EDN: MNWSFN
УДК 517.55+512.7

Parametrizations of Limit Positions for the Discriminant Locus of a Trinomial System

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Received 28.01.2023, received in revised form 20.03.2023, accepted 24.04.2023

Abstract. The paper deals with the discriminant of the reduced system of n trinomial algebraic equations. We study zero loci of truncations of the discriminant on facets of its Newton polytope. The basis of the study is the properties of the parametrization of the discriminant set of the system and the general combinatorial construction of the tropicalization of algebraic varieties.

Keywords: algebraic equation, discriminant, Newton polytope, truncation of the polynomial, discriminant set, parametrization.

Citation: Irina A. Antipova, Ekaterina A. Kleshkova, Parametrizations of Limit Positions for the Discriminant Locus of a Trinomial System, J. Sib. Fed. Univ. Math. Phys., 2023, 16(3), 318–329. EDN: MNWSFN.

1. Introduction and preliminaries

Consider a *reduced system* of n trinomial algebraic equations

$$Q_i := y^{\omega^{(i)}} + x_i y^{\sigma^{(i)}} - 1 = 0, \quad i = 1, \dots, n \tag{1}$$

with unknowns $y = (y_1, \dots, y_n) \in (\mathbb{C} \setminus 0)^n$ and variable complex coefficients $x = (x_1, \dots, x_n)$, where $y^{\omega^{(i)}} := y_1^{\omega_1^{(i)}} \cdot \ldots \cdot y_n^{\omega_n^{(i)}}$, $y^{\sigma^{(i)}} := y_1^{\sigma_1^{(i)}} \cdot \ldots \cdot y_n^{\sigma_n^{(i)}}$ are monomials in variables $y_1, \dots, y_n$ with integer exponents. The coefficients of the system (1) vary in the vector space $\mathbb{C}_x^n$. We assume that the matrix ω formed by column vectors $\omega^{(i)}$ is non-degenerate. The universal trinomial system in which all monomials have independent variable coefficients can be reduced to the form (1) by means of monomial transformations of the coefficients in view of the polyhomogeneity property of its solution [1, 3].

Denote by ∇° the set in $\mathbb{C}_x^n$ of all $x = (x_i)$ such that the polynomial mapping $Q = (Q_1, \dots, Q_n)$ has multiple zeros in the complex algebraic torus $(\mathbb{C} \setminus 0)^n$, i.e.

$$\nabla^\circ := \left\{ x \in \mathbb{C}_x^n : Q_1(y^0) = \dots = Q_n(y^0) = \frac{\partial Q}{\partial y}(y^0) = 0,\, y^0 \in (\mathbb{C} \setminus 0)^n \right\},$$

where $\dfrac{\partial Q}{\partial y}$ is the Jacobian of the mapping Q.

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Definition 1. *The discriminant locus ∇ of the system (1) is defined to be the closure of ∇° in the space of coefficients. If ∇ is a hypersurface, then its defining polynomial $\Delta(x)$ is said to be the discriminant of the system (1).*

This approach to the definition of the discriminant of the polynomial system was proposed in [1] as an extension of the concept of the A-discriminant developed in the book [5]. Further, we denote the discriminant of the system (1) by $\Delta_n(x)$ to clarify the number of equations and the dimension of the space of coefficients.

Our main objects of interest are limit positions of the discriminant locus ∇ in the toric compactification of the space $(\mathbb{C}\setminus 0)^n$ associated with the Newton polytope of the discriminant $\Delta_n(x)$. Recall that the *Newton polytope* $\mathcal{N}_{\Delta_n}$ of the polynomial $\Delta_n(x)$ is defined to be the convex hull of its support in $\mathbb{R}^n$. Each monomial $x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$ is visualized by the point $\alpha = (\alpha_1,\ldots,\alpha_n)$ of the lattice $\mathbb{Z}^n$. The *support* of a polynomial is defined to be the set of exponents of all its monomials with non-zero coefficients. The limit positions under study are determined by *truncations* of the discriminant $\Delta_n(x)$ to facets of the Newton polytope $\mathcal{N}_{\Delta_n}$.

Definition 2. *The truncation of the polynomial $\Delta_n(x)$ to a face h of the Newton polytope $\mathcal{N}_{\Delta_n}$ is the sum of all monomials of $\Delta_n(x)$ whose exponents belong to the face h.*

In the classical case $n = 1$, it is known that the Newton polytope of the discriminant of the algebraic equation of degree m is combinatorially equivalent to the $(m-1)$-dimensional cube [5, Theorem 2.2, Chapter 12]. The classical discriminant is well-studied and, in particular, a new approach to the proof of factorization identities for its truncations was proposed in recent papers [8,9]. The factorization identities were proven in [5] by means of sophisticated techniques of the theory of A-determinants. Truncations are factorized into the product of discriminants of lower degree equations.

Let us introduce a matrix σ whose columns are exponents $\sigma^{(1)},\ldots,\sigma^{(n)}$ of the system (1), matrices $\Psi := \omega^*\sigma$ and $\tilde{\Psi} := \Psi - |\omega|E_n$, where ω^* is the adjoint matrix to the ω, the E_n is the identity matrix and $|\omega|$ is the determinant of ω. The matrices σ and ω determine the support of the system (1). As it follows from [2], rows of matrices $(-\Psi)$ and $\tilde{\Psi}$ (we denote them $-\psi_1,\ldots,-\psi_n$ and $\tilde{\psi}_1,\ldots,\tilde{\psi}_n$, correspondingly) could define the inner normal directions for facets of the polytope $\mathcal{N}_{\Delta_n}$. We assume that matrices Ψ and $\tilde{\Psi}$ do not contain zero elements. This is a sufficient condition for the set ∇ to be a hypersurface.

