = 0, \ i \geqslant 4.$$

Let $D'(e_2) = \sum\limits_{j=1}^{n} b_j e_j, \ n \in \mathbb{N}$. Consider

$$0 = D'([e_2, e_1]) = [D'(e_2), e_1] = \left[\sum_{j=1}^{n} b_j e_j, e_1\right] = b_1 e_3 + \sum_{j=3}^{n} b_j e_{j+1}, \ n \in \mathbb{N}.$$

From the last equality we have $b_1 = 0, \ b_j = 0, \ 3 \leqslant j \leqslant n$. Hence $D'(e_2) = b_2 e_2$. Since $D'(e_i) = 0, \ i \geqslant 3$, then considering equality

$$0 = D'([e_1, e_2]) = [e_1, D'(e_2)] = b_2 \sum_{s=1}^{m} \beta_{t_s} e_{t_s},$$

we obtain

$$b_2 \cdot \beta_{t_s} = 0, \quad 1 \leqslant s \leqslant m. \tag{6}$$

In algebra F_2^∞ at least one of the parameters $\beta_{t_s} (1 \leqslant s \leqslant m)$ is nonzero, otherwise if all are $\beta_{t_s} = 0 (1 \leqslant s \leqslant m)$, then algebra coincides with algebra of naturally graded thin Leibniz algebras L_2. So there will always be $\beta_{t_{s_0}} \neq 0$, then we have $b_2 = 0$, as a consequence $D'(e_2) = 0$. □

5. Almost inner derivations of solvable Lie algebra whose nilradical is natural graded filifform Lie algebra

In this section we consider almost inner derivations of solvable Lie algebra whose nilradical is natural graded filifform Lie algebra. The multiplication table of natural graded filifform Lie algebra has the next form:

$$\mathfrak{n}_{n,1},\ (n \geqslant 4): \quad [e_i, e_1] = -[e_1, e_i] = e_{i+1},\ 2 \leqslant i \leqslant n-1.$$

Theorem 5.1 ([15]). *There are three of solvable Lie algebras of dimension $(n+1)$ whose nilradical is isomorphic to $\mathfrak{n}_{n,1}$ $(n \geqslant 4)$. The isomorphism classes in the basis $\{e_1, e_2, \dots, e_n, x\}$ are represented by the following algebras:*

$$S_{n+1}(\alpha, \beta) = \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = ((i-2)\alpha + \beta)e_i, & 2 \leqslant i \leqslant n, \\ [e_1, x] = -[x, e_1] = \alpha e_1. \end{cases}$$

The mutually non-isomorphic algebras:

1) $S_{n+1,n}(\beta) := S_{n+1}(1, \beta)$ depending on the value of β, in this case there are three different classes: a) $S_{n+1}(1, 0)$, b) $S_{n+1}(1, n-2)$, c) $S_{n+1}(1, \beta)$, $\beta \notin \{0, n-2\}$;

2) $S_{n+1,2} := S_{n+1}(0, 1)$;

3) $S_{n+1,3}: \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = (i-1)e_i, & 2 \leqslant i \leqslant n, \\ [e_1, x] = -[x, e_1] = e_1 + e_2. \end{cases}$

4) $S_{n+1,4}(\alpha_3, \alpha_4, \dots, \alpha_{n-1}): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_i, x] = -[x, e_i] = e_i + \sum\limits_{l=i+2}^{n} \alpha_{l+1-i} e_l, & 2 \leqslant i \leqslant n, \end{cases}$ *where at least one $\alpha_i \neq 0$ and the first non-vanishing parameter $\{\alpha_3, \alpha_4, \dots, \alpha_{n-1}\}$ can be assumed to be equal to 1.*

The following theorem is the main result in this section.

Theorem 5.2. *Let $\mathfrak{g}$ is solvable Lie algebra with nilradical $\mathfrak{n}_{n,1}$. Then any almost inner derivation solvable Lie algebra with nilradical $\mathfrak{n}_{n,1}$ is inner.*

Proof. Consider the following cases:

Case 1. Let $\mathfrak{g} = S_{n+1}(1, 0)$ be the solvable Lie algebra and let $a = \sum\limits_{i=1}^{n} a_i e_i + a_x x \in \mathfrak{g}$. For basis e_i, x $(i = 1, \dots, n)$ define $ad_a(e_i)$, $ad_a(x)$:

$$ad_a(e_1) = [a, e_1] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_1\right] = -a_x e_1 + \sum_{k=2}^{n-1} a_k e_{k+1},$$

$$ad_a(e_2) = [a, e_2] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_2\right] = -a_1 e_3,$$

$$ad_a(e_i) = [a, e_i] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, e_i\right] = -(i-2)a_x e_i - a_1 e_{i+1}, \quad 3 \leqslant i \leqslant n,$$

$$ad_a(x) = [a, x] = \left[\sum_{k=1}^{n} a_k e_k + a_x x, x\right] = a_1 e_1 + \sum_{k=3}^{n}(k-2) a_k e_k.$$

