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On the Sharp Estimates for Maximal Operators

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Abstract. The paper deals with boundedness problem for the maximal operators associated with hypersurfaces in the space of square integrable functions. A necessary condition for boundedness is given in the case of one nonvanishing principal curvature. A criterion for the boundedness is obtained for a particular class of convex hypersurfaces.

Keywords: maximal operator, Fourier transform, hypersurface, boundedness.

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1. Introduction and preliminaries

Let S be a smooth hypersurface in $\mathbb{R}^{n+1}$ and let $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ be a smooth non-negative function with compact support. We consider the associated averaging operator A_t given by

$$A_t f(x) = \int\limits_S f(x - ty)\psi(y)d\sigma(y),$$

where $d\sigma$ denotes the surface measure on S. Let $Mf(x)$ be the associated maximal operator, i.e.

$$Mf(x) = \sup_{t>0} |A_t f(x)|. \tag{1}$$

We are interested in obtaining L^p-boundedness of M, i.e., we would like to determine for all $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|Mf\|_{L^p} \leqslant C_p \|f\|_{L^p}, \tag{2}$$

where $\|\cdot\|_{L^p}$ is the norm of the Lebesgue space $L^p(\mathbb{R}^{n+1})$.

We further work under the following transversality assumption on S:

Assumption 1.1 (Transversality)**.** *The affine tangent plane $x + T_xS$ to S through x does not pass through the origin for every $x \in S$.*

We denote by $p(S)$ the minimal Lebesgue exponent such that the maximal operator M is L^p-bounded for any $p > p(S)$, but unbounded for $p < p(S)$, assuming that no mitigating effect through the vanishing of the density p occurs. Stein [1] proved that if S is the Euclidean unit sphere centered at the origin in $\mathbb{R}^{n+1}$, $n \geqslant 2$, then the corresponding spherical maximal operator is bounded on $L^p(R^{n+1})$ for every $p > \dfrac{n+1}{n}$, i.e., the a priori estimate (2) is valid and the

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maximal function M is unbounded for $p \leqslant \dfrac{n+1}{n}$. In particular, in this case $p(S) = \dfrac{n+1}{n}$. Later an analogous result in dimension $n = 1$ was proven by J. Bourgain [2]. The key property of spheres which allows to prove such results is the non-vanishing of the Gaussian curvature on spheres. The problem of boundedness of the maximal operators in L^p associated with hypersurfaces for which the Gaussian curvature vanishes at some points is open. In [3] it was proved that if S is a convex hypersurface of finite linear type (see Definition 3.2) and $p \geqslant 2$ then the condition

$$d(x,H)^{-\frac{1}{p}} \in L^1_{loc}(S) \tag{3}$$

is necessary and sufficient for the boundedness of the maximal operators in $L^p(\mathbb{R}^{n+1})$, where H is any hyperplane not passing through the origin and $d(x,H)$ is the distance from $x \in S$ to H. Moreover, in [4] (see Theorem 3.6) the same authors proved the necessity of condition (3). Ikromov I. A., Kempe M., Müller D [5] proved that if $S \subset \mathbb{R}^3$ is a smooth hypersurface, then for $p > h \geqslant 2$ (where $h(S)$ is the hypersurface height introduced in the classical work by A. N. Varchenko [6]) the maximal operator is bounded. If S is an analytic hypersurface, then for $p \leqslant h$ the maximal operator is unbounded. However, for $p = h$ the question of the boundedness of the maximal operator remained open for smooth hypersurfaces.

In this paper we assume that at each point at least one of the principal curvatures is nonzero. From the results of S. D. Sogge [7] follows that the associate maximal operator is bounded in L^p for $p > 2$. But the question of boundedness of the maximal operator for $p = 2$ remains open. This article is devoted to obtaining a necessary and sufficient condition for the boundedness of the maximal operator in the space L^2. These results generalize the result of [3] for the class of convex functions with an isolated zero at the origin.