Now, we formulate the main result of our study without detailing all the structures used.

Theorem 1. *Let h be a facet of the Newton polytope $\mathcal{N}_{\Delta_n}$ with a normal direction $\mu \in \left\{-\psi_1,\ldots,-\psi_n,\tilde{\psi}_1,\ldots,\tilde{\psi}_n\right\}$. The zero locus of the truncation $\Delta_n(x)|_h$ contains the set*

$$\left\{x\in(\mathbb{C}\setminus 0)^n : \Delta_{n-1}(z)|_{z=u(x)} = 0\right\}, \tag{2}$$

where $\Delta_{n-1}(z)$ is the discriminant of the reduced system of $n-1$ trinomials, and $z = u(x) : (\mathbb{C}\setminus 0)^n_x \to (\mathbb{C}\setminus 0)^{n-1}_z$ is a mapping given by monomial functions of coefficients $x = (x_1,\ldots,x_n)$ of the system (1).

The constructive proof of Theorem 1 is given in Section 4. It is important to note that the proposed scheme is meaningful in case when matrices θ, $\tilde{\theta}$, $\varkappa$, $\tilde{\varkappa}$ (see (9), (10), (16), (17)) do not contain zero elements.

2. Parametrization of the discriminant locus for the system (1)

The parametrization of the dehomogenized discriminant locus of a system of n Laurent polynomials in n variables was comprehensively studied in the paper [1]. It is applied for computing the tropical discriminant of the system [2] and is a key tool in the proof of Theorem 1 .

Let us introduce two copies of the space $\mathbb{C}^n$. The first one is the space $\mathbb{C}^n_x$ with coordinates $x = (x_1, \dots, x_n)$, and the second one is $\mathbb{C}^n_s$ with coordinates $s = (s_1, \dots, s_n)$. We interpret the $\mathbb{C}^n_s$ as the space of homogeneous coordinates in $\mathbb{CP}^{n-1}_s$. Consider the multivalued algebraic mapping $x = x(s) : \mathbb{CP}^{n-1}_s \to \mathbb{C}^n_x$ with components

$$x_i = -\frac{|\omega| s_i}{\langle \tilde{\psi}_i, s\rangle} \prod_{k=1}^{n} \left(\frac{\langle \tilde{\psi}_k, s\rangle}{\langle \psi_k, s\rangle} \right)^{\frac{\psi_k^{(i)}}{|\omega|}}, \quad i = 1, \dots, n, \tag{3}$$

where brackets $\langle , \rangle$ denote the inner product of vectors. The number of branches in (3) equals to the absolute value of the determinant $|\omega|$, however, some branches may coincide. If the discriminant locus of the system (1) is an irreducible hypersurface depending on all variables $x_1, \dots, x_n$, then the mapping (3) parametrizes it with the multiplicity equal to the index $|\mathbb{Z}^n : H|$ of the sublattice $H \subset \mathbb{Z}^n$ generated by the columns of the matrix $(\omega|\sigma)$, i.e. by all exponents of the system (1).

3. Tropical discriminant

We start this section with some basic concepts of the tropical geometry following the book [10]. Consider the *tropical semiring* $(\mathbb{R} \cup \infty, \oplus, \odot)$, where arithmetic operations of addition and multiplication are defined as follows:

$$x \oplus y := \min(x, y), \quad x \odot y := x + y.$$

Consider a field $\mathbb{K}$ with a valuation $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$. For a polynomial

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u$$

in the ring $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$, the *tropicalization* is defined to be the function

$$\mathrm{trop}(f)(w) := \min_{u \in \mathbb{Z}^n} \left(\mathrm{val}(c_u) + \langle u, w\rangle\right). \tag{4}$$

The tropical polynomial $\mathrm{trop}(f)(w)$ is a piecewise linear function that is obtained by replacing all coefficients c_u by their valuations $\mathrm{val}(c_u)$ and performing all operations in the tropical semiring.

The Laurent polynomial f over the field $\mathbb{K}$ determines the algebraic hypersurface

$$V(f) = \{y \in \mathbb{T}^n : f(y) = 0\},$$

where $\mathbb{T}^n := (\mathbb{K} \setminus 0)^n$ is the algebraic torus.

Definition 3. *The tropicalization of the algebraic hypersurface $V(f)$ is defined to be the set*

$$\mathrm{trop}(V(f)) := \{w \in \mathbb{R}^n : \textit{the minimum in (4) is attained at least twice}\}.$$

Therefore, the trop$(V(f))$ is the locus in $\mathbb{R}^n$, where the function trop$(f)(w)$ fails to be linear.

