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The Fredholm Navier-Stokes Type Equations for the de Rham Complex over Weighted Hölder Spaces

Ksenija V. Gagelgans*
Alexander A. Shlapunov†
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. We consider a family of initial problems for the Navie–Stokes type equations generated by the de Rham complex in $\mathbb{R}^n \times [0,T]$, $n \geqslant 2$, with a positive time T over a scale weighted anisotropic Hölder spaces. As the weights control the order of zero at the infinity with respect to the space variables for vectors fields under the consideration, this actually leads to initial problems over a compact manifold with the singular conic point at the infinity. We prove that each problem from the family induces Fredholm open injective mappings on elements of the scales. At the step 1 of the complex we may apply the results to the classical Navier–Stokes equations for incompressible viscous fluid.

Keywords: Navier-Stokes type equations, de Rham complex, Fredholm operator equations.

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The Navier–Stokes equations describe the dynamics of incompressible viscous fluid that is of great importance in applications, see, for instance, [1, 2]. Essential contributions has been published in the research articles [3–6], as well as surveys and books [1, 2, 7], etc. Actually, the problem is solved in the frame of the concept of weak solutions, see, J. Leray [3, 4], E. Hopf [6], O. A. Ladyzhenskaya [2], but no general uniqueness theorem for weak solutions has been known except the two-dimensional case. As far as we know, there are no general results on the global solvability in time for the problem in spaces of sufficiently regular vector fields where the uniqueness theorems for it are available, too. We point out an important direction related to the problem of the existence of regular solutions to the Navier–Stokes equation: S. Smale [8] developed the concept of Fredholm non-linear mappings of Banach spaces applicable to a wide class of non-linear equations of Mathematical Physics (cf. [9] for the steady version of the Navier–Stokes equations).

Recently, the Navier–Stokes type equations were considered in the frame of elliptic differential complexes, see [10–13] over scales of Bochner-Sobolev type spaces parametrized by smoothness index $s \in \mathbb{Z}_+$ where the Sobolev embedding theorems provide point-wise smoothness for sufficiently large s.

On the other hand, results of paper [14] demonstrate that considering the Navier–Stokes type equations over the whole space $\mathbb{R}^n \times [0, +\infty)$ it is important to control the order of zero at the infinity with respect to the space variables for the corresponding solutions. Namely, [14] provides

*ksenija.sidorova2017@yandex.ru https://orcid.org/0000-0001-6709-3334
†ashlapunov@sfu-kras.ru https://orcid.org/0000-0001-7270-8751

an instructive example of a non-linear problem in $\mathbb{R}^n \times [0, T)$, structurally similar to the Cauchy problem for the Navier-Stokes equations and 'having the same energy estimate', but, according to some considerations including numerical simulations, admitting singular solutions of special type for smooth data if $n \geqslant 5$. An essential role in the arguments of this paper plays the fact that certain asymptotic behaviour of the initial data at the infinity with respect to the space variables prevents blow-up behaviour in a finite time interval for the considered solutions, cf. also comments by [15, formulas (4), (5)] related to the data of the Navier–Stokes equations for incompressible fluid.

One of the possibilities to deal with the asymptotic was indicated in [16] where the Navier–Stokes equations for incompressible viscous fluid were considered in $\mathbb{R}^n \times [0, T]$, $n \geqslant 3$, for a positive time T over a scale weighted anisotropic Hölder spaces with the weights controlling the order of decreasing at the infinity with respect to the space variables for the vectors fields under the consideration. This actually leads to an initial problem where the space variables belong to a compact manifold with the singular conic point at the infinity, cf. [17].

In the present paper we extend the results of [16] to a family of initial problems for the Navier–Stokes type equations generated by the de Rham complex in $\mathbb{R}^n \times [0, T]$, $n \geqslant 2$, with a positive time T over a scale of weighted anisotropic Hölder spaces. It is worth to say that the problem, discussed in [14], is included to the consideration. Using the recent developments of the Hodge theory for the de Rham complex over these spaces, see [18, 19], we involve weight indexes $\delta > n/2$, that corresponds to the asymptotic $|x|^{-\delta-|\alpha|}$, $x \in \mathbb{R}^n$, as $|x| \to +\infty$, for the related solutions and their partial derivatives of order $\alpha \in \mathbb{Z}_+$. Namely, we consider the Navier–Stokes type equations in the framework of the theory of operator equations in Banach space and we prove that each initial problem from the family induces Fredholm open injective mappings on elements of the scales. At the step 1 of the complex we may apply the results to the classical Navier–Stokes equations for incompressible viscous fluid.

We do not discuss existence theorems here but we hope that the use of the weighted Hölder spaces with proper weight indexes may exclude the blow-up behaviour of solutions to the Navier–Stokes type equations considered in [14].

1. Function spaces, embedding theorems and a non-linear problem

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the coordinates $x = (x_1, \dots, x_n)$. To introduce weighted Hölder spaces over $\mathbb{R}^n$ we set

$$w(x) = \sqrt{1 + |x|^2}, \ \ w(x, y) = \max\{w(x), w(y)\} \sim \sqrt{1 + |x|^2 + |y|^2}$$

for $x, y \in \mathbb{R}^n$. Let $\delta \in \mathbb{R}$. (Note that δ is tacitly assumed to be nonnegative.) For $s = 0, 1, \dots,$ denote by $C^{s,0,\delta}$ the space of all s times continuously differentiable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{C^{s,0,\delta}} = \sum_{|\alpha| \leqslant s} \sup_{x \in \mathbb{R}^n} (w(x))^{\delta + |\alpha|} |\partial^\alpha u(x)|.$$

For $0 < \lambda \leqslant 1$, we introduce

$$\langle u \rangle_{\lambda,\delta} = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y \\ |x-y| \leqslant |x|/2}} (w(x, y))^{\delta+\lambda} \frac{|u(x) - u(y)|}{|x - y|^\lambda}.$$

and we define $C^{0,\lambda,\delta}$ to consist of all continuous functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{C^{0,\lambda,\delta}} = \|u\|_{C^{0,\lambda}(\overline{U})} + \|u\|_{C^{0,0,\delta}} + \langle u\rangle_{\lambda,\delta},$$

where U is a small neighbourhood of the origin in $\mathbb{R}^n$ and $C^{0,\lambda}(\overline{U})$ is the standard Hölder space over the compact $\overline{U}$. Finally, for $s \in \mathbb{Z}_{\geqslant 0}$, we introduce $C^{s,\lambda,\delta}$ to be the space of all s times continuously differentiable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{C^{s,\lambda,\delta}} = \sum_{|\alpha|\leqslant s} \|\partial^\alpha u\|_{C^{0,\lambda,\delta+|\alpha|}},$$

see [20] for similar weighted Sobolev spaces over $\mathbb{R}^n$.

The normed spaces $C^{s,\lambda,\delta}$ constitute a scale of Banach spaces parametrised by $s \in \mathbb{Z}_{\geqslant 0}$, $\lambda \in [0,1]$ and $\delta \in \mathbb{R}$. The properties of the scale (e.g. natural coninuous and compact embeddings) are well known, see, for instance, [16, 18].