2. Main result

Let $S = \{(x, y, z(x, y))\} \subset \mathbb{R}^3$ be a given surface and H be its tangent plane at the origin. Denote by $d(Y,H)$ the distance from the point $Y := (x, y, z(x, y))$ of the surface to its tangent plane H. Moreover, let the surface S in some neighborhood of the origin be given by the following formula:

$$z(x, y) = x^2(1 + A(x) + B(y)) + \varphi(y) + C, \tag{4}$$

where $C \neq 0$ is a constant $A(0)=B(0)=B'(0)=0$ and $\varphi(y)$ is a convex function, and derivatives of all orders vanish at the origin, i.e. $0=\varphi(0)=\varphi^{'}(0)=\ldots\varphi^n(0)=\ldots$. Moreover, let $\varphi^{''}(x)\geqslant 0$ for all $x \in U$. If there exists $x_1 > 0$ such that $\varphi'(x_1) = 0$, then for $x \in [0, x_1]$ we have: $\varphi'(x) = 0$ and therefore $\varphi(x) \equiv 0$ on the segment $[0, x_1]$. Then, it is easy to show that the maximal operator (1) is unbounded in L^2. Similarly, if for some $x_2 < 0$ $\varphi'(x_2) = 0$, then the maximal operator defined as (1) is unbounded in L^2. Therefore, in what follows, we will assume that for $x > 0$, $\varphi'(x) > 0$ and $\varphi'(x) < 0$ for $x < 0$. It follows that φ on the segment $[0, \delta]$ (where $\delta > 0$ is some positive number) is strictly increasing and strictly decreasing on the segment $[-\delta, 0]$, in particular $\varphi(y) > 0$ for any $y \neq 0$. Thus, for each $u > 0$ we have $\varphi^{-1}(u) = \{z_1, z_2\}$, where $z_2 > 0$ and $z_1 < 0$. We define the function γ by the formula

$$\gamma(u) = z_2 - z_1 = |z_2| + |z_1|. \tag{5}$$

Next, let the maximal operator be defined as in (1), where $supp\, \psi \subset U \subset S$.

Then our main result reads:

Theorem 2.1. *If S is a convex surface given by (4) and the function φ is a flat function at the origin, then there exists a neigborhood U of $(0,0,C)$ such that following conditions are equivalent:*

1. *The maximal operator M defined as (1) is bounded in the space $L^2(\mathbb{R}^3)$.*
2. *The following inclusion*

$$\frac{1}{(d(Y,H))^{\frac{1}{2}}} \in L^1(S\bigcap U)$$

holds true.

3. *$\ln\varphi(y)$ is integrable on $[-\varepsilon,\varepsilon]$, where $\varepsilon>0$ is sufficiently small.*
4. *$\sum\limits_{n=1}^{\infty}\gamma(2^{-n})$ is a convergent series.*

3. Preliminaries

Definition 3.1. *A function $\phi(x)$ is called convex if for any vectors x,$y \in U\subset\mathbb{R}^n$ (where U is some convex neighborhood of the origin) and for any non-negative numbers α, β satisfying the condition $\alpha+\beta=1$, the following inequality holds:*

$$\phi(\alpha x+\beta y)\leqslant\alpha\phi(x)+\beta\phi(y).$$

Definition 3.2. *A function $\phi(x)$ is called a function of finite linear type in the direction of the unit vector $\xi\in\mathbb{R}^n$ at the origin if there exists $N\geqslant 2$ such that $D_\xi^N(\phi(0))\neq 0$, where $D_\xi\phi$ is the derivative of the function ϕ in the direction of the vector ξ at the origin of the coordinate system. If Φ is of finite linear type for every unit vector $\xi\in\mathbb{R}^n$, then the hypersurface $S=\{(x,\Phi(x))\}$ is said to be of finite linear type at the point $(0,\Phi(0))$ (cf. the definition of the finite linear type in [8]).*

Remark 1. *If $n=2$, and Φ is convex and has a finite linear type in the direction of some unit vector $\xi\in\mathbb{R}^2$ and does not have a finite linear type, then, after a possible linear transformation, it can be written in the form*

$$\Phi(x,y)=(C_0+R(x,y))x^{2k}+\sum_{\alpha=0}^{2k-1}x^\alpha R_\alpha(y),$$

where R,R_α are ($\alpha=0,1,\dots,2k-1$)-smooth functions and R satisfies $R(0,0)=0$ and R_α are flat functions, $C_0>0$.

We give the following proposition, whose proof is analogous to the proof of G. Shulz' theorem [9] (see also [10]).