Let $D \in AID(\mathfrak{g})$. For basis e_i and x exists b_{e_i} and b_x respectively such that $D(e_i) = [b_{e_i}, e_i]$ $(1 \leqslant i \geqslant n)$ and $D(x) = [b_x, x]$. Then

$$D(e_1) = [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \delta_1 x, e_1\right] = -\delta_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1},$$
$$D(e_2) = [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \delta_2 x, e_2\right] = -b_{2,1}e_3.$$

By multiplication of algebra $S_{n+1}(1,0)$ for all $3 \leqslant i \leqslant n$ we obtain:

$$D(e_i) = D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = -(i-2)\delta_1 e_i - b_{2,1}e_{i+1}.$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$.

Consider the following:

$$D([e_1, x]) = [D(e_1), x] + [e_1, D(x)] = (\delta_x - \delta_1)e_1 + \sum_{k=2}^{n-1}((k-1)b_{1,k} - b_{x,k})e_{k+1}.$$

On the other hand $D([e_1, x]) = D(e_1) = -\delta_1 e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_{k+1}$. Comparing coefficients we have:

$$\begin{cases} \delta_x = 0, \\ b_{x,2} = 0, \\ b_{x,j} = (j-2)b_{1,j}, \quad 3 \leqslant j \leqslant n-1. \end{cases}$$

Hence $D(x) = b_{x,1}e_1 + \sum\limits_{k=3}^{n-1}(k-2)b_{1,k}e_k + b_{x,n}e_n$.

Now consider

$$0 = D([e_2, x]) = [D(e_2), x] + [e_2, D(e_x)] = (-b_{2,1} + b_{x,1})e_3.$$

From this we get $b_{x,1} = b_{2,1}$. Then

$$\begin{aligned} D(e_1) &= -\delta_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1}, \\ D(e_2) &= -b_{2,1}e_3, \\ D(e_i) &= -(i-2)\delta_1 e_i - b_{2,1}e_{i+1}, \\ D(x) &= b_{2,1}e_1 + \sum_{k=3}^{n-1}(k-2)b_{1,k}e_k + b_{x,n}e_n. \end{aligned}$$

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = (b_{2,1}+\delta_1)e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

Case 2. Let $\mathfrak{g} = S_{n+1}(1, n-2)$. Analogously as Case 1 we have

$$ad_a(e_1) = [a, e_1] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_1\right] = -a_x e_1 + \sum_{i=2}^{n-1} a_i e_{i+1},$$
$$ad_a(e_j) = [a, e_j] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_j\right] = -(n+i-4)a_x e_j - a_1 e_{j+1}, \quad 2 \leqslant j \leqslant n,$$
$$ad_a(x) = [a, x] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, x\right] = a_1 e_1 + \sum_{k=2}^{n}(n+k-4)a_k e_k$$

and

$$D(e_1) = [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \gamma_1 x, e_1\right] = -\gamma_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1},$$
$$D(e_2) = [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \gamma_2 x, e_2\right] = -(n-2)\gamma_2 e_2 - b_{2,1}e_3,$$
$$D(e_i) = D([e_{i-1}, e_1]) = -((i-2)\gamma_1 + (n-2)\gamma_2)e_i - b_{2,1}e_{i+1}, \quad 3 \leqslant i \leqslant n.$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$. Consider the following

$$D([e_1, x]) = [D(e_1), x] + [e_1, D(x)] = (\gamma_x - \gamma_1)e_1 + \sum_{k=2}^{n-1}((n+k-3)b_{1,k} - b_{x,k})e_{k+1}.$$

On the other hand $D([e_1, x]) = D(e_1) = -\gamma_1 e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_{k+1}$. From this we have

$$\begin{cases} \gamma_x = 0, \\ b_{x,k} = (n+k-4)b_{1,k}, \quad 2 \leqslant k \leqslant n-1. \end{cases}$$

Hence $D(x) = b_{x,1}e_1 + \sum\limits_{k=2}^{n-1}(n+k-4)b_{1,k}e_k + b_{x,n}e_n$.

Consider the next equality

$$\begin{aligned} (n-2)(-(n-2)\gamma_2 e_2 - b_{21}e_3) &= D([e_2, x]) = [D(e_2, x)] + [e_2, D(x)] = \\ &= -(n-2)^2\gamma_2^2 e_2 + (b_{x,1} - (n-1)b_{21})e_3. \end{aligned}$$

From this we get

$$\begin{cases} (n-2)^2\gamma_2 = (n-2)^2\gamma_2 \\ b_{x,1} - (n-1)b_{2,1} = -(n-2)b_{2,1} \end{cases} \Rightarrow \begin{cases} \gamma_2 = 0, n \neq 2 \\ b_{x,1} = b_{2,1} \end{cases}.$$

Hence $D(x) = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1}(n+k-4)b_{1,k}e_k + b_{x,n}e_n$.