The general construction of tropicalization for an algebraic variety that admits a rational parametric representation whose components are Laurent monomials in linear forms is proposed in [7]. Let $\mathbb{K}$ be a field with trivial valuation. Consider a $m \times d$ matrix $U = (u_{ij})$ over $\mathbb{K}$ and a $p \times m$ matrix $V = (v_{ij})$ with integer entries. The matrix U determines m linear forms in the ring $\mathbb{K}[s_1, \dots, s_d]$:

$$l_i(s) = u_{i1}s_1 + \ldots + u_{id}s_d, \ \ i = 1, \dots, m. \tag{5}$$

The matrix V encodes p Laurent monomials in $\mathbb{K}[z_1^{\pm}, \dots, z_m^{\pm}]$:

$$z_1^{v_{j1}} \cdot \ldots \cdot z_m^{v_{jm}}, \ \ j = 1, \dots, p. \tag{6}$$

The composition of monomials (6) and forms (5) defines a rational mapping $\alpha : \mathbb{K}^d \to \mathbb{K}^p$ with components

$$\alpha_j(s) = \prod_{i=1}^{m} l_i(s)^{v_{ij}}.$$

We denote by Y the closure of the image of α. According to [7, Theorem 3.1], the tropicalization of the variety Y is a polyhedral fan trop(Y) that coincides with the image of the Bergman fan $\mathfrak{B}_M$ of the matroid M associated with the matrix U under the linear map defined by V. The Bergman fan being a geometric model of the matroid M, is the tropicalization of the linear variety given by the linear map $s \to Us$ (see [4, 10]).

Consider the rational map $\mathbb{CP}_s^{n-1} \to \mathbb{C}_w^n$ that is obtained by raising the components of the map (3) to the power $|\omega|$. It defines the rational variety $\tilde{\nabla}$, and by the construction given above the tropicalization trop$(\tilde{\nabla})$ is encoded by the pair of block matrices

$$U_{3n\times n} = \left(-|\omega|E_n \,\middle|\, \Psi^T \middle|\, \tilde{\Psi}^T\right)^T, \ \text{ and } \ V_{n\times 3n} = \left(|\omega|E_n \,\middle|\, -\Psi^T \middle|\, \tilde{\Psi}^T\right). \tag{7}$$

The study of the tropical discriminant in case of the general polynomial system has been carried out in [2]. In Section 5, using an example, we demonstrate how a tropical variety constructed on the basis of the parametrization of the discriminant set reveals normals to facets of the Newton polytope of the discriminant, while the matroid associated with the matrix U suggests the parametrization of 'hidden' limit positions of the discriminant locus.

4. Zero loci of truncations

Here we present a proof of the main result.

The relation between the discriminant $\Delta_n(x)$ and its truncation $\Delta_n(x)|_h$ to the facet h is established by means of the function

$$H_h^{\tau}(x) := \tau^d \Delta_n\left(\frac{x_1}{\tau^{\mu_1}}, \dots, \frac{x_n}{\tau^{\mu_n}}\right),$$

where $\mu_1, \dots, \mu_n$ are entries of the normal vector μ to the facet h, d is the weighted degree of all monomials of the truncation $\Delta_n(x)|_h$ with respect to the weight μ.

Lemma 1. *The function $H_h^{\tau}(x)$ being a homogenization of the discriminant of the system (1) with respect to the weight μ has the following property*

$$H_h^{\tau}(x) \underset{\tau\to 0}{\longrightarrow} \Delta_n(x)|_h.$$

Proof. The weighted degree of all monomials of the $\Delta_n(x)$ that do not belong to the truncation $\Delta_n(x)|_h$ is strictly less than d. Therefore, as a result of passing to the limit when $\tau \to 0$, all monomials disappear, except for those whose exponents belong on the face h. Lemma 1 is proved. $\square$

Proof of Theorem 1. First, we study the truncation $\Delta_n(x)|_h$ to the facet h having a normal vector $\mu = -\psi_j$. Recall that ψ_j is the jth row of the matrix Ψ and introduce the following constructions. Let $\delta_k^{(m)}$ denote a 2×2 minor of the matrix Ψ formed by the intersection of rows ψ_k, ψ_j, and columns $\psi^{(m)}$, $\psi^{(j)}$. Therefore, we have

$$\delta_k^{(m)} = \pm\left(\psi_k^{(m)}\psi_j^{(j)} - \psi_k^{(j)}\psi_j^{(m)}\right), \tag{8}$$

where the sign on the right-hand side depends on the choice of k and m. Define the square matrix θ of the order $n-1$ that contains the entries

$$\begin{aligned}
\theta_k^{(m)} &:= \delta_k^{(m)}, & k,m &\in \{1,\dots,j-1\},\\
\theta_k^{(m)} &:= -\delta_k^{(m+1)}, & k \in \{1,\dots,j-1\},\ m &\in \{j,\dots,n-1\},\\
\theta_k^{(m)} &:= -\delta_{k+1}^{(m)}, & k \in \{j,\dots,n-1\},\ m &\in \{1,\dots,j-1\},\\
\theta_k^{(m)} &:= \delta_{k+1}^{(m+1)}, & k,m &\in \{j,\dots,n-1\},
\end{aligned} \tag{9}$$

and matrices

$$\xi := |\omega|\psi_j^{(j)}E_{n-1} \text{ and } \tilde{\theta} := \theta - \xi. \tag{10}$$

Introduce a system of $n-1$ trinomials

$$y^{\xi^{(i)}} + z_i y^{\theta^{(i)}} - 1 = 0, \quad i = 1,\dots,n-1, \tag{11}$$

where $y = (y_1,\dots,y_{n-1})$, $\xi^{(i)}$ and $\theta^{(i)}$ are columns of matrices ξ and θ respectively.