Proposition 3.3. *Suppose $rank(Hess\Phi(0,0))=1$. Then, after appropriate linear changing of the coordinates, the function Φ in a sufficiently small neighborhood of the origin can be represented in the form:*

$$\Phi(x,y)=b(x,y)(y-\psi(x))^2+\varphi(x),$$

where b, φ and ψ are smooth functions, such that $\psi'(0)=\psi(0)=0$ and $b(0,0)\neq 0$.

If a function $\varphi(x)$ has a zero of finite order at $x=0$, then it can be written as $\varphi(x)=x^n\beta(x)$, where β is smooth function with $\beta(0)\neq 0$. In this case it follows from A. Greenleaf's Theorem [11] (also see [12]) that for $p>1+\dfrac{n}{n+2}$ the maximal operator is bounded in L^p. But this is not a sharp bound for p, (see [13]).

Proposition 3.4. *Let the function* Φ *be given in the form*

$$\Phi = b(x,y)(y-\psi(x))^2 + \varphi(x).$$

If the function Φ *is convex and* φ *is a flat function, then so is* ψ.

Proof. For the proof see [14]. □

Now we formulate the following theorems due to A. Iosevich and E. Sawyer [3] (see also [15]), which contain important methods for proving the boundedness of maximal operators.

Theorem 3.5. *Suppose* τ *is a distribution supported in a ball* B *of radius* C_1 *with* $|\hat{\tau}(\xi)| \leqslant C_1$, $\max\{|x| : x \in supp\,\tau\} \leqslant C_2$. *Suppose moreover that*

$$\Big(\int_1^2 |\hat{\tau}_t(\xi)|^2 dt\Big)^{\frac{1}{2}} \leqslant C_1(1+|\xi|)^{-\frac{1}{2}}\gamma(|\xi|),$$

and

$$\Big(\int_1^2 |\nabla\hat{\tau}_t(\xi)|^2 dt\Big)^{\frac{1}{2}} \leqslant C_2(1+|\xi|)^{-\frac{1}{2}}\gamma(|\xi|),$$

where $\hat{\tau}_t(\xi) = \hat{\tau}(t\xi)$ *and* γ *is bounded and nonincreasing on* $[0,\infty)$, *satisfying the condition* $\sum\limits_{n=0}^{\infty}\gamma(2^n) < \infty$. *For* $t>0$ *define,*

$$M_\tau f(x) = \sup_{t>0}|f * \tau_t(x)|.$$

Then

$$\|M_\tau f\|_{L^2} \leqslant C\sqrt{C_1C_2}\|f\|_{L_2}.$$

Theorem 3.6. *Let* S *be a smooth hypersurface,* ψ *a smooth cutoff function,* $Mf(x)$ *be defined as in* (1). *Suppose* H *is a hyperplane not passing through the origin and* $d(x,H)$ *is the distances from* $x \in S$ *to* H. *If the maximal operator* M *is bounded on* $L^p(\mathbb{R}^{n+1})$ *for some* $p>1$, *then* $d^{-\frac{1}{p}}(x,H)$ *is locally integrable on* S. *In particular, if* $S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x)+c\}$, *where* Φ *is homogeneous of degree* m *and* c *is a some nonzero constant, and if* M *is bounded on* $L^p(\mathbb{R}^{n+1})$ *for some* $p>1$, *then* $p > \frac{m}{n}$ *and* $\Phi(\omega)^{-1} \in L^{\frac{1}{p}}(S^{n-1})$.

4. Proof of the main result

The proof of $1 \Rightarrow 2$ follows from Theorem 3.6.

We prove $2 \Rightarrow 3$. Let the surface S be defined by (4), and H is its tangent plane and $Y = (x, y, z(x,y))$. Since

$$\frac{\partial z}{\partial x}\Big|_{(0,0)} = \frac{\partial z}{\partial y}\Big|_{(0,0)} = 0$$

according to the formula

$$z - z_0 = \frac{\partial z}{\partial x}(x-x_0) + \frac{\partial z}{\partial y}(y-y_0),$$

the tangent plane is defined by the equation $z = C$ and so the distance from point Y of the surface to the tangent plane is equal to

$$d(Y,H) = |\, x^2(1+A(x)+B(y)) + \varphi(y)\,| \,.$$

Consider the integral

$$I = \int\limits_{-\epsilon}^{\epsilon}\int\limits_{-\epsilon}^{\epsilon} \frac{dxdy}{\sqrt{\mid x^2(1+A(x)+B(y))+\varphi(y)\mid}}. \tag{6}$$