Next, denote by Λ^q the bundle of exterior forms of degree $0 \leqslant q \leqslant n$ over $\mathbb{R}^n$. We write $C^\infty_{\Lambda^q}(\mathbb{R}^n)$ for the space of all differential forms of degree q with C^∞ coefficients on $\mathbb{R}^n$. These space constitute the so-called de Rham complex $C^\infty_{\Lambda^\cdot}(\mathbb{R}^n)$ on $\mathbb{R}^n$ whose differential is given by the exterior derivative d. To display d acting on q-forms one uses the designation $du := d_q u$ for $u \in C^\infty_{\Lambda^q}(\mathbb{R}^n)$ (see for instance [21]); it is convenient to set $d_q = 0$ if $q < 0$ or $q \geqslant n$. As usual, denote by d^*_q the formal adjoint for d_q. Then, as it is known, we have

$$d_{q+1} \circ d_q = 0, \quad d^*_q d_q + d_{q-1} d^*_{q-1} = -E_{m(q)} \Delta, \; 0 \leqslant q \leqslant n, \tag{1}$$

where E_m is the unit matrix of type $(m \times m)$ and $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \dots + \partial^2_{x_n}$ is the usual Laplace operator in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. For a differential operator A acting on sections of the vector bundle Λ^q over $\mathbb{R}^n$, we denote by $C^{s,\lambda,\delta}_{\Lambda^q} \cap \mathcal{S}_A$ the space of all differential q-forms u with components from $C^{s,\lambda,\delta}$, satisfying $Au = 0$ in the sense of the distributions in $\mathbb{R}^n$. This space is obviously closed subspace of $C^{s,\lambda,\delta}_{\Lambda^q}$ and so this is Banach space under the induced norm. Let us introduce anisotropic Hölder spaces which suit well to parabolic theory and are weighted at $x = \infty$ (see [16, 17, 22] and elsewhere).

More generally, given a Banach space $\mathcal{B}$, we denote by $C^{s,0}([0,T],\mathcal{B})$ the Banach space of all mappings $v : [0,T] \to \mathcal{B}$ with finite norm

$$\|v\|_{C^{s,0}([0,T],\mathcal{B})} = \sum_{j=0}^{s} \sup_{t\in[0,T]} \|(d/dt)^j v\|_{\mathcal{B}},$$

where $s \in \mathbb{Z}_{\geqslant 0}$. We also let

$$\langle v\rangle_{\lambda,[0,T],\mathcal{B}} = \sup_{\substack{t',t''\in[0,T]\\ t'\neq t''}} \frac{\|v(t') - v(t'')\|_{\mathcal{B}}}{|t' - t''|^\lambda}$$

and let $C^{s,\lambda}([0,T],\mathcal{B})$ stand for the space of all functions $v \in C^{s,0}([0,T],\mathcal{B})$ with finite norm

$$\|v\|_{C^{s,\lambda}([0,T],\mathcal{B})} = \sum_{j=0}^{s} \Big(\sup_{t\in[0,T]} \|(d/dt)^j v\|_{\mathcal{B}} + \langle (d/dt)^j v\rangle_{\lambda,[0,T],\mathcal{B}} \Big).$$

The Hölder spaces in question will be parametrised several parameters s, λ, δ, and T. By abuse of notation we introduce the special designation $\mathbf{s}(s,\lambda,\delta)$ for the quintuple

$\mathbf{s}(s,\lambda,\delta) := \Big(2s,\lambda,s,\frac{\lambda}{2},\delta\Big)$. Let $C_T^{\mathbf{s}(0,0,\delta)} = C^{0,0}([0,T],C^{0,0,\delta})$ be the space of all continuous functions on $\mathbb{R}^n \times [0,T]$ with finite norm

$$\|u\|_{C_T^{\mathbf{s}(0,0,\delta)}} = \sup_{(x,t)\in\mathbb{R}^n\times[0,T]} (w(x))^{\delta}|u(x,t)|,$$

and, for $0<\lambda\leqslant 1$,

$$C_T^{\mathbf{s}(0,\lambda,\delta)} = C^{0,0}([0,T],C^{0,\lambda,\delta}) \cap C^{0,\lambda/2}([0,T],C^{0,0,\delta})$$

is the space of all continuous functions on $\mathbb{R}^n \times [0,T]$ with finite norm

$$\|u\|_{C_T^{\mathbf{s}(0,\lambda,\delta)}} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{C^{0,\lambda,\delta}} + \sup_{\substack{t',t''\in[0,T]\\ t'\neq t''}} \frac{\|u(\cdot,t')-u(\cdot,t'')\|_{C^{0,0,\delta}}}{|t'-t''|^{\lambda/2}}. \qquad (2)$$

Then $C_T^{\mathbf{s}(s,0,\delta)} = \bigcap_{j=0}^{s} C^{j,0}([0,T],C^{2(s-j),0,\delta})$ is the space of functions on $\mathbb{R}^n\times[0,T]$ with continuous derivatives $\partial_x^\alpha\partial_t^j u$, for $|\alpha|+2j\leqslant 2s$, and with finite norm

$$\|u\|_{C_T^{\mathbf{s}(s,0,\delta)}} = \sum_{|\alpha|+2j\leqslant 2s} \|\partial_x^\alpha\partial_t^j u\|_{C_T^{\mathbf{s}(0,0,\delta+|\alpha|)}}.$$

Similarly,

$$C_T^{\mathbf{s}(s,\lambda,\delta)} = \bigcap_{j=0}^{s}\Big(C^{j,0}([0,T],C^{2(s-j),\lambda,\delta}) \cap C^{j,\lambda/2}([0,T],C^{2(s-j),0,\delta})\Big)$$

is the space of functions on $\mathbb{R}^n\times[0,T]$ with continuous partial derivatives $\partial_x^\alpha\partial_t^j u$, for $|\alpha|+2j\leqslant 2s$, and with finite norm

$$\|u\|_{C_T^{\mathbf{s}(s,\lambda,\delta)}} = \sum_{|\alpha|+2j\leqslant 2s} \|\partial_x^\alpha\partial_t^j u\|_{C_T^{\mathbf{s}(0,\lambda,\delta+|\alpha|)}}.$$

We also need a function space whose structure goes slightly beyond the scale of function spaces $C_T^{\mathbf{s}(s,\lambda,\delta)}$. Namely, given any integral $k\geqslant 0$, we denote by $C_T^{k,\mathbf{s}(s,\lambda,\delta)}$ the space of all continuous functions u on $\mathbb{R}^n\times[0,T]$ whose derivatives $\partial_x^\beta u$ belong to $C_T^{\mathbf{s}(s,\lambda,\delta+|\beta|)}$ for all multi-indices β satisfying $|\beta|\leqslant k$, with finite norm

$$\|u\|_{C_T^{k,\mathbf{s}(s,\lambda,\delta)}} = \sum_{|\beta|\leqslant k} \|\partial_x^\beta u\|_{C_T^{\mathbf{s}(s,\lambda,\delta+|\beta|)}}.$$

For $k=0$, this space just amounts to $C_T^{\mathbf{s}(s,\lambda,\delta+|\beta|)}$, and so we omit the index $k=0$. The normed spaces $C_T^{k,\mathbf{s}(s,\lambda,\delta)}$ are obviously Banach spaces.