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n + (\gamma_1 + \gamma_2)x \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

Case 3. Let $\mathfrak{g} = S_{n+1}(1, \beta)$. Similar as Case 1 we get

$$ad_a(e_1) = [a, e_1] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_1\right] = -a_x e_1 + \sum_{i=2}^{n-1} a_i e_{i+1},$$
$$ad_a(e_j) = [a, e_j] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, e_j\right] = -(j-2+\beta)a_x e_j - a_1 e_{j+1}, \quad 2 \leqslant j \leqslant n,$$
$$ad_a(x) = [a, x] = \left[\sum_{i=1}^{n} a_i e_i + a_x x, x\right] = a_1 e_1 + \sum_{k=2}^{n}(k-2+\beta)a_k e_k$$

and

$$D(e_1) = [b_{e_1}, e_1] = \left[\sum_{k=1}^{n} b_{1,k}e_k + \gamma_1 x, e_1\right] = -\gamma_1 e_1 + \sum_{k=2}^{n-1} b_{1,k}e_{k+1}.$$
$$D(e_2) = [b_{e_2}, e_2] = \left[\sum_{k=1}^{n} b_{2,k}e_k + \gamma_2 x, e_2\right] = -\gamma_2\beta e_2 - b_{2,1}e_3,$$
$$D(e_i) = D([e_{i-1}, e_1]) = [D(e_{i-1}), e_1] + [e_{i-1}, D(e_1)] = -((i-2)\gamma_1 + \beta\gamma_2)e_i - b_{2,1}e_{i+1}, \ 3 \leqslant i \leqslant n.$$

Let $D(x) = \sum\limits_{k=1}^{n} b_{x,k} + \delta_x x$. Now we check the conditions of derivation:

From $D([e_1, x])$ we have

$$\begin{cases} \gamma_x = 0, \\ b_{x,k} = (k-1+\beta)b_{1,k}, \quad 2 \leqslant k \leqslant n-1. \end{cases}$$

From $D([e_2, x])$ we obtain $b_{x,1} = b_{2,1}$. Hence $D(x) = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1}(k-1+\beta)b_{1,k}e_k + b_{x,n}e_n$.

For every element $y = \sum\limits_{i=1}^{n} y_i e_i + y_{n+1}x \in \mathfrak{g}$ we take element $b = b_{2,1}e_1 + \sum\limits_{k=2}^{n-1} b_{1,k}e_k + b_{x,n}e_n + (\gamma_1 + \gamma_2)x \in \mathfrak{g}$ such that $D(y) = ad_b(y)$, and this means that almost inner derivations D is inner.

For the remaining algebras $S_{n+1,2},\ S_{n+1,3},\ S_{n+1,4}(\alpha_3, \ldots, \alpha_{n-1})$ is proved in a similar way. □

6. Almost inner derivations of solvable Leibniz algebra whose nilradical is null filiform algebra

Recall the definition of null-filiform Leibniz algebras.

Definition 6.1 ([5]). *An n-dimensional Leibniz algebra is said to be null-filiform if* $\dim L^i = n+1-i,\ 1 \leqslant i \leqslant n+1$.

Theorem 6.1 ([5]). *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:*

$$NF_n: \quad [e_i, e_1] = e_{i+1},\ 1 \leqslant i \leqslant n-1,$$

where $\{e_1, e_2, \ldots, e_n\}$ *is a basis of the algebra* NF_n.

From this theorem it is easy to see that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra. Note that this notion has no sense in Lie algebras case, because they are at least two-generated.

We present the following well-known results that we will use to study the main result.

Theorem 6.2 ([11]). *Let R be a solvable Leibniz algebra whose nilradical is* NF_n. *Then there exists a basis* $\{e_1, e_2, \ldots, e_n, x\}$ *of the algebra R such that the multiplication table of R with respect to this basis has the following form:*

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [x, e_1] = e_1, & \\ [e_i, x] = -ie_i, & 1 \leqslant i \leqslant n. \end{cases} \tag{7}$$

Theorem 6.3 ([11]). *Let R be a solvable Leibniz algebra such that* $R = NF_k \oplus NF_s + Q$, *where* $NF_k \oplus NF_s$ *is the nilradical of R and* $\dim Q = 1$. *Let us assume that* $\{e_1, e_2, \ldots, e_k\}$ *is a basis of* NF_k, $\{f_1, f_2, \ldots, f_s\}$ *is a basis of* NF_s *and* $\{x\}$ *is a basis of Q. Then the algebra R is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R(\alpha): \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant k-1, & [f_i, f_1] = f_{i+1}, \quad 1 \leqslant i \leqslant s-1, \\ [x, e_1] = e_1, & [x, f_1] = \alpha f_1, \quad \alpha \neq 0, \\ [e_i, x] = -ie_i, \quad 1 \leqslant i \leqslant k, & [f_i, x] = -i\alpha f_i, \quad 1 \leqslant i \leqslant s. \end{cases} \tag{8}$$