Let us use the new change of variables

$$\begin{cases} \sqrt{1+A(x)+B(y)}x = \xi \\ y = \tau. \end{cases}$$

Now, we calculate the Jacobian of this system and make sure that the Jacobian is nonzero. Moreover, the Jacobian of this system at zero is equal to $J(0,0)=1$ and by the inverse mapping theorem there are such smooth functions $y=\tau$, $x=A(\xi,\tau)$ that the integral (6) has a lower estimate:

$$I \geqslant \int_{-\varepsilon}^{\varepsilon} d\tau \int_{-\delta}^{\delta} \frac{a(\tau,\xi)d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}},$$

where $a(\tau,\xi)=A'_{\xi}(\tau,\xi)>0$, ε and δ are some positive numbers. Consider the inner integral and do some transformations.

$$\int_{-\delta}^{\delta} \frac{a(\tau,\xi)d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} = \int_{-\delta}^{\delta} \frac{(a(\tau,\xi)-a(\tau,0))d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} + a(\tau,0)\int_{-\delta}^{\delta} \frac{d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} = I_1+I_2.$$

Let us now estimate the integral $I_1 = \int\limits_{-\delta}^{\delta} \frac{(a(\tau,\xi)-a(\tau,0))d\xi}{\sqrt{\xi^2+\varphi(\tau)}}$. By the Lagrange mean value theorem we have $a(\tau,\xi)-a(\tau,0)=a'_{\xi}(\tau,\zeta)\xi$. Then for the integral I_1 we get

$$\mid I_1\mid \leqslant \int_{-\delta}^{\delta} \frac{\mid a'_{\xi}(\tau,\zeta)\xi \mid d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} \leqslant \big|\max_{\zeta} a'_{\xi}(\tau,\zeta)\big| \int_{-\delta}^{\delta} \frac{\mid \xi\mid d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} =$$

$$= 2\big|\max_{\zeta} a'_{\xi}(\tau,\zeta)\big| \int_{0}^{\delta} \frac{\xi d\xi}{\sqrt{\mid \xi^2+\varphi(\tau)\mid}} \leqslant C,$$

where C is some constant that does not depend on τ. So $I_1=O(1)$. For the integral I_2 we make the change of variables $\xi = z\sqrt{\mid \varphi(\tau)\mid}$ and we get that

$$I_2 = 2a(\tau,0)\int\limits_{0}^{\frac{\delta}{\sqrt{\mid\varphi(\tau)\mid}}} \frac{\sqrt{\mid \varphi(\tau)\mid}}{\sqrt{\mid \varphi(\tau)\mid}}\cdot\frac{dz}{\sqrt{\mid z^2\pm 1\mid}} =$$

$$= 2a(\tau,0)\int\limits_{0}^{\frac{\delta}{\sqrt{\mid\varphi(\tau)\mid}}} \frac{dz}{\sqrt{\mid z^2\pm 1\mid}} = O(1)+2a(\tau,0)\int\limits_{2}^{\frac{\delta}{\sqrt{\mid\varphi(\tau)\mid}}} \frac{dz}{z}.$$

Thus, for the integral I_2 we get

$$I_2 = 2a(\tau,0)ln\mid \varphi(\tau)\mid + O(1).$$

Considering all this, for the integral (6) we obtain the following equality:

$$I = 2a(0,0)\int_{-\varepsilon}^{\varepsilon} ln\mid\varphi(\tau)\mid d\tau + O(1).$$

The last equality shows that if $ln \mid \varphi \mid \notin L^1(-\varepsilon, \varepsilon)$, then $d^{-\frac{1}{2}}(Y, H) \notin L^1_{loc}(S \cap U)$.

The proof of $(3 \Rightarrow 4)$ is given in [14].