We note that the function classes introduced above can be thought of as "physically" admissible solutions to the Navier–Stokes equations (at least for proper numbers δ). By the construction, if $1\leqslant p<+\infty$ and $\delta>n/p$ then there exists a constant $c(\delta,p)>0$ depending on δ and p, such that

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leqslant c(\delta,p)\,\|u\|_{C_T^{\mathbf{s}(0,0,\delta)}} \qquad (3)$$

for all $t\in[0,T]$ and all $u\in C_T^{\mathbf{s}(0,0,\delta)}$.

Also, the following embedding theorem is rather expectable, see [16, 22].

Theorem 1.1. *Also, if $s,s'\in\mathbb{Z}_{\geqslant 0}$, $\delta,\delta'\in\mathbb{R}_{\geqslant 0}$, $\lambda,\lambda'\in[0,1]$ and $k\in\mathbb{Z}_+$ such that $s+\lambda\geqslant s'+\lambda'$ and $\delta\geqslant\delta'$, then the space $C_T^{k,\mathbf{s}(s,\lambda,\delta)}$ is embedded continuously into $C_T^{k,\mathbf{s}(s',\lambda',\delta')}$. The embedding is compact if $s+\lambda>s'+\lambda'$ and $\delta>\delta'$.*

We also need a standard lemma on the multiplication of functions, see [16].

Lemma 1.1. *Let s, k be nonnegative integers and $\lambda \in [0,1]$. If $u \in C^{k,\mathbf{s}(s,\lambda,\delta)}_T$ and $v \in C^{k,\mathbf{s}(s,\lambda,\delta')}_T$, then the product uv belongs to $C^{k,\mathbf{s}(s,\lambda,\delta+\delta')}_T$ and*

$$\|uv\|_{C^{k,\mathbf{s}(s,\lambda,\delta+\delta')}_T} \leqslant c\,\|u\|_{C^{k,\mathbf{s}(s,\lambda,\delta)}_T}\|v\|_{C^{k,\mathbf{s}(s,\lambda,\delta')}_T} \tag{4}$$

with $c>0$ a constant independent of u and v.

However we need scales of weighted Hölder spaces, that fit the refined structure of the Navier–Stokes type equations. First, for $s,k \in \mathbb{Z}_{\geqslant 0}$ and $0<\lambda<\lambda'<1$, we introduce

$$\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_T := C^{k+1,\mathbf{s}(s,\lambda,\delta)}_T \cap C^{k,\mathbf{s}(s,\lambda',\delta)}_T.$$

When given the norm $\|u\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_T} := \|u\|_{C^{k+1,\mathbf{s}(s,\lambda,\delta)}_T} + \|u\|_{C^{k,\mathbf{s}(s,\lambda',\delta)}_T}$, this is obviously a Banach space. The following lemma explains why this scale is important for our exposition, see [16].

Lemma 1.2. *Let s be a positive integer, $k\in\mathbb{Z}_{\geqslant 0}$, $0<\lambda<\lambda'<1$ and $\delta>\delta'$. Then the embedding $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_T \hookrightarrow \mathcal{F}^{k+1,\mathbf{s}(s-1,\lambda,\lambda',\delta')}_T$ is compact.*

Consider the induced vector bundle $\Lambda^q(t)$ over $\mathbb{R}^n\times[0,+\infty)$ consisting of the differential forms with coefficients depending on both the variable $x\in\mathbb{R}^n$ and on the real parameter $t\in[0,+\infty)$. In the sequel we consider the following Cauchy problem. Given any sufficiently regular differential forms $f=\sum\limits_{\#I=q} f_I(x,t)dx_I$ and $u_0=\sum\limits_{\#I=q} u_{I,0}(x)dx_I$ on $\mathbb{R}^n\times[0,T]$ and $\mathbb{R}^n$, respectively, find a pair (u,p) of sufficiently regular differential forms $u=\sum\limits_{\#I=q} u_I(x,t)dx_I$ and $p=\sum\limits_{\#I=q-1} p_I(x,t)dx_I$ on $\mathbb{R}^n\times[0,T]$ satisfying

$$\left\{\begin{array}{rcll} \partial_t u-\mu\Delta u+\mathcal{N}_q u+a\,d_{q-1}p &=& f, & (x,t)\in\mathbb{R}^n\times(0,T),\\ a\,d^*_{q-1}\,u &=& 0, & (x,t)\in\mathbb{R}^n\times(0,T),\\ a\,d^*_{q-2}\,p &=& 0, & (x,t)\in\mathbb{R}^n\times(0,T),\\ u &=& u_0, & (x,t)\in\mathbb{R}^n\times\{0\} \end{array}\right. \tag{5}$$

with positive fixed numbers T and μ, a parameter a that, equals to 0 or 1, and a non-linear term $\mathcal{N}_q u$ that is specified by the following assumptions (see [12] or [10] for more general problems in the context of elliptic differential complexes):

$$\mathcal{N}_q u = M^{(q)}_1(d_q\oplus d^*_{q-1}u,u)+d_{q-1}M^{(q)}_2(u,u) \tag{6}$$

with two bilinear differential operators with constant coefficients and of zero order:

$$M^{(q)}_1(v,u): C^\infty_{\Lambda^{q+1}\oplus\Lambda^{q-1}}(\mathbb{R}^n)\times C^\infty_{\Lambda^q}(\mathbb{R}^n)\to C^\infty_{\Lambda^q}(\mathbb{R}^n), \tag{7}$$

$$M^{(q)}_2(v,u): C^\infty_{\Lambda^q}(\mathbb{R}^n)\times C^\infty_{\Lambda^q}(\mathbb{R}^n)\to C^\infty_{\Lambda^{q-1}}(\mathbb{R}^n). \tag{8}$$

Of course, we have to assume that $d^*_{q-1}u_0=0$ on $\mathbb{R}^n$ if $a=1$, and, as we want to provide the uniqueness for solutions to (5), we have to set $p=0$ if $a=0$.

For $n=1$, $q=0$ and $\mathcal{N}_0 u = u'\,u$ relations (5) reduce obviously to the Cauchy problem for Burgers' equation, [23].

If we denote by $\star$ the $\star$-Hodge operator and by $\wedge$ the exterior product of differential forms then for $n=3$, $q=1$, $a=1$ we may identify 1-forms with n-vector-fields, the operator d_0 with the gradient operator ∇, the operator $(-d_0^*)$ with the divergence operator and the operator d_1 with the rotation operator. Then for the non-linearity

$$\mathcal{N}_1 u = (u \cdot \nabla) u = \star(\star d_1 u \wedge u) + d_0 |u|^2/2, \tag{9}$$

written in the Lamb form, relations (5) are usually referred to as but the Navier-Stokes equations for incompressible fluid with given dynamical viscosity μ of the fluid under the consideration, density vector of outer forces f, the initial velocity u_0 and the search-for velocity vector field u and the pressure p of the flow, see for instance [1]. In [12] these equations with $a = 1$ were considered in Bochner–Sobolev type spaces; as it was explained there, for $q = 0$ and $q = n$ the equations become degenerate in a sense, so, if $a = 1$ we will consider the equations for $1 \leqslant q \leqslant n-1$, only.