$$R(\beta_2, \beta_3, \dots, \beta_s, \gamma) : \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant k-1, \\ [f_i, f_1] = f_{i+1}, \quad 1 \leqslant i \leqslant s-1, \\ [x, e_1] = e_1, \\ [f_i, x] = \sum\limits_{j=i+1}^{s} \beta_{j-i+1} f_j, \quad 1 \leqslant i \leqslant s, \\ [e_i, x] = -ie_i, \quad 1 \leqslant i \leqslant k, \\ [x, x] = \gamma f_s. \end{cases} \tag{9}$$

in the second family of algebras the first non-zero element of the set $(\beta_2, \beta_3, \dots, \beta_s, \gamma)$ can be assumed equal to 1.

Theorem 6.4 ([11]). *Let L be a solvable Leibniz algebra such that $L = NF_{n_1} \oplus NF_{n_2} \oplus ... \oplus NF_{n_s} \dot{+} Q$, where $NF_{n_1} \oplus NF_{n_2} \oplus \dots \oplus NF_{n_s}$ is nilradical of L and $dim Q = 1$. There exists $p, q \in \mathbb{N}$ with $p \neq 0$ and $p + q = s$, a basis $\{e_1^i, e_2^i, \dots, e_{n_i}^i\}$ of NF_{n_i}, for $1 \leqslant i \leqslant p$, a basis $\{f_1^k, f_2^k, \dots, f_{n_k}^k\}$ of NF_{p+k}, for $1 \leqslant k \leqslant q$, and a basis $\{x\}$ of Q such that the multiplication table of the algebra is given by*

$$R_{p,q} = \begin{cases} [e_i^j, e_1^j] = e_{i+1}^j, \ 1 \leqslant i \leqslant n_j - 1, \ [f_i^k, f_1^k] = f_{i+1}^k, \ 1 \leqslant i \leqslant n_k - 1, \\ [x, e_1^j] = \delta^j e_1^j, \ \delta^j \neq 0, \ [f_i^k, x] = \sum\limits_{m=i+1}^{n_k} \beta_{m-i+1}^k f_m^k, \ 1 \leqslant i \leqslant n_k, \\ [e_i^j, x] = -i\delta^j e_i^j, \ 1 \leqslant i \leqslant n_j, \ [x, x] = \sum\limits_{m=1}^{k} \gamma^m f_{n_m}. \end{cases} \tag{10}$$

6.1. Almost inner derivations of solvable Leibniz algebra whose nilradical is NF_n

In the subsection consider almost inner derivations on solvable Leibniz algebra whose nilradical is NF_n.

Let $\mathcal{L}$ solvable Leibniz algebra whose nilradical is NF_n with multiplication the form (7). Then we have the next is one of the main results in this section.

Theorem 6.5. *Let $\mathcal{L}$ solvable Leibniz algebra with nilradical NF_n. Then any almost inner derivations solvable Leibniz algebra $\mathcal{L}$ is inner*

Proof. The solvable algebra $\mathcal{L}$ is a two-generated algebra, i.e. generated by e_1, x. Let $D \in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for basis e_1 there exists b_{e_1} such that $D(e_1) = R_{b_{e_1}}$. Let $D' \in AID(\mathcal{L})$ and let $D' = D - R_{b_{e_1}}$, then we get $D'(e_1) = 0$. Then by multiplication (7) we have

$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] + [e_{i-1}, D'(e_1)] = 0, \quad 2 \leqslant i \leqslant n.$$

Let $D'(x) = \sum\limits_{i=1}^{n} a_i e_i + a_{n+1} x$. Consider

$$\begin{aligned} 0 &= D'(e_1) = D'([x, e_1]) = [D'(x), e_1] + [x, D'(e_1)] = \left[\sum_{i=1}^{n} a_i e_i + a_{n+1} x, e_1 \right] = \\ &= a_{n+1} e_1 + a_1 e_2 + a_2 e_3 + \dots + a_{n-1} e_n. \end{aligned}$$

Hence we have

$$a_1 = a_2 = \dots = a_{n-1} = a_{n+1} = 0$$

and $D'(x) = a_n e_n$.

On the other hand by definition of almost inner derivations for basis x exists $\xi_x \in \mathcal{L}$, such that $D'(x) = [x, \xi_x]$. Further

$$a_n e_n = D'(x) = [x, \xi_x] = [x, \xi_{x,1}e_1 + \xi_{x,2}e_2 + \cdots + \xi_{x,n}e_n + \xi_{x,n+1}x] = \xi_{x,1}e_1.$$

Hence we get $a_n = \xi_{x,1} = 0$. Then $D'(x) = 0$. □

6.2. Almost inner derivations of solvable Leibniz algebra whose nilradical is $NF_k \oplus NF_s$

In this subsection consider almost inner derivations on solvable Leibniz algebra whose nilradical is $NF_k \oplus NF_s$. Let $\mathcal{L} = R(\alpha)$ first solvable Leibniz algebra in Theorem 6.2 with table multiplication (8). Then we get the following results. The following theorem is one the results in this section.