Finally, we prove $4 \Rightarrow 1$. It is known that $\hat{\tau}(\xi)$ can be written as the following multiple integral:

$$\hat{\tau}(\xi) = \int_{\mathbb{R}^2} e^{i(\xi_3 z(x,y) + \xi_1 x + \xi_2 y)} \varphi(x, y, z(x, y)) \sqrt{1 + |\nabla z(x, y)|^2} dxdy, \tag{7}$$

where $z(x, y)$ is the convex surface and given by formula (4). We may assume that $\xi_3 \neq 0$ otherwise $\hat{\tau}(\xi)$ is the Fourier transform of a smooth function with compact support and so $\hat{\tau}(\xi) = O(|\xi|^{-N})$, where N is a large number. We can choose N as large as we need. We define the following phase function

$$\Phi(x, y, s) = z(x, y) + s_1 x + s_2 y,$$

where $s_j = \dfrac{\xi_j}{\xi_3}$, $j = 1, 2$. If $|s| := |s_1| + |s_2| > 1$ then there exists a positive number $\delta > 0$ such that inequality $0 < \delta|s| \leqslant |\nabla\Phi(x, y, s)|$ is satisfied for any point $(x, y) \in supp(\varphi(\cdot, z(\cdot)))$. By using the integration by parts formula N times we have $\hat{\tau}(\xi) = O(|\xi|^{-N})$. It is a much better estimate what we want. Therefore, we consider the case $\max\{|s_1|, |s_2|\} \leqslant 1$ and write (7) as follows:

$$J(\lambda, s) := \int_{\mathbb{R}^2} e^{i\lambda\Phi(x,y,s)} a(x, y) dxdy, \tag{8}$$

where $a(x, y) = \varphi(x, y, z(x, y))\sqrt{1 + |\nabla z(x, y)|^2}$ is an amplitude function. Further we study the behavior of the oscillatory integral (8). Therefore, first, we consider the following one-dimensional oscillatory integral:

$$J(\lambda, y, s) = \int e^{i\lambda(x^2(1+A(x)+B(y)) - s_1 x)} a(x, y) dx. \tag{9}$$

4.1. The form of the critical point and critical value

First, consider the following function $f(x, s) = x^2(1 + A(x)) - s_1 x$. We apply the parametric Morse lemma to the function $f(x, s)$ on the critical point. For this, we find critical points of the function f, i.e. we solve the following equation:

$$\frac{\partial f}{\partial x} = x(2 + 2A(x) + xA'(x)) - s_1 = 0. \tag{10}$$

Lemma 4.1. *The solution to equation* (10) *in a neigborhood of the origin* $(0, 0)$ *has the form* $x_0^c = s_1 B_0(s_1)$, *where* B_0 *is a smooth function defined in a sufficiently small neighborhood of the origin and* $B_0(0) = \dfrac{1}{2}$.

Proof. We use the change of variables $x = s_1 y$ in equation (10), i.e.

$$s_1 y(2 + 2A(s_1 y) + s_1 y A'(s_1 y)) - s_1 = 0. \tag{11}$$

Assume $s_1 \neq 0$. Then equation (11) is equivalent to the following form:

$$y(2 + 2A(s_1 y) + s_1 y A'(s_1 y)) - 1 = 0. \tag{12}$$

We investigate solution to equation (12) in a neighborhood of the point $(0, \frac{1}{2})$ with respect to (s_1, y). Due to the implicit function theorem, it has a smooth solution $y = B_0(s_1)$, and $B_0(0) = \dfrac{1}{2}$. Thus $x_0^c = s_1 B_0(s_1)$ is a unique solution to equation (10). □

Therefore, by the Morse lemma, we can rewrite the function $f(x,s)$ in the following form

$$f(x,s) = (x - s_1B_0(s_1))^2G(x,s_1) + s_1^2B_2(s_1),$$

where $G(0,0) = 1$. After simple transformations, the phase function can be written as

$$x^2(1 + A(x) + B(y)) - s_1x = x^2(1 + A(x)) - s_1x + x^2B(y) =$$

$$= (x - s_1B_0(s_1))^2(G(x,s_1) + B(y)) + 2(x - s_1B_0(s_1))s_1B_0(s_1)B(y) +$$
$$+ s_1^2B_0^2(s_1)B(y) + s_1^2B_2(s_1).$$

Denoting $x - s_1B_0(s_1)$ by $\tilde{x}$ we get

$$x^2(1 + A(x) + B(y)) - s_1x = \tilde{x}^2(G(\tilde{x} + s_1B_0(s_1), s_1) + B(y)) +$$

$$+ 2\tilde{x}s_1B_0(s_1)B(y) + s_1^2B_2(s_1).$$

If we denote $\tilde{G}(\tilde{x}, s_1) = G(\tilde{x} + s_1B_0(s_1), s_1) + B(y)$ then (9) takes the form

$$J(\lambda, y, s) = \int e^{i\lambda(\tilde{x}^2\tilde{G}(\tilde{x},s_1) + 2\tilde{x}s_1B_0(s_1)B(y) + s_1^2B_2(s_1))}\tilde{a}(\tilde{x}, y)d\tilde{x}.$$