Let us comment the example by [14] by P. Plecháč and V. Šverák.

Example 1.1. If $\mu = 1$, $q = 1$, $a = 0$, b is a real parameter, and

$$\mathcal{N}_1 u = (u \cdot \nabla) u \, b + \frac{(1-b)\, \nabla |u|^2 + (\mathrm{div} u) u}{2} = \star(\star d_1 u \wedge u)\, b + \frac{d_0 |u|^2 - (d_0^* u) u}{2} \tag{10}$$

then (5) becomes the non-linear problem in $\mathbb{R}^n \times [0, T)$ considered in [14]. Actually, they consider the 'radial vector fields'

$$u = -v(r,t) x, \tag{11}$$

with functions v of variables t and $r = |x|$. Under the hypothesis of this example the fields are solutions to (5) for $f = 0$ and $u_0 = -v(r,0)x$ if

$$v'_t = v''_{rr} + \frac{n+1}{r} v'_r + (n+2) v^2 + 3 r v v'_r. \tag{12}$$

Next, for v satisfying (12) they consider the self-similar solutions

$$v(r,t) = \frac{1}{2\varkappa(T-t)} w\Big(\frac{r}{\sqrt{2\varkappa(T-t)}} \Big) \tag{13}$$

with functions $w(y)$ binded by the following relations, see [14, (1.9)–(1.11)]:

$$w'' + \frac{n+1}{y} w' - \varkappa\, y\, w' + (n+2) w^2 + 3 y w w' - 2\varkappa\, w = 0, \; y \in (0, +\infty), \tag{14}$$

$$w(0) = \gamma \geqslant 0, \; w'(0) = 0, \; w(y) = y^{-2} \text{ as } y \to +\infty, \tag{15}$$

with a positive parameter $\varkappa$. Based on some analysis of solutions to the steady equation related to (12) and numerical simulations, they made conclusion that for $n > 4$ self-similar solutions (13) may produce singular solutions in finite time to this particular version of (5) for regular data via formula (11) if $\gamma > 0$. However it might be, the numerical simulations can not be arguments in analysis. On the other hand, they showed that certain asymptotic behaviour of the initial data at the infinity with respect to the space variables prevents blow-up behaviour in a finite time interval for the considered type of solutions, at least in the dimension $n = 3$. This gives some hope that the use of the weighted Hölder spaces with proper weight indexes may exclude the blow-up behaviour of solutions to the Navier-Stokes type equations, at least for the non-linearity (10).

Thus, we will investigate the Navier–Stokes type equations (5) over the scale of the weighted Hölder spaces $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}$. With this purpose, for a linear operator $A : X \to Y$ between Banach spaces X, Y with a domain $\mathcal{D}_A \subset X$ we denote by $X_{\mathcal{D}_A}$ the Banach space endowed with the so-called graph norm

$$\|u\|_{X_{\mathcal{D}_A}} = \|u\|_X + \|Au\|_Y \text{ for all } u \in \mathcal{D}_A.$$

Thus, we introduce $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}$ to be the space of all exterior differential q-forms u with the coefficients from $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T}$ endowed with the natural norm. Let also $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$ be a subset of the space $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}$ with the property that $d_q \oplus d^*_{q-1} u \in C^{k,\mathbf{s}(s,\lambda,\delta+1)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}}$; we endow this space with the graph norm

$$\|u\|_{C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}} = \|u\|_{C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}} + \|d_q \oplus d^*_{q-1}\|_{C^{k,\mathbf{s}(s,\lambda,\delta+1)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}}}.$$

Similarly, let $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$ be a subset of $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ with the property that $d_q \oplus d^*_{q-1} u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta+1)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}}$; we endow this space with the graph norm

$$\|u\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}} = \|u\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}} + \|d_q \oplus d^*_{q-1}\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta+1)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}}}.$$

Let us continue with a suitable linearization of (5) over the defined scales.

2. The Navier-Stokes type equations as Fredholm mappings

First, we recall the notion of Fredholm mappings in Banach spaces, see [8]. It is said that a linear bounded operator $A_0 : X \to Y$ has the Fredholm property if its kernel and co-kernel are finite-dimensional subspaces of X and Y, respectively, and its range $R(A)$ is closed in Y. Then a non-linear mapping $A : X \to Y$ is Fredholm if its Fréchet derivative $A'_{|v}$ is a linear bounded Fredholm operator at each point $v \in X$. The Fredholm property provides many uselful information on the operator equation $Au = f$ in the Banach spaces X and Y, see [8] (cf. also [9] for the steady Navier–Stokes equations).

We continue this section with the following linear Cauchy problem for $n \geqslant 2$. Given any $0 \leqslant q \leqslant n$ and any sufficiently regular differential forms

$$w = \sum_{\#I=q} w_I(x,t)dx_I, \;\; f = \sum_{\#I=q} f_I(x,t)dx_I, \;\; u_0 = \sum_{\#I=q} u_{I,0}(x)dx_I$$

on $\mathbb{R}^n \times [0,T]$ and $\mathbb{R}^n$, respectively, find a pair (u,p) of sufficiently regular differential forms $u = \sum\limits_{\#I=q} u_I(x,t)dx_I$ and $p = \sum\limits_{\#I=q-1} p_I(x,t)dx_I$ on $\mathbb{R}^n \times [0,T]$ satisfying

$$\left\{ \begin{array}{rcll} \partial_t u - \mu\Delta u + \mathcal{B}_q(u,w) + a\, d_{q-1} p & = & f, & (x,t) \in \mathbb{R}^n \times (0,T), \\ a\, d^*_{q-1}\, u & = & 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ a\, d^*_{q-2}\, p & = & 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & u_0, & (x,t) \in \mathbb{R}^n \times \{0\} \end{array} \right. \tag{16}$$

where $a\, d^*_{q-1} u_0 = 0$ in $\mathbb{R}^n$ and $\mathcal{B}_q(u,w)$ is given by

$$M^{(q)}_1(d_q \oplus d^*_{q-1} u, w) + d_{q-1} M^{(q)}_2(u,w) + M^{(q)}_1(d_q \oplus d^*_{q-1} w, u) + d_{q-1} M^{(q)}_2(w,u) \tag{17}$$

Again, as we want to provide the uniqueness for solutions to (16), we have to set $p = 0$ if $a = 0$.

We are moving towards expectable uniqueness and existence theorem in the weighted spaces (note that in the standard Sobolev and Hölder spaces are well known, see, for instance, [25]). However, it depends drastically on the paparemeter a.

Theorem 2.1. *Let $n \geqslant 2$, $0 \leqslant q \leqslant n$, $a = 0$. Assume that $s, k \in \mathbb{N}$, $0 < \lambda < \lambda' < 1$, $\delta > n/2$, and $w \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$. Then for any pair*

$$F = (f, u_0) \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q} \tag{18}$$

there is a unique solution $u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,}$ to (16) and, moreover,

$$\|u\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}} \leqslant c(w) \|F\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q}}$$

with a positive constant $c(w)$ independent on F.