Now we are in a position to study a parametrization of the zero locus of the truncation $\Delta_n(x)|_h$. To this end, according to Lemma 1, we consider the sets $\{x : H_h^\tau(x) = 0\}$, $\tau \neq 0$ that admit the parametrization

$$x_i = -\tau^{\psi_j^{(i)}} \frac{|\omega|s_i}{\langle\tilde{\psi}_i, s\rangle} \prod_{k=1}^{n}\left(\frac{\langle\tilde{\psi}_k, s\rangle}{\langle\psi_k, s\rangle}\right)^{\frac{\psi_k^{(i)}}{|\omega|}}, \quad i = 1,\dots,n. \tag{12}$$

In the projective space with coordinates $s = (s_1 : \ldots : s_n)$, we define a plane γ_j given by the equation

$$\langle\psi_j, s\rangle = 0,$$

and use the parametrization (12) to get restrictions of the monomials $x_i \cdot x_j^{-\psi_j^{(i)}/\psi_j^{(j)}}$ to γ_j for $i \neq j$. Note that

$$\left.\left(\frac{|\omega|s_j}{\langle\tilde{\psi}_j, s\rangle}\right)^{-\psi_j^{(i)}/\psi_j^{(j)}}\right|_{\gamma_j} = (-1)^{-\psi_j^{(i)}/\psi_j^{(j)}},$$

and, for $i \neq j$,

$$\left.\frac{|\omega|s_i}{\langle\tilde{\psi}_i, s\rangle}\right|_{\gamma_j} = \frac{|\omega|\psi_j^{(j)}s_i}{\sum\limits_{m\neq j}\left(\psi_i^{(m)}\psi_j^{(j)} - \psi_i^{(j)}\psi_j^{(m)}\right)s_m - |\omega|\psi_j^{(j)}s_i}.$$

Moreover, for $k \neq j$

$$\left.\frac{\langle\tilde{\psi}_k, s\rangle}{\langle\psi_k, s\rangle}\right|_{\gamma_j} = \frac{\sum\limits_{m\neq j}\left(\tilde{\psi}_k^{(m)}\psi_j^{(j)} - \tilde{\psi}_k^{(j)}\psi_j^{(m)}\right)s_m}{\sum\limits_{m\neq j}\left(\psi_k^{(m)}\psi_j^{(j)} - \psi_k^{(j)}\psi_j^{(m)}\right)s_m}.$$

Therefore, in terms of notations (9) and (10) restrictions of monomials $x_i \cdot x_j^{-\psi_j^{(i)}/\psi_j^{(j)}}$ to the plane γ_j admit representations

$$\begin{aligned} \left. x_i \cdot x_j^{-\psi_j^{(i)}/\psi_j^{(j)}}\right|_{\gamma_j} &= -\frac{\xi_j^{(j)} s_i}{\langle\tilde{\theta}_i, s\rangle}\prod_{k=1}^{n-1}\left(\frac{\langle\tilde{\theta}_k, s\rangle}{\langle\theta_k, s\rangle}\right)^{\frac{\theta_k^{(i)}}{\xi_j^{(j)}}}, \qquad i = 1,\dots,j-1,\\ \left. x_i \cdot x_j^{-\psi_j^{(i)}/\psi_j^{(j)}}\right|_{\gamma_j} &= -\frac{\xi_j^{(j)} s_{i-1}}{\langle\tilde{\theta}_{i-1}, s\rangle}\prod_{k=1}^{n-1}\left(\frac{\langle\tilde{\theta}_k, s\rangle}{\langle\theta_k, s\rangle}\right)^{\frac{\theta_k^{(i-1)}}{\xi_j^{(j)}}}, \quad i = j+1,\dots,n, \end{aligned} \tag{13}$$

where $s = (s_1 : \ldots : s_{n-1})$ are homogeneous coordinates of $\mathbb{CP}^{n-2}$. The right-hand sides of formulae (13) determine the parametrization of the discriminant locus for the system (11). Thus, the zero locus of the truncation $\Delta_n(x)|_h$ contains the set

$$\left\{x \in (\mathbb{C}\setminus 0)^n : \Delta_{n-1}(z)|_{z=u(x)} = 0\right\}, \tag{14}$$

where $\Delta_{n-1}(z)$ is the discriminant of the system (11) and $z = u(x)$ is a monomial mapping with components

$$u_i(x) = \begin{cases} x_i \cdot x_j^{-\psi_j^{(i)}/\psi_j^{(j)}}, & i = 1,\dots,j-1,\\ x_{i+1}\cdot x_j^{-\psi_j^{(i+1)}/\psi_j^{(j)}}, & i = j,\dots,n-1. \end{cases} \tag{15}$$