Theorem 6.6. *Let $\mathcal{L} = R(\alpha)$ solvable Leibniz algebra with nilradical $NF_k \oplus NF_s$. Then any almost inner derivation solvable Leibniz algebra $\mathcal{L}$ is inner.*

Proof. The solvable algebra $\mathcal{L}$ is a three-generated algebra, i.e.generated by e_1, f_1, x. Let $D \in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for element e_1 there exists b_{e_1} such that $D(e_1) = R_{b_{e_1}}$. Let $D' \in AID(\mathcal{L})$ and let $D' = D - R_{b_{e_1}}$, then we get $D'(e_1) = 0$. Then by multiplication (8) we have

$$D'(e_i) = D'([e_{i-1}, e_1]) = [D'(e_{i-1}, e_1)] + [e_{i-1}, D'(e_1)] = 0, \quad 2 \leqslant i \leqslant k.$$

Let $D'(x) = \sum\limits_{i=1}^{k} \epsilon_{x,i}e_i + \sum\limits_{j=1}^{s} \phi_{x,j}f_j + a_x x$. Consider

$$\begin{aligned} 0 &= D'(x) = D'([x, e_1]) = [D'(x), e_1] = \left[\sum_{i=1}^{k} \epsilon_{x,i}e_i + \sum_{j=1}^{s} \phi_{x_j}f_j + a_x x, e_1\right] = \\ &= a_x e_1 + \epsilon_{x,1}e_2 + \epsilon_{x,2}e_3 + ... + \epsilon_{x,k-1}e_k. \end{aligned}$$

We have

$$\epsilon_{x,1} = \cdots = \epsilon_{x,k-1} = a_x = 0.$$

Hence $D'(x) = \epsilon_{x,k}e_k + \sum\limits_{j=1}^{s} \phi_{x,j}f_j$. By definition AID (Almost Inner Derivation) for basis x exists the element $b_x \in \mathcal{L}$ such that $D'(x) = [x, b_x]$. Then we obtain

$$\epsilon_{x,k}e_k + \sum_{j=1}^{s} \phi_{x,j}f_j = D'(x) = [x, b_x] = [x, \epsilon_{b_x,1}e_1 + \phi_{b_x,1}f_1] = \epsilon_{b_x,1}e_1 + \alpha\phi_{b_x,1}f_1.$$

Hence

$$\begin{cases} \epsilon_{b_{x,1}} = \epsilon_{x,k} = 0 \\ \phi_{x,1} = \alpha\phi_{b_{x,1}} \\ \phi_{x,j} = 0, \quad 2 \leqslant j \leqslant s \end{cases}.$$

Then $D'(x) = \phi_{x,1}f_1$.

Let $D'(f_1) = \sum\limits_{i=1}^{k} \epsilon_{f_1,i}e_i + \sum\limits_{j=1}^{s} \phi_{f_1,j}f_j + a_{f_1}x$. By definition AID for basis f_1 exists the element $b_{f_1} \in \mathcal{L}$ such that

$$D'(f_1) = [f_1, b_{f_1}] = [f_1, \phi_{b_{f_1},1}f_1 + a_{b_{f_1},x}x] = -\alpha a_{b_{f_1}}f_1 + \phi_{b_{f_1},1}f_2.$$

Comparing the coefficients at the basis elements we get

$$\begin{cases} \epsilon_{f_1,i} = 0, \quad 1 \leqslant i \leqslant k \\ \phi_{f_1,1} = -\alpha a_{b_{f_1}} \\ \phi_{f_1,2} = \phi_{f_1,1} \\ \phi_{f_1,j} = 0, \quad 3 \leqslant j \leqslant s \\ a_{f_1} = 0 \end{cases}.$$

Hence $D'(f_1) = \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2$.