Proof. We use the theory of operator equations in Banach spaces and method of integral representation. Namely, Let ψ_μ be the standard fundamental solution of the convolution type to the heat operator $H_\mu = \partial_t - \mu\Delta$ in $\mathbb{R}^{n+1}$, $n \geqslant 1$,

$$\psi_\mu(x,t) = \frac{\theta(t)}{(4\pi\mu t)^{n/2}} \, e^{-\frac{|x|^2}{4\mu t}},$$

where $\theta(t)$ is the Heaviside function. We set

$$\psi_{\mu,q}(x,y,t) = \sum_{|I|=q} \psi_\mu(x-y,t) \, (\star dy_I) \, dx_I,$$

and for q-forms v and u_0 over $\mathbb{R}^n \times [0,T]$ and $\mathbb{R}^n$, respectively, denote by

$$\begin{aligned} (\Psi_\mu v)(x,t) &= \int_0^t \!\! \int_{\mathbb{R}^n} v(y,s) \wedge \psi_{\mu,q}(x,y,t-s) \, ds, \\ (\Psi_{\mu,q,0} u_0)(x,t) &= \int_{\mathbb{R}^n} u_0(y) \wedge \psi_{\mu,q}(x,y,t) \end{aligned}$$

the so-called volume parabolic potential and Poisson parabolic potential, respectively, defined for $(x,t) \in \mathbb{R}^n \times (0,T)$.

Lemma 2.1. *Let $s, k \in \mathbb{Z}_{\geqslant 0}$, $0 < \lambda < 1$ and $\delta > 0$. The parabolic potentials $\Psi_{\mu,q}$ and $\Psi_{\mu,q,0}$ induce bounded linear operators*

$$\Psi_{\mu,q,0} : C^{2s+k,\lambda,\delta}_{\Lambda^q}(\mathbb{R}^n) \to C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q} \cap \mathcal{S}_{H_\mu},$$

$$\Psi_{\mu,q} : C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q} \to C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{H_\mu}}, \ \ \Psi_{\mu,q} : C^{k,\mathbf{s}(s-1,\lambda,\delta+2)}_{T,\Lambda^q} \to C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}.$$

Proof. As the potentials act on the differential forms coefficient-wise, the statement follows from [16, Lemmas 4.5 and 4.8]. □

Now we set

$$W_q u = M_1^{(q)}((d_q \oplus d^*_{q-1})u, w) + M_1^{(q)}((d_q \oplus d^*_{q-1})w, u).$$

Lemma 2.2. *If $k \geqslant \mathbb{N}$ and $\delta > 1$ then following the operators are compact:*

$$\Psi_{\mu,q} B_q(w,\cdot) : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}, \ \ \Psi_{\mu,q} W_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q \mathcal{D}_{d\oplus d^*}} \tag{19}$$

Proof. According to embedding Theorem 1.1, multiplication Lemma 1.1, and Lemma 2.1, the operators

$$\Psi_{\mu,q} B_q(w,\cdot) : C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,} \to C^{k-1,\mathbf{s}(s+1,\lambda,2\delta-1)}_{T,\Lambda^q}, \tag{20}$$

$$\Psi_{\mu,q} W_q : C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to C^{k,\mathbf{s}(s+1,\lambda,2\delta-1)}_{T,\Lambda^q}, \tag{21}$$

$$d_q \oplus d^*_{q-1} \Psi_{\mu,q} W_q : C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to C^{k-1,\mathbf{s}(s+1,\lambda,2\delta)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}}, \tag{22}$$

are continuous if $k \geqslant 1$, $\delta > 0$. As the embeddings

$$C^{k,\mathbf{s}(s+1,\lambda,2\delta-1)}_{T,\Lambda^q} \to C^{k+2,\mathbf{s}(s,\lambda,2\delta-1)}_{T,\Lambda^q}, \quad C^{k-1,\mathbf{s}(s+1,\lambda,2\delta)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}} \to C^{k+1,\mathbf{s}(s,\lambda,2\delta)}_{T,\Lambda^{q+1}\oplus\Lambda^{q-1}} \tag{23}$$

are continuous, we see that the operator $\Psi_{\mu,q} W_q$ maps the space $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$ continuously to $C^{k+1,\mathbf{s}(s,\lambda,2\delta-1)}_{T,\Lambda^q \mathcal{D}_{d\oplus d^*}}$ and the operator $\Psi_{\mu,q} B_q(w,\cdot)$ maps the space $C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}$ continuously to $C^{k+1,\mathbf{s}(s,\lambda,2\delta-1)}_{T,\Lambda^q}$. In particular, the operators

$$\Psi_{\mu,q} B_q(w,\cdot) : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k+1,\mathbf{s}(s,\lambda,\lambda',2\delta-1)}_{T,\Lambda^q},$$

$$\Psi_{\mu,q} W_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to \mathcal{F}^{k+1,\mathbf{s}(s,\lambda,\lambda',2\delta-1)}_{T,\Lambda^q \mathcal{D}_{d\oplus d^*}}$$

are continuous, too, for $k \in \mathbb{N}$, $\delta > 0$. If $k \in \mathbb{N}$, $\delta > 1$ then $2\delta - 1 > \delta$ and hence, by Lemma 1.2, the operators (19) are compact. □

Next we reduce the Cauchy problem (16) to an operator Frredholm equation.

Lemma 2.3. *Let $0 \leqslant q \leqslant n$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, +\infty)$, and $w \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$. Then the operator*

$$I + \Psi_{\mu,q} B_q(w,\cdot) : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \tag{24}$$

is continuously invertible.

Proof. By Lemma 2.2, the operator

$$\Psi_{\mu,q} B_q(w,\cdot) : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$$

is compact. Hence the mapping (24) is a Fredholm linear operator of index zero by the famous Fredholm theorem; in particular, it is continuously invertible if and only if it is injective.

Assume that $u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ and

$$u + \Psi_{\mu,q} B_q(w,u) = 0.$$

Then the properties of the fundamental solution Ψ_μ mean that u is a solution to the following Cauchy problem:

$$\begin{cases} H_\mu u + B_q(w,u) &= 0 \quad (x,t) \in \mathbb{R}^n \times (0,T), \\ u &= 0, \quad (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases}$$

In particular, (1) and an integration by parts yields for all $t \in [0,T]$:

$$\partial_t \|u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)} + \mu \sum_{j=1}^n \|\partial_j u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)} = (B_q(w,u)(\cdot,t), u(\cdot,t))^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}. \tag{25}$$