Next, we follow the similar way to study the truncation $\Delta_n(x)|_h$ to the facet h with the normal vector $\mu = \tilde{\psi}_j$. Define the square matrix $\varkappa$ of the order $n-1$ that contains the entries

$$\begin{aligned} \varkappa_k^{(m)} &:= \delta_k^{(m)} - |\omega|\psi_k^{(m)}, && k, m \in \{1,\dots,j-1\},\\ \varkappa_k^{(m)} &:= -\delta_k^{(m+1)} - |\omega|\psi_k^{(m+1)}, && k \in \{1,\dots,j-1\},\ \ m \in \{j,\dots,n-1\},\\ \varkappa_k^{(m)} &:= -\delta_{k+1}^{(m)} - |\omega|\psi_{k+1}^{(m)}, && k \in \{j,\dots,n-1\},\ \ m \in \{1,\dots,j-1\},\\ \varkappa_k^{(m)} &:= \delta_{k+1}^{(m+1)} - |\omega|\psi_{k+1}^{(m+1)}, && k, m \in \{j,\dots,n-1\}, \end{aligned} \tag{16}$$

and matrices

$$\eta := |\omega|\tilde{\psi}_j^{(j)} E_{n-1},\ \ \tilde{\varkappa} := \varkappa - \eta. \tag{17}$$

In this case, the sets $\{x : H_h^\tau(x) = 0\}$, $\tau \neq 0$ admit parametrization

$$x_i = -\tau^{\tilde{\psi}_j^{(i)}}\frac{|\omega| s_i}{\langle\tilde{\psi}_i, s\rangle}\prod_{k=1}^{n}\left(\frac{\langle\tilde{\psi}_k, s\rangle}{\langle\psi_k, s\rangle}\right)^{\frac{\psi_k^{(i)}}{|\omega|}}, \quad i = 1,\dots,n. \tag{18}$$

In the projective space with coordinates $s = (s_1 : \ldots : s_n)$, we define a plane $\tilde{\gamma}_j$ given by the equation

$$\langle\tilde{\psi}_j, s\rangle = 0,$$

and use the parametrization (18) to get restrictions of monomials $x_i \cdot x_j^{-\tilde{\psi}_j^{(i)}/\tilde{\psi}_j^{(j)}}$ to $\tilde{\gamma}_j$ for $i \neq j$. We obtain the following result

$$
\begin{aligned}
x_i \cdot x_j^{-\tilde{\psi}_j^{(i)}/\tilde{\psi}_j^{(j)}}\Big|_{\tilde{\gamma}_j} &= -\frac{\eta_j^{(j)} s_i}{\langle \tilde{\varkappa}_i, s\rangle} \prod_{k=1}^{n-1} \left(\frac{\langle \tilde{\varkappa}_k, s\rangle}{\langle \varkappa_k, s\rangle}\right)^{\frac{\varkappa_k^{(i)}}{\eta_j^{(j)}}}, \qquad i = 1, \ldots, j-1, \\
x_i \cdot x_j^{-\tilde{\psi}_j^{(i)}/\tilde{\psi}_j^{(j)}}\Big|_{\tilde{\gamma}_j} &= -\frac{\eta_j^{(j)} s_{i-1}}{\langle \tilde{\varkappa}_{i-1}, s\rangle} \prod_{k=1}^{n-1} \left(\frac{\langle \tilde{\varkappa}_k, s\rangle}{\langle \varkappa_k, s\rangle}\right)^{\frac{\varkappa_k^{(i-1)}}{\eta_j^{(j)}}}, \quad i = j+1, \ldots, n.
\end{aligned} \tag{19}
$$

The right-hand sides of (19) determine the parametrization of the discriminant set for the trinomial system with the support $(\eta|\varkappa)$. Thus, the zero locus of the truncation $\Delta_n(x)|_h$ contains the set

$$\left\{x \in (\mathbb{C} \setminus 0)^n : \Delta_{n-1}(z)|_{z=u(x)} = 0\right\}, \tag{20}$$

where $\Delta_{n-1}(z)$ is the discriminant of the system of $n-1$ trinomials with the support $(\eta|\varkappa)$ and $z = u(x)$ is a monomial mapping with entries

$$
u_i(x) = \begin{cases} x_i \cdot x_j^{-\tilde{\psi}_j^{(i)}/\tilde{\psi}_j^{(j)}}, & i = 1, \ldots, j-1, \\ x_{i+1} \cdot x_j^{-\tilde{\psi}_j^{(i+1)}/\tilde{\psi}_j^{(j)}}, & i = j, \ldots, n-1. \end{cases} \tag{21}
$$

The proof of Theorem 1 is completed. □

Remark 1. The described procedure can be applied to the truncation $\Delta_n|_{h_j^0}$ that lies in a coordinate plane $\alpha_j = 0$. The limit set $\left\{x : H^\tau_{h_j^0}(x) = 0\right\}$ as $\tau \to 0$ approaches the discriminant set $\{\Delta_{n-1}(x_1, \ldots [j] \ldots, x_n) = 0\}$ of the system of $n-1$ trinomials of the form

$$y_i^{|\omega|} + x_i y^{\psi^{(i)}[j]} - 1 = 0, \;\; i = 1, \ldots [j] \ldots, n, \tag{22}$$

where $\psi^{(i)}[j] := (\psi_1^{(i)}, \ldots [j] \ldots, \psi_n^{(i)})$, $y^{\psi^{(i)}[j]} := y_1^{\psi_1^{(i)}} \cdot \ldots [j] \ldots \cdot y_n^{\psi_n^{(i)}}$.