Consider the following:

1)

$$\begin{aligned} D'(f_2) &= D'([f_1, f_1]) = [D'(f_1), f_1] + [f_1, D'(f_1)] = \\ &= [\phi_{f_1,1} f_1 + \phi_{f_1,2} f_2, f_1] + [f_1, \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2] = \\ &= 2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3. \end{aligned}$$

On other hand by definition AID for basis f_2 exists b_{f_2} such that

$$2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3 = D'(f_2) = [f_2, b_{f_2}] = [f_2, \phi_{b_{f_2},1} f_1 + a_{b_{f_2}} x] = -2\alpha a_{b_{f_2}} f_2 + \phi_{b_{f_2},1} f_3.$$

From here we get

$$\phi_{f_1,1} = -\alpha a_{b_{f_2}}, \quad \phi_{f_1,2} = \phi_{b_{f_2},1}. \tag{11}$$

2)

$$\begin{aligned} D'(f_3) = D'([f_2, f_1]) &= [D'(f_2), f_1] + [f_2, D'(f_1)] = \\ &= [2\phi_{f_1,1} f_2 + \phi_{f_1,2} f_3, f_1] + [f_1, \phi_{f_1,1} f_1 + \phi_{f_1,2} f_2] = \\ &= 3\phi_{f_1,1} f_3 + \phi_{f_1,2} f_4. \end{aligned}$$

On other hand by definition AID for f_3 exists b_{f_3} such that

$$3\phi_{f_1,1} f_3 + \phi_{f_1,2} f_4 = D'(f_3) = [f_3, b_{f_3}] = [f_3, \phi_{b_{f_3},1} f_1 + a_{b_{f_3}} x] = -3\alpha a_{b_{f_3}} f_3 + \phi_{b_{f_3},1} f_4.$$

From here we have

$$\phi_{f_1,1} = -\alpha a_{b_{f_3}}, \quad \phi_{f_1,2} = \phi_{b_{f_3},1}. \tag{12}$$

Continuing this process we obtain

$$D'(f_j) = D'([f_{j-1}, f_1]) = j\phi_{f_1,1} f_j + \phi_{f_1,2} f_{j+1}, \quad 4 \leqslant j \leqslant s$$

and by definition AID for $4 \leqslant j \leqslant s$:

$$D'(f_j) = [f_j, b_{f_j}] = -j\alpha a_{b_{f_j}} f_j + \phi_{b_{f_j},1} f_{j+1},$$

and we have that

$$\phi_{f_1,1} = -\alpha a_{b_{f_j}}, \quad \phi_{f_1,2} = \phi_{b_{f_j},1}, \quad 4 \leqslant j \leqslant s.$$

So, we have that $b := b_{f_1} = b_{f_2} = \cdots = b_{f_s}$, $1 \leqslant j \leqslant s$, i.e.

$$D'(f_j) = [f_j, b], \ 1 \leqslant j \leqslant s.$$

Let $T \in AID(\mathcal{L})$, then for basis f_i, $1 \leqslant i \leqslant s$ exists element $b \in \mathcal{L}$ such that $T(f_i) = [f_i, b]$.

Since $D' = D - R_{b_{e_1}}$, then

$$D'(f_1) = D(f_1) - R_{b_{e_1}}(f_1) = [f_1, b] - [f_1, b_{e_1}] = [f_1, b] - [f_1, b] = 0.$$

Thus, according to the multiplication (8) for all $2 \leqslant i \leqslant s$ we have

$$D'(f_i) = D'([f_{i-1}, f_1]) = [D'(f_{i-1}), f_1] + [f_{i-1}, D'(f_1)] = 0$$

and $D'(x) = \phi_{x,1} f_1$.

Now from the following equality

$$0 = \alpha D'(f_1) = D'([x, f_1]) = [D'(x), f_1] = \phi_{x,1} f_2.$$

we get that $\phi_{x,1} = 0$. Then $D'(x) = 0$. □

Let $\mathcal{L} = R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ solvable Leibniz algebra with product table (9). The following result holds. The following theorem is one the results in this section.

Theorem 6.7. *Let $\mathcal{L} = R(\beta_2, \beta_3, \ldots, \beta_s, \gamma)$ solvable Leibniz algebra with nilradical $NF_k \oplus NF_s$. Then any almost inner derivations solvable Leibniz algebra $\mathcal{L}$ is inner.*

Proof. The solvable algebra $\mathcal{L}$ is a three-generated algebra, i.e. generated by e_1, f_1, x. Let $D \in AID(\mathcal{L})$. Then, by the definition of almost inner derivation, for basis f_1 there exists b_{f_1} such that $D(f_1) = R_{b_{f_1}}(f_1)$. Let $D' \in AID(\mathcal{L})$ and let $D' = D - R_{b_{f_1}}$, then we get $D'(f_1) = 0$. Then by multiplication (8) we have

$$D'(f_i) = D'([f_{i-1}, f_1]) = [D'(f_{i-1}, f_1)] + [f_{i-1}, D'(f_1)] = 0, \quad 2 \leqslant i \leqslant s.$$

Let $D'(x) = \sum\limits_{i=1}^{s} \epsilon_{x,i} e_i + \sum\limits_{j=1}^{s} \phi_{x,j} f_j + a_x x$. By the definition of AID for basis x exists $b_x \in \mathcal{L}$ such that

$$D'(x) = [x, b_x] = [x, \epsilon_{b_x,1} e_1 + a_{b_x} x] = \epsilon_{b_x,1} e_1 + a_{b_x} \gamma f_s,$$

and we have that

$$\begin{cases} \epsilon_{x,1} = \epsilon_{b_x,1} \\ \epsilon_{x,i} = 0, \quad 2 \leqslant i \leqslant k \\ \phi_{x,i} = 0, \quad 1 \leqslant i \leqslant s-1 \\ \phi_{x,s} = a_{b_x} \gamma \\ a_x = 0 \end{cases}.$$

Then $D'(x) = \epsilon_{x,1} e_1 + \phi_{x,s} f_s = \epsilon_{b_x,1} e_1 + a_{b_x} \gamma f_s$.