Now, we find critical points of the function

$$F(\tilde{x}, s_1) = \tilde{x}^2\tilde{G}(\tilde{x}, s_1) + 2\tilde{x}s_1B_0(s_1)B(y).$$

Lemma 4.2. *The critical point of the function $F(\tilde{x}, s_1)$ has the form:*

$$\tilde{x}^c = s_1B(y)B_0(s_1)g(s_1, B(y)).$$

Proof. The proof is carried out in the same way as in the proof of Lemma 4.1. □

In [16] it is proved that if $x^c(y)$ is a critical point by variable x for function $F(x,y)$ then the following equality is true:

$$F''(x^c(y), y) = \frac{Hess F(x,y)}{\frac{\partial^2 F}{\partial x^2}}. \tag{13}$$

Using the stationary phase method to the last integral $J(\lambda, y, s)$ we have

$$J(\lambda, y, s) = C\frac{e^{i\lambda(s_1^2B_0^2(s_1)B(y)(1 + B(y)\tilde{g}(s_1, B(y)) + \varphi(y) - s_2y) + isgn(\lambda)\frac{\pi}{2})}}{\lambda^{\frac{1}{2}}}\tilde{a}(\tilde{x}^c, y) + R(\lambda, s_2, y),$$

where $R(\lambda, s_2, y)$ is a remainder term satisfying the condition

$$|R(\lambda, s_2, y)| \leqslant \frac{C}{\lambda^{\frac{3}{2}}}.$$

Hence, our two-dimension oscillatory integral (8) can be written as

$$J(\lambda, s) = \frac{1}{\lambda^{\frac{1}{2}}}\int e^{i\lambda(s_1^2B_0^2(s_1)B(y)(1 + B(y)\tilde{g}(s_1, B(y)) + \varphi(y) - s_2y)}a_1(y, s_2)dy + O(\lambda^{-\frac{3}{2}}),$$

where $a_1(y, s_2) = C\tilde{a}(\tilde{x}^c, y)$.

Let

$$\Phi(y, s_2) = s_1^2B_0^2(s_1)B(y)(1 + B(y)\tilde{g}(s_1, B(y)) + \varphi(y). \tag{14}$$

Lemma 4.3. *The function* (14) *is convex and satisfies the following estimate*

$$\Phi(y, s_2) \geqslant \varphi(y). \tag{15}$$

Proof. The convexity of the $\Phi(y, s_2)$ follows from (13). Now we prove (15). From the convexity of (4), we obtain that $B''(y) \geqslant 0$. But it should be noted that, in general, the opposite is not true. A good example is $B(y) = y$. Let y_1 be such a point that $B'(y_1) = 0$. Then, the monotonicity of $B'(y)$ implies that on the interval $[0, y_1]$ we have $B'(y) = 0$ and $B(y) = 0$. In this case the boundedness of the maximal operator is proved in [14]. So, assume that $B'(y) \geqslant 0$. Finally, integrating twice the inequality $B''(y) \geqslant 0$ we get $B(y) \geqslant 0$. From the definition of the function (14) follows (15). □

From the results of A. Nagel, A. Seeger and S. Wainger [17] we have

$$|\hat{\tau}(\xi)| \leqslant C \frac{\Phi^{-1}(\frac{1}{\lambda}, s_2)}{\lambda^{\frac{1}{2}}}.$$

Using (15) in Lemma 4.3 for the last estimate we get

$$|\hat{\tau}(\xi)| \leqslant C \frac{\varphi^{-1}(\frac{1}{\lambda})}{\lambda^{\frac{1}{2}}}.$$

Finally, applying Theorem 3.5 we obtain the proof of Theorem 2.1.

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О точных оценках максимальных операторов

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Аннотация. В статье рассматривается проблема ограниченности максимальных операторов, ассоциированных с гиперповерхностями в пространстве квадратично интегрируемых функций. Дано необходимое условие ограниченности в случае одной ненулевой главной кривизны. Критерий для ограниченности получается для определенного класса выпуклых гиперповерхностей.

Ключевые слова: максимальный оператор, преобразование Фурье, гиперповерхность, ограниченность.