Using the structure of the operator $B_q(w,\cdot)$, see see that there are positive constants $c_q^{(j)}$ independent on t and u such that

$$\begin{gathered}|(B_q(w,u)(\cdot,t),u(\cdot,t))^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}| \leqslant c_q^{(1)}\|\nabla w\|_{C^{0,\mathbf{s}(0,0,\delta+1)}_{T,\Lambda^{q+1}}}\| \leqslant u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}+\\ +c_q^{(2)}\|w\|_{C^{0,\mathbf{s}(0,0,\delta)}_{T,\Lambda^q}}\|\nabla u(\cdot,t)\|_{L^2_{\Lambda^q}(\mathbb{R}^n)}\|u(\cdot,t)\|_{L^2_{\Lambda^q}(\mathbb{R}^n)} \leqslant \\ \leqslant \frac{\mu}{2}\|\nabla u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)} + \Big(\frac{2(c_q^{(2)})^2}{\mu}\|w\|^2_{C^{0,\mathbf{s}(0,0,\delta)}_{T,\Lambda^q}} + c_q^{(1)}\|\nabla w\|_{C^{0,\mathbf{s}(0,0,\delta+1)}_{T,\Lambda^q}}\Big)\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}\end{gathered} \tag{26}$$

for all $t\in[0,T]$. Now, combining (25) and (26) we conclude that for all $t\in[0,T]$

$$\partial_t\|u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)} \leqslant C_\mu \|u(\cdot,t)\|^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}$$

with a positive constant C_μ independent on u. Finally, the Gronwall lemma, see, for instance, [24, Ch. XII, p. 353, formula (1.2)] yields that $u\equiv 0$, that was to be proved. □

Finally, according to Lemma 2.1, the form

$$v^{(0)} = \Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0,$$

associated with the pair (18), belongs to the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$. Then, there is a unique form $u\in\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$, satisfying

$$u+\Psi_{\mu,q}B_q(w,u) = v^{(0)}. \tag{27}$$

By the properties of the fundamental solution Ψ_μ, we have

$$\left\{\begin{array}{lcll} H_\mu u + B_q(w,u) & = & f & (x,t)\in\mathbb{R}^n\times(0,T),\\ u & = & u_0, & (x,t)\in\mathbb{R}^n\times\{0\}.\end{array}\right. \tag{28}$$

As we have seen in the proof of Lemma 2.3, problem (28) has no more than one solution in the weighted Hölder spaces and that was to be proved. □

Next, we consider the case where $a=1$. At the degree $q=1$ the following theorem was proved in [16, Corollary 5.9].

Theorem 2.2. *Let $n\geqslant 2$, $1\leqslant q\leqslant n-1$, $a=1$. Assume that $s,k\in\mathbb{N}$, $0<\lambda<\lambda'<1$, $n/2<\delta<n$, $\delta\neq(n-1)$ and $w\in\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$. Then for any pair*

$$F=(f,u_0)\in\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}\times C^{2s+k+1,\lambda,\delta}_{\Lambda^q}\cap\mathcal{S}_{d^*} \tag{29}$$

there is a unique solution

$$U=(u,p)\in\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}\times\mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}},$$

to (16) *and, moreover,*

$$\|U\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}\times\mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}}} \leqslant c(w)\|F\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}\times C^{2s+k+1,\lambda,\delta}_{\Lambda^q}}$$

with a positive constant $c(w)$ independent on F.

Proof. Let

$$e(x) = \begin{cases} \dfrac{1}{\pi} \ln |x|, & \text{for} \quad n = 2, \\ \dfrac{1}{\sigma_n} \dfrac{|x|^{2-n}}{2-n}, & \text{for} \quad n \geqslant 3, \end{cases}$$

be the standard two-sided fundamental solution of the convolution type to the Laplace operator in $\mathbb{R}^n$ and σ_n the area of the unit sphere in $\mathbb{R}^n$. We set

$$e_q(x,y) = \sum_{|I|=q} e(x-y)\,(\star dy_I)\,dx_I,$$

and then, for $f \in C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^{q+1}}$,

$$(\Phi_q\, f)(x,t) = \int_{\mathbb{R}^n} f(y,t) \wedge \phi_q(x,y) \tag{30}$$

where $\phi_q(x,y) = (d_{n-q-1})^*_y e_q(x,y)$, $n \geqslant 2$.

Lemma 2.4. *Let $n \geqslant 2$, $q \geqslant 0$, $s \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, $0 < \lambda < 1$, $\delta > 0$, $\delta + 1 - n \notin \mathbb{Z}_+$. The differential d induces a bounded linear operator*

$$d_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta+1)}_{T,\Lambda^{q+1}} \cap \mathcal{S}_d.$$

The related operator equation a normally solvable map; more precisely,

1. *the operator d_q is an isomorphism if $0 < \delta < n-1$ and its inverse is given by the integral operator Φ_q;*
2. *if there is $m \in \mathbb{Z}_+$ such that $n - 1 + m < \delta < n + m$ then d defines an injection and its (closed) range $R^{k,\mathbf{s}(s,\lambda,\lambda',\delta+1)}_{T,\Lambda^{q+1}}$ consists of $f \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta+1)}_{T,\Lambda^{q+1}} \cap \mathcal{S}_{d_{q+1}}$ satisfying for all $t \in [0,T]$ and all $h \in H_{\leqslant m+1,\Lambda^q}$*

   $$(f(\cdot,t), d_q h)_{L^2(\mathbb{R}^n,\Lambda^{q+1})} = 0,$$

   and the left inverse of d is given by the integral operator Φ_q.

Proof. Follows immediately from the definition of the scale $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ and [19, Theorem 3.4] where the range of the operator

$$d_q \oplus d^*_{q-1} : C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to C^{k,\mathbf{s}(s,\lambda,\delta+1)}_{T,\Lambda^{q+1}} \times C^{k,\mathbf{s}(s,\lambda,\delta+1)}_{T,\Lambda^{q-1}}$$

was described. □

Now, if $0 < \delta < n-1$ then $\Phi_q d_q$ maps the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}}$ continuously to $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$ for $\delta + 1 - n \notin \mathbb{Z}_+$. If $n - 1 < \delta < n$, then,

$$(d_q v(\cdot,t), d_q h)_{L^2(\mathbb{R}^n,\Lambda^{q+1})} = (v(\cdot,t), d^*_q d_q h)_{L^2(\mathbb{R}^n,\Lambda^{q+1})} = 0 \tag{31}$$

for all $t \in [0,T]$ and all $h \in H_{\leqslant 1,\Lambda^q}$ and any $v \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$. Applying Lemma 2.4 with $m = 0$ we see that the operator

$$\Phi_q d_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \tag{32}$$

is a continuous if $n/2 < \delta < n$, $\delta + 1 - n \notin \mathbb{Z}_+$, too; in particular, $\Phi_q d_q u = u$ for all $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$. Actually, the operator $\Phi_q d_q$ represents the Leray-Helmholtz type L^2-projection on the subspace $L^2_{\Lambda^q}(\mathbb{R}^n) \cap \mathcal{S}_{d^*}$ of $L^2_{\Lambda^q}(\mathbb{R}^n)$, see [18, Corollary 1] for the isotropic weighted Hölder spaces or [22, Corollary 2] for the anisotropic ones.

Lemma 2.5. *Let $1 \leqslant q \leqslant n-1$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, n)$, $\delta \neq (n-1)$ and $w \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$. Then the operator*

$$I+\Phi_q\, d_q\, \Psi_{\mu,q} W_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \tag{33}$$

is continuously invertible.

Proof. By Lemma 2.2 and the discussion above, the operator

$$\Phi_q\, d_q\, \Psi_{\mu,q} W_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$$

is compact. Hence the mapping (33) is a Fredholm linear operator of index zero by the famous Fredholm theorem; in particular, it is continuously invertible if and only if it is injective.