Let $D'(e_1) = \sum\limits_{i=1}^{k} \epsilon_{e_1,i} e_i + \sum\limits_{j=1}^{s} \phi_{e_1,j} f_j + a_{e_1} x$. By the definition of AID for basis e_1 exists element $b_{e_1} \in \mathcal{L}$ such that

$$D'(e_1) = [e_1, b_{e_1}] = [e_1, \epsilon_{b_{e_1}} e_1 + a_{b_{e_1}} x] = -a_{b_{e_1}} e_1 + \epsilon_{b_{e_1},1} e_2.$$

Comparing we get

$$\begin{cases} \epsilon_{e_1,1} = -a_{b_{e_1}} \\ \epsilon_{e_1,2} = \epsilon_{b_{e_1},1} \\ \epsilon_{e_1,i} = 0, \quad 3 \leqslant i \leqslant k \\ \phi_{e_1,i} = 0, \quad 1 \leqslant i \leqslant s, \\ a_{e_1} = 0 \end{cases}.$$

Then $D'(e_1) = \epsilon_{e_1,1} e_1 + \epsilon_{e_1,2} e_2 = -a_{b_{e_1}} e_1 + \epsilon_{b_{e_1},1} e_2$. Further for all $2 \leqslant i \leqslant k$ we have

$$\begin{aligned} D'(e_i) &= D'([e_{i-1}, e_1]) = [D'(e_{i-1}), e_1] + [e_{i-1}, D'(e_1)] = \\ &= [(i-1)\epsilon_{e_1,1} e_{i-1} + e_{e_1,2} e_i, e_1] + [e_{i-1}, \epsilon_{e_1,1} e_1 + \epsilon_{e_1,2} e_2] = \\ &= i\epsilon_{e_1,1} e_i + \epsilon_{e_1,2} e_{i+1}. \end{aligned}$$

Hence $D'(e_j) = j\epsilon_{e_1,1}e_j + \epsilon_{e_1,2}e_{j+1}, \quad 1 \leqslant j \leqslant k$. By the definition of AID for e_i exists elements b_{e_i} such that

$$D'(e_i) = [e_i, b_{e_i}] = [e_i, \epsilon_{e_i,1}e_1 + a_{e_i}x] = -ia_{e_i}e_i + \epsilon_{e_i,1}e_{i+1}, \quad 1 \leqslant i \leqslant k.$$

So, for all $1 \leqslant i \leqslant k$ we have

$$\begin{cases} \epsilon_{e_1,1} = -a_{e_i} \\ \epsilon_{e_2,1} = \epsilon_{e_i,1} \end{cases}.$$

From the last equality we obtain $b_{e_1} = b_{e_2} = \cdots = b_{e_k} =: b, \quad 1 \leqslant i \leqslant k$, i.e. for any $T \in AID(\mathcal{L})$ such that $T(e_i) = [e_i, b], \ 1 \leqslant i \leqslant k$.

Since $D' = D - R_{b_{f_1}}$, then

$$D'(e_1) = (D - R_{b_{f_1}})(e_1) = D(e_1) - R_{b_{f_1}}(e_1) = [e_1, b] - [e_1, b_{f_1}] = [e_1, b] - [e_1, b] = 0.$$

Hence $D'(e_i) = 0, \quad 2 \leqslant i \leqslant k$.

Now consider the following:

$$0 = D'(e_1) = D'([x, e_1]) = [D'(x), e_1] = [\epsilon_{x,1}e_1 + \phi_{x,s}f_s, e_1] = \epsilon_{x,1}e_2.$$

From here we have $\epsilon_{x,1} = 0$. Hence $D'(x) = \phi_{x,s}f_s = a_{b_x}\gamma f_s$.

Consider the following cases.

Case 1. Let $\gamma = 0$. Then $D'(x) = 0$ and $AID(\mathcal{L}) = InDer(\mathcal{L})$.