5. Example: 'hidden' facets and truncations

Consider a system of equations

$$
\begin{cases} y_1 + a{y_1}^2 y_2 y_3 - 1 = 0, \\ y_2 + b y_1 {y_2}^2 y_3 - 1 = 0, \\ y_3 + c y_1 y_2 {y_3}^2 - 1 = 0 \end{cases} \tag{23}
$$

with unknowns y_1, y_2, y_3 and variable coefficients a, b, c. The matrix of exponents of the system is

$$(\omega|\sigma) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right).$$

Its columns generate the lattice $\mathbb{Z}^3$. Since matrices

$$\Psi = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } \tilde{\Psi} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

do not contain zero elements, the discriminant set ∇ is a hypersurface. The rational mapping $\mathbb{CP}_s^2 \to \mathbb{C}^3_{a,b,c}$ given by formulae

$$
\begin{aligned}
a &= -\frac{s_1}{s_1+s_2+s_3}\left(\frac{s_1+s_2+s_3}{2s_1+s_2+s_3}\right)^2\left(\frac{s_1+s_2+s_3}{s_1+2s_2+s_3}\right)\left(\frac{s_1+s_2+s_3}{s_1+s_2+2s_3}\right),\\
b &= -\frac{s_2}{s_1+s_2+s_3}\left(\frac{s_1+s_2+s_3}{2s_1+s_2+s_3}\right)\left(\frac{s_1+s_2+s_3}{s_1+2s_2+s_3}\right)^2\left(\frac{s_1+s_2+s_3}{s_1+s_2+2s_3}\right),\\
c &= -\frac{s_3}{s_1+s_2+s_3}\left(\frac{s_1+s_2+s_3}{2s_1+s_2+s_3}\right)\left(\frac{s_1+s_2+s_3}{s_1+2s_2+s_3}\right)\left(\frac{s_1+s_2+s_3}{s_1+s_2+2s_3}\right)^2,
\end{aligned}
\tag{24}
$$

parametrizes ∇ with the multiplicity one. Here $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ are homogeneous coordinates in $\mathbb{CP}_s^2$.

Let us study the tropicalization $\tau(\nabla)$ of the rational variety $\nabla \subset \mathbb{C}^3$. As it was pointed out in Section 3, the mapping (24) is encoded by two matrices

$$
U = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\\ 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 & -2 & -1 & -1 & 1 & 1 & 1\\ 0 & 1 & 0 & -1 & -2 & -1 & 1 & 1 & 1\\ 0 & 0 & 1 & -1 & -1 & -2 & 1 & 1 & 1 \end{pmatrix}.
$$

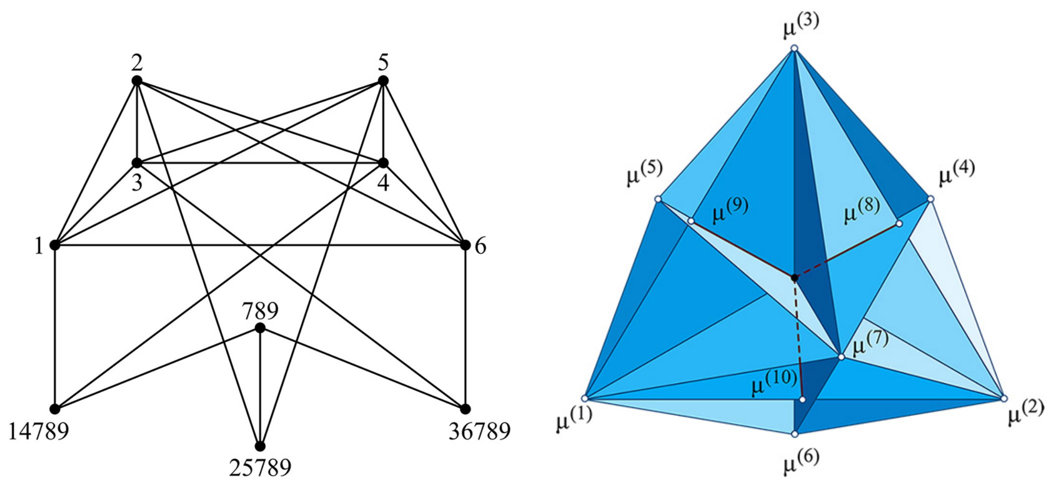
Consider the matroid M on the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ associated with the set of rows of the matrix U. The tropical linear space related to the matroid M is the Bergman fan $\mathfrak{B}(M)$. It is a two-dimensional fan in $\mathbb{R}^9/\mathbb{R}\mathbf{1}$ or a graph depicted in Fig. 1a. The graph has ten vertices, corresponding to seven flats of the rank one (1, 2, 3, 4, 5, 6, 789) and three circuits (14789, 25789, 36789) of the matroid M. The edges 12, 13, 15, 16, 23, 24, 26, 43, 35, 45, 46, 56 of the graph correspond to flats of the rank two.