First, we note that, as the scalar H_μ commutes with the differential operator d_q we conclude that the operators $\Phi_q d_q$ and H_μ commute, too. Then any element u from the kernel of the operator (33) is a solution to the following Cauchy problem:

$$\begin{cases} H_\mu u + \Phi_q\, d_q\, W_q(w,u) & = & 0 & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & 0, & (x,t) \in \mathbb{R}^n \times \{0\}. \end{cases}$$

Next, according, to [22, Corollary 2], the operator $\Phi_q d_q$ represents the Leray-Helmholtz type L^2-projection on the subspace $L^2_{\Lambda^q}(\mathbb{R}^n) \cap \mathcal{S}_{d^*}$ of $L^2_{\Lambda^q}(\mathbb{R}^n)$ and hence

$$(B_q(w,u)(\cdot,t), u(\cdot,t))^2_{L^2_{\Lambda^q}(\mathbb{R}^n)} = (\Phi_q d_q W_q u(\cdot,t), u(\cdot,t))^2_{L^2_{\Lambda^q}(\mathbb{R}^n)}$$

for all $t \in [0,T]$ and all $u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$. Therefore, the injectivity of the operator (33) follows in the same way as for the operator (24). □

To finish the proof of the theorem, we note that the form

$$v^{(0)} = \Phi_q d_q \Big(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 \Big),$$

associated with the pair (29), belongs to the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$ if $\delta \in (n/2, n)$, $\delta + 1 - n \notin \mathbb{Z}_+$. Then, according to Lemma 2.4, there is a unique form $u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$, satisfying

$$u + \Phi_q\, d_q\, \Psi_{\mu,q} W_q u = v^{(0)}. \tag{34}$$

By the discussion above, see (31) and Lemma 2.4,

$$\Phi_q\, d_q\, \Psi_{\mu,q} W_q u = \Phi_q\, d_q\, \Psi_\mu B_q(w,u), \tag{35}$$

$$d_q (I - \Phi_q\, d_q) \Big(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 - \Psi_{\mu,q} B_q(w,u) \Big) = 0$$

if $\delta \in (n/2, n)$, $\delta + 1 - n \notin \mathbb{Z}_+$. Hence, applying statement (1) of Lemma 2.4 we see that there is a unique form $\tilde{p} \in \mathcal{F}^{k-1,\mathbf{s}(s,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$ satisfying

$$d_{q-1} \tilde{p} = (I - \Phi_q\, d_q) \Big(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 - \Psi_\mu B_q(w,u) \Big). \tag{36}$$

Taking in account (34), (35), (36), we conclude that

$$u + \Psi_{\mu,q} B_q(w,u) + d_{q-1} \tilde{p} = \Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0.$$

Then the form $p = H_\mu \tilde{p}$ belongs to the space $\mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$. Again, the properties of the fundamental solution Ψ_μ mean that the pair (u,p) is a solution to the Cauchy problem (16). Moreover u is a solution to

$$\left\{ \begin{array}{lcll} H_\mu u + \Phi_q d_q B_q(w,u) & = & \Phi_q d_q f, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & u_0, & (x,t) \in \mathbb{R}^n \times \{0\}. \end{array} \right. \tag{37}$$

As we have noted in the proof of Lemma 2.5, problem (37) has no more than one solution in the weighted Hölder spaces and then the uniqueness of the solution (u,p) to (16) follows from Lemma 2.4 because $d_{q-1}p = (I - \Phi_q\, d_q)\Big(f - B_q(w,u)\Big)$, that was to be proved. □

Now we may pass to the non-linear problem (5).

Corollary 2.1. *Let $n \geqslant 2$, $0 \leqslant q \leqslant n$, $a = 0$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, +\infty)$. Then the non-linear mapping*

$$\Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \tag{38}$$

is continuous and compact and the mapping

$$I + \Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \tag{39}$$

is continuous, Fredholm, injective and open.

Proof. Since the bilinear form $\mathcal{B}_q$ is symmetric and $\mathcal{B}_q(u,u) = 2\mathcal{N}_q(u)$, we easily obtain

$$\mathcal{N}_q(u') - \mathcal{N}_q(u'') = \mathcal{B}_q(u'', u' - u'') + (1/2)\,\mathcal{B}_q(u' - u'', u' - u''). \tag{40}$$

Then, using Theorem 1.1, multiplication Lemma 1.1, and Lemma 2.1, cf. (20), we see that

$$\|\Psi_{\mu,q}\mathcal{N}_q(u') - \Psi_{\mu,q}\mathcal{N}_q(u'')\|_{C^{k-1,\mathbf{s}(s+1,\lambda,2\delta-1)}_{T,\Lambda^q}} \leqslant$$

$$C_1\, \|u''\|_{C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}}\, \|u' - u''\|_{C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}} + C_2\, \|u' - u''\|^2_{C^{k,\mathbf{s}(s,\lambda,\delta)}_{T,\Lambda^q}},$$

with positive constants C_j independent on u', u''. $C^{k+1,\mathbf{s}(s,\lambda,2\delta-1)}_{T,\Lambda^q}$. In particular, the operator

$$\Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to \mathcal{F}^{k-1,\mathbf{s}(s+1,\lambda,\lambda',2\delta-1)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}},$$

is continuous, for $k, s \in \mathbb{N}$. If $\delta > 1$ then $2\delta - 1 > \delta$ and hence, by Lemma 1.2, the operator is compact:

$$\Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q\mathcal{D}_{d\oplus d^*}}.$$

Equality (40) makes it evident that the Frechét derivative $(I + \Psi_{\mu,q}\mathcal{N}_q)'_{|w}$ of the nonlinear mapping $(I + \Psi_{\mu,q}\mathcal{N}_q)$ at an arbitrary point $w \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ coincides with the continuous linear mapping $(I + \Psi_{\mu,q}B_q(w,\cdot)$. By Lemma 2.3, $(I + \Psi_{\mu,q}B_q(w,\cdot))$ is an invertible continuous linear mapping of the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ and hence the non-linear mapping (42) is Fredholm one. Both the openness and the injectivity of the mapping (42) follow now from the implicit function theorem for Banach spaces, see for instance [26, Theorem 5.2.3, p. 101]. □

Corollary 2.2. *Let $n \geqslant 2$, $1 \leqslant q \leqslant n-1$, $a = 1$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, n)$, $\delta \neq (n-1)$. Then the non-linear mapping*

$$\Phi_q\, d_q\, \Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \tag{41}$$

is continuous and compact and the mapping

$$I+\Phi_q\, d_q\, \Psi_{\mu,q}\mathcal{N}_q : \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \to \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*} \tag{42}$$

is continuous, Fredholm, injective and open.

Proof. Taking into the account the continuity of the operator (32) for $\delta \in (n/2, n)$, $\delta+1-n \notin \mathbb{Z}_+$ we may argue as in the proof of Corollary 2.1, replacing the scale $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$ with the scale $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$ and formula (20) with formulas (21), (22), to conclude that the non-linear operator (41) is compact and continuous, too. Thus, as in the proof of Corollary 2.1, the statement follows now from the implicit function theorem for Banach spaces, see [26, Theorem 5.2.3, p. 101]. □

Let us formulate the corresponding statement for equations (5).