Case 2. Let $\gamma \neq 0$. Then by the definition of AID for $e_1 + x$ exists $b_{e_1+x} \in \mathcal{L}$ such that

$$\begin{aligned} D'(e_1 + x) &= [e_1 + x, b_{e_1+x}] = \Big[e_1 + x, \sum_{i=1}^{k} \epsilon_{e_1+x,i}e_i + \sum_{j=1}^{s} \phi_{e_1+x,j}f_j + a_{e_1+x}x\Big] = \\ &= (\epsilon_{e_1+x,1} - a_{e_1+x})e_1 + \epsilon_{e_1+x,1}e_2 + \alpha\phi_{e_1+x,1}f_1. \end{aligned}$$

On the other hand

$$a_{b_x}\gamma f_s = D'(x) = D'(x) + D'(e_1) = D'(e_1 + x) = [e_1 + x, b_{e_1+x}].$$

Comparing the coefficients at the basic elements, we obtain the following

$$\begin{cases} \epsilon_{e_1+x,1} = a_{e_1+x} \\ \epsilon_{e_1+x,1} = 0 \\ \alpha\phi_{e_1+x,1} = 0 \\ \gamma a_{b_x} = 0 \end{cases}.$$

The last equation implies $a_{b_x} = 0$, hence $D'(x) = 0$. $\square$

References

[1] K.K.Abdurasulov, B.A.Omirov, Maximal solvable extensions of finite-dimensional nilpotent Lie algebras, arXiv:2111.07651v2 [math.RA] 17 Nov 2021.

[2] J.Adashev, T.Kurbanbaev, Almost inner derivations of some nilpotent Leibniz algebras, *Journal of Siberian Federal University. Mathematics and Physics*, 2020, 13(6), 733–745. DOI: 10.17516/1997-1397-2020-13-6-733–745

[3] Sh.A.Ayupov, K.K.Kudaybergenov, Local derivations on finite-dimensional Lie algebras, *Linear Algebra and its Applications*, **493**(2016), 381–398. DOI: 10.1016/j.laa.2015.11.034

[4] Sh.A.Ayupov, K.K.Kudaybergenov, Local automorphisms on finite-dimensional Lie and Leibniz algebras, In: Algebra, Complex Analysis, and Pluripotential Theory, vol. 264, 2018.

[5] Sh.A.Ayupov, O B.A.mirov, On some classes of nilpotent Leibniz algebras, *Siberian Math. J.*, **42**(2001), no. 1, 15–24.

[6] Sh.Ayupov, B.Yusupov, 2-Local derivations of infinite-dimensional Lie algebras, *Journal of Algebra and Its Applications*, **19**(2020), no. 05, 2050100.

[7] D.Burde, K.Dekimpe, B.Verbeke, Almost inner derivation of Lie algebras, *Journal of Algebra and Its Applications*, **17**(2018), no. 11. DOI: 10.1142/S0219498818502146

[8] D.Burde, K.Dekimpe, B.Verbeke, Almost inner derivations of Lie algebras II, *International Journal of Algebra and Computation*, **31**(2021), no. 02, 341–364

[9] E.M.Cañete, Kh.A.Khudoyberdiyev, The classification of 4-dimensional Leibniz algebras, *Linear Algebra and its Applications*, **439**(2013), 273–288. DOI: 10.1016/J.LAA.2013.02.035

[10] M.Goze, Y.Khakimdjanov, Nilpotent Lie algebras, Mathematics and its Applications, vol. 361, Kluwer Academic Publishers Group, Dordrech, 1996.

[11] J.M.Casas, M.Ladra, B.A.Omirov, I.A.Karimjanov, Classification of solvable Leibniz algebras with null-filiform nilradical, *Linear and Multilinear Algebra*, **61**(2012), no. 6, 758–774.

[12] A.Fialowski, Classification of Graded Lie Algebras with Two Generators, *Moscow University Mathematics Bulletin*, **38**(1983), no. 2, 76–79

[13] C.S.Gordon, E.N.Wilson, Isospectral defermations of compact solvmanifolds, *J. Diferential Geom.*, **19**(1984), no. 1, 214–256.

[14] B.A.Omirov, *Thin Leibniz Algebras*, Mathematical Notes, **80**(2006), no. 2, 244–253.

[15] L.Šnobl, P.Winternitz, A class of solvable Lie algebras and their Casimir invariants, *J. Phys. A*, **38**(2005), no. 12, 2687–2700.

Почти внутренние дифференцирования некоторых алгебр Лейбница

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Аннотация. Настоящая работа посвящена почти внутренним дифференцированиям тонких и разрешимых алгебр Лейбница. А именно мы рассматриваем тонкую алгебру Ли, разрешимую алгебру Ли с нильрадикалом естественной градуированной филиформной алгеброй Ли, натуральную градуированную тонкую алгебру Лейбница, тонкую нелиевскую алгебру Лейбница и разрешимую алгебру Лейбница с нильрадикалом нуль-филиформная алгебра. Доказано, что любые почти внутренние дифференцирования всех этих алгебр являются внутренними дифференцированиями.

Ключевые слова: алгебра Ли, алгебра Лейбница, разрешимая алгебра, нильрадикал, тонкая алгебра Ли, тонкая алгебра Лейбница, дифференцирования, внутренние дифференцирования, почти внутренние дифференцирования.