The image of $\mathfrak{B}(M)$ under V is a two-dimensional fan $\tau(\nabla) \subset \mathbb{R}^3$ (see Fig. 1b). It is the tropical variety $\tau(\nabla)$ that consists of all codimension one cones of the normal fan of the Newton polytope $\mathcal{N}_{\Delta_3}$ for the discriminant of the system (23). Seven rays $\mu^{(1)}, \ldots, \mu^{(7)}$ of the fan $\tau(\nabla)$ generated by columns V are predicted explicitly by the parametrization (24) and determine normal directions

$$
\begin{aligned}
&\mu^{(1)} = (1, 0, 0), && \mu^{(5)} = (-1, -2, -1),\\
&\mu^{(2)} = (0, 1, 0), && \mu^{(6)} = (-1, -1, -2),\\
&\mu^{(3)} = (0, 0, 1), && \mu^{(7)} = (1, 1, 1).\\
&\mu^{(4)} = (-2, -1, -1), &&
\end{aligned}
$$

There are three more 'hidden' rays that are revealed as a result of intersection of two-dimensional cones of the fan $\tau(\nabla)$. More precisely, images of cones 14789 and 23 intersect along the ray $\mathbb{R}_{\geqslant 0}(0, 1, 1)^T$ ($\mu^{(10)}$ in Fig. 1b); images of 25789 and 13 intersect along $\mathbb{R}_{\geqslant 0}(1, 0, 1)^T$ ($\mu^{(9)}$ in Fig. 1b); images of 36789 and 12 intersect along $\mathbb{R}_{\geqslant 0}(1, 1, 0)^T$ ($\mu^{(8)}$ in Fig. 1b). Therefore, all inner normals to facets of the Newton polytope of the discriminant for the system (23) are found. The discriminant of (23) is as follows

$$
\begin{aligned}
\Delta_3(a,b,c) = {} & 4a^5b^2 - 8a^5bc + 4a^5c^2 - 27a^4b^4 + 36a^4b^3c - 6a^4b^3 - 2a^4b^2c^2 + 6a^4b^2c + a^4b^2 + 36a^4bc^3 + \\
& + 6a^4bc^2 - 2a^4bc - 27a^4c^4 - 6a^4c^3 + a^4c^2 + 36a^3b^4c - 6a^3b^4 - 256a^3b^3c^3 - 52a^3b^3c^2 + 16a^3b^3c - \\
& - 2a^3b^3 - 52a^3b^2c^3 - 16a^3b^2c^2 + 2a^3b^2c + 36a^3bc^4 + 16a^3bc^3 + 2a^3bc^2 - 6a^3c^4 - 2a^3c^3 + 4a^2b^5 - \\
& - 2a^2b^4c^2 + 6a^2b^4c + a^2b^4 - 52a^2b^3c^3 - 16a^2b^3c^2 + 2a^2b^3c - 2a^2b^2c^4 - 16a^2b^2c^3 - 6a^2b^2c^2 + 6a^2bc^4 + \\
& + 2a^2bc^3 + 4a^2c^5 + a^2c^4 - 8ab^5c + 36ab^4c^3 + 6ab^4c^2 - 2ab^4c + 36ab^3c^4 + 16ab^3c^3 + 2ab^3c^2 + \\
& + 6ab^2c^4 + 2ab^2c^3 - 8abc^5 - 2abc^4 + 4b^5c^2 - 27b^4c^4 - 6b^4c^3 + b^4c^2 - 6b^3c^4 - 2b^3c^3 + 4b^2c^5 + b^2c^4.
\end{aligned}
$$



a) b)

Fig. 1. a) The Bergman fan $\mathfrak{B}(M)$. b) The tropical variety $\tau(\nabla)$

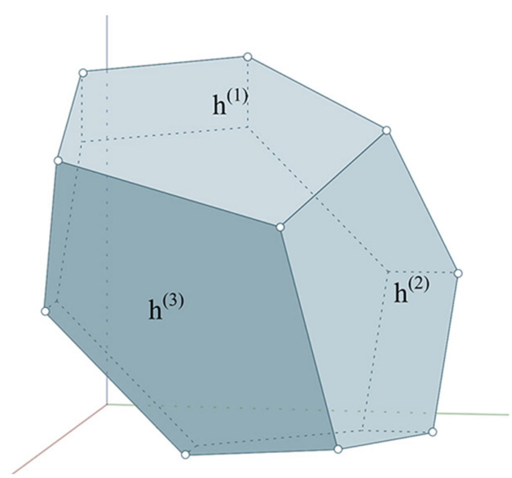


Fig. 2. The Newton polytope $\mathcal{N}_{\Delta_3}$

The Newton polytope $\mathcal{N}_{\Delta_3}$ has 10 facets (Fig. 2) enumerated by $h^{(j)}$, $j = 1, \ldots, 10$, in accordance with normal vectors $\mu^{(j)}$.

The truncations of the discriminant $\Delta_3(a,b,c)$ to coordinate faces $h^{(1)}$, $h^{(2)}$, $h^{(3)}$ are as follows

$$\begin{aligned}\Delta_3|_{h^{(1)}} &= b^2c^2(-27b^2c^2+4b^3-6b^2c-6bc^2+4c^3+b^2-2bc+c^2),\\ \Delta_3|_{h^{(2)}} &= a^2c^2(-27a^2c^2+4a^3-6a^2c-6ac^2+4c^3+a^2-2ac+c^2),\\ \Delta_3|_{h^{(3)}} &= a^2b^2(-27a^2b^2+4a^3-6a^2b-6ab^2+4b^3+a^2-2ab+b^2).\end{aligned} \tag{25}$$

Polynomials in brackets in (25) are irreducible and coincide with the discriminant $\Delta_2(z_1,z_2)$ of the system

$$\begin{cases} y_1+z_1{y_1}^2y_2-1=0,\\ y_2+z_2y_1{y_2}^2-1=0,\end{cases}$$

under condition of a suitable determination of variable coefficients.