Corollary 2.3. *Let $n \geqslant 2$, $1 \leqslant q \leqslant n-1$, $a = 1$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, n)$, $\delta \neq (n-1)$. Then, for any pair*

$$(f^{(0)}, u_0^{(0)}) \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q} \cap \mathcal{S}_{d^*}$$

admitting the solution $(u^{(0)}, p^{(0)})$ to (5) in $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \times \mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}}$, there is a number $\varepsilon > 0$ with the property that for all data

$$(f, u_0) \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q} \cap \mathcal{S}_{d^*}$$

satisfying the estimate

$$\|f - f^{(0)}\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}} + \|u_0 - u_0^{(0)}\|_{C^{2s+k+1,\lambda,\delta}_{\Lambda^q}} < \varepsilon \tag{43}$$

equations (5) have a unique solution in $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \times \mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}}$.

Proof. Indeed, as we have seen Lemma 2.1 and the properties of the fundamental solution Ψ_μ and the Leray–Helmholtz type projection $\Phi_q d_q$ imply that the solution $(u^{(0)}, p^{(0)})$ to (5) in related to the data $(f^{(0)}, u_0^{(0)})$ satisfies also the operator equation in the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$:

$$(I+\Phi_q\, d_q\, \Psi_{\mu,q}\mathcal{N}_q)u^{(0)} = \Phi_q\, d_q\, (\Psi_{\mu,q} f^{(0)} + \Psi_{\mu,q,0} u_0^{(0)}).$$

Estimate (43) and Corollary 2.2 provide that the norm

$$\|\Phi_q\, d_q(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 - \Psi_{\mu,q} f^{(0)} - \Psi_{\mu,q,0} u_0^{(0)})\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}} \times \mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}}}$$

is sufficiently small for the operator equation

$$(I+\Phi_q\, d_q\, \Psi_{\mu,q}\mathcal{N}_q)u = \Phi_q\, d_q\, (\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0) \tag{44}$$

to admit the unique solution in the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$.

By the discussion in the proof of Theorem 2.2, see (31) and Lemma 2.4,

$$\Phi_q\, d_q\, \Psi_{\mu,q}\mathcal{N}_q u = \Phi_q\, d_q\, \Psi_\mu M_1^{(q)}(d_q \oplus d^*_{q-1} u, u), \tag{45}$$

$$d_q(I - \Phi_q\, d_q)\Big(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 - \Psi_{\mu,q}\mathcal{N}_q u\Big) = 0$$

if $\delta \in (n/2, n)$, $\delta + 1 - n \notin \mathbb{Z}_+$. Hence, applying statement (1) of Lemma 2.4 we see that there is a unique form $\tilde{p} \in \mathcal{F}^{k-1,\mathbf{s}(s,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$ satisfying

$$d_{q-1}\tilde{p} = (I - \Phi_q\, d_q)\Big(\Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0 - \Psi_\mu \mathcal{N}_q u\Big). \tag{46}$$

Taking in account (34), (35), (36), we conclude that

$$u + \Psi_{\mu,q}\mathcal{N}_q u + d_{q-1}\tilde{p} = \Psi_{\mu,q} f + \Psi_{\mu,q,0} u_0$$

Then the form $p = H_\mu \tilde{p}$ belongs to the space $\mathcal{F}^{k-1,\mathbf{s}(s-1,\lambda,\lambda',\delta-1)}_{T,\Lambda^{q-1},\mathcal{D}_{d\oplus d^*}} \cap \mathcal{S}_{d^*}$. Again, the properties of the fundamental solution Ψ_μ mean that the pair (u,p) is a solution to the Cauchy problem (5). Moreover for any solution (u',p') to (5) in the class $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$ is a solution to

$$\left\{ \begin{array}{lcll} H_\mu u' + \Phi_q d_q \mathcal{N}_q u' & = & \Phi_q d_q f, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u & = & u_0, & (x,t) \in \mathbb{R}^n \times \{0\}. \end{array} \right. \tag{47}$$

As the solutions to (47) and (44) in the class $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q,\mathcal{D}_{d\oplus d^*}}$ coincide, Corollary 2.2 yields that problem (47) has no more than one solution in the weighted Hölder spaces, i.e. $u' = u$. Then the uniqueness of the solution (u,p) to (5) follows from Lemma 2.4 because $d_{q-1}p = (I - \Phi_q\, d_q)\big(f - \mathcal{N}_q u\big)$, that was to be proved. $\square$

Finally, in a similar way we obtain the statement corresponding to $a = 0$.

Corollary 2.4. *Let $n \geqslant 2$, $0 \leqslant q \leqslant n$, $a = 0$, s and k be positive integers, $0 < \lambda < \lambda' < 1$, $\delta \in (n/2, +\infty)$. Then, for any pair $(f^{(0)}, u_0^{(0)}) \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q}$ admitting the solution $u^{(0)}$ to* (5) *in the space $\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$, there is a number $\varepsilon > 0$ with the property that for all data $(f, u_0) \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q} \times C^{2s+k+1,\lambda,\delta}_{\Lambda^q}$ satisfying the estimate*

$$\|f - f^{(0)}\|_{\mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}} + \|u_0 - u_0^{(0)}\|_{C^{2s+k+1,\lambda,\delta}_{\Lambda^q}} < \varepsilon \tag{48}$$

equations (5) *have a unique solution $u \in \mathcal{F}^{k,\mathbf{s}(s,\lambda,\lambda',\delta)}_{T,\Lambda^q}$.*

Thus, we see that there is crucial difference between problem (5) in the "local" situation where $a = 0$ and the "non-local" situation where $a = 1$. As in the second case the problem is equivalent to a "pseudo-differential" Cauchy problem (47), we observe some restrictions on possible asymptotic behaviour of solutions at the infinity with respect to the space variables and some additional loss of smoothness of the solutions. The reason is that we deal with scales of parabolic Hölder spaces, where the dilation principle is partially neglected with regard to the weight because we need to provide some continuity of the integral operators Φ_q and $\Phi_q d_q$.

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Уравнения Фредгольма типа Навье-Стокса для комплекса де Рама над весовыми пространствами Гёльдера

Ксения В. Гагельганс
Александр А. Шлапунов
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Мы рассматриваем семейство задач Коши для уравнений типа Навье-Стокса, порожденных комплексом де Рама в $\mathbb{R}^n \times [0,T]$, $n \geqslant 2$, с положительным временем T над шкалой весовых анизотропных пространств Гёльдера. Поскольку веса определяют порядок нуля на бесконечности по пространственным переменным для рассматриваемых векторных полей, это фактически приводит к задачам Коши над компактным многообразием с конической особой точкой на бесконечности. Доказано, что каждая задача семейства индуцирует открытые инъективные отображения Фредгольма на элементах шкал. На шаге 1 комплекса можно применить результаты к классическим уравнениям Навье–Стокса для несжимаемой вязкой жидкости.

Ключевые слова: уравнения типа Навье–Стокса, комплекс де Рама, операторные уравнения Фредгольма.