The factorized truncations of the polynomial $\Delta_3(a,b,c)$ to facets $h^{(4)}$, $h^{(5)}$, $h^{(6)}$ are as follows

$$\begin{aligned}\Delta_3|_{h^{(4)}} &= -a^3(256b^3c^3+27ab^4-36ab^3c+2ab^2c^2-36abc^3+27ac^4-4a^2b^2+8a^2bc-4a^2c^2),\\ \Delta_3|_{h^{(5)}} &= -b^3(256a^3c^3+27a^4b-36a^3bc+2a^2bc^2-36abc^3+27bc^4-4a^2b^2+8ab^2c-4b^2c^2),\\ \Delta_3|_{h^{(6)}} &= -c^3(256a^3b^3+27a^4c-36a^3bc+2a^2b^2c-36ab^3c+27b^4c-4a^2c^2+8abc^2-4b^2c^2).\end{aligned}$$

Implementing the construction proposed in Theorem 1 for these truncations, we obtain the following representations

$$\begin{aligned}\Delta_3(a,b,c)|_{h^{(4)}} &= a^6\cdot\Delta_2(z)|_{z=u^{(1)}(a,b,c)},\\ \Delta_3(a,b,c)|_{h^{(5)}} &= b^6\cdot\Delta_2(z)|_{z=u^{(2)}(a,b,c)},\\ \Delta_3(a,b,c)|_{h^{(6)}} &= c^6\cdot\Delta_2(z)|_{z=u^{(3)}(a,b,c)},\end{aligned}$$

where

$$\Delta_2(z)=256z_1^3z_2^3+27z_1^4-36z_1^3z_2+2z_1^2z_2^2-36z_1z_2^3+27z_2^4-4z_1^2+8z_1z_2-4z_2^2$$

is the discriminant of the system

$$\begin{cases} {y_1}^2+z_1y_1^3y_2-1=0,\\ {y_2}^2+z_2y_1{y_2}^3-1=0,\end{cases} \tag{26}$$

and

$$\begin{aligned}u^{(1)}(a,b,c) &= (ba^{-1/2},ca^{-1/2}),\\ u^{(2)}(a,b,c) &= (ab^{-1/2},cb^{-1/2}),\\ u^{(3)}(a,b,c) &= (ac^{-1/2},ac^{-1/2}).\end{aligned}$$

As applied to the truncation $\Delta_3(a,b,c)|_{h^{(7)}}$, the construction degenerates. If we consider parametrizations of sets $\{(a,b,c): H^{\tau}_{h^{(7)}}(a,b,c)=0\}$, $\tau\neq 0$, and, according to (19), express the restrictions of monomials $b\cdot a^{-1}$, $c\cdot a^{-1}$ to the plane $\langle\mu^{(7)},s\rangle=0$, then we get the equations

$$\frac{b}{a}=\frac{c}{a}=1,$$

that define the zero locus of the truncation $\Delta_3|_{h^{(7)}}$. The truncation itself is as follows

$$\Delta_3|_{h^{(7)}}=(b-c)^2(a-c)^2(a-b)^2. \tag{27}$$

The limit positions of the discriminant locus associated with facets $h^{(8)}$, $h^{(9)}$, $h^{(10)}$ can be investigated by means of the the tropical fan $\tau(\nabla)$. For example, consider the ray $\mathbb{R}_{\geqslant 0}(1,1,0)^T$

that defines the normal to the facet $h^{(8)}$. It is the intersection of cones associated with flats 36789 and 12. This means that the zero locus of the truncation $\Delta_3|_{h^{(8)}}$ can be obtained by the restriction of the parametrization for the family $\{(a,b,c) : H^{\tau}_{\mu^{(8)}}(a,b,c) = 0\}$, $\tau \neq 0$ to the planes

$$\gamma_1 : s_1 = s_2 = 0 \text{ and } \gamma_2 : s_3 = 0,\ s_1 + s_2 + s_3 = 0.$$

As a result, we get that the zero locus of the truncation $\Delta_3|_{h^{(8)}}$ consists of three components $c = 0$, $a = b$ and $c = -\frac{1}{4}$ which agrees with the expression

$$\Delta_3|_{h^{(8)}} = c^4(a-b)^2(4c+1).$$

Similarly, we study limit positions of the discriminant locus associated with facets $h^{(9)}$, $h^{(10)}$. They are given by polynomials

$$\Delta_3|_{h^{(9)}} = b^4(a-c)^2(4b+1),$$
$$\Delta_3|_{h^{(10)}} = a^4(b-c)^2(4a+1).$$

All the discriminants from the example are computed using the computer algebra system for polynomial computations Singular [6].

The research is supported by the Krasnoyarsk Mathematical Center funded by the Ministry of Science and Higher Education of the Russian Federation (Agreement no. 075-02-2023-936).

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Параметризации предельных положений дискриминантного множества системы триномов

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Аннотация. Рассматривается дискриминант приведенной системы n триномиальных алгебраических уравнений. Исследуются срезки дискриминанта на гиперграни его многогранника Ньютона. Основой исследования являются свойства параметризации дискриминантного множества системы и общая комбинаторная конструкция тропикализации алгебраических многообразий.

Ключевые слова: алгебраическое уравнение, дискриминант, многогранник Ньютона, срезка, дискриминантное множество, параметризация.