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Examples of Computing Power Sums of Roots of Systems of Equations

Evgeniya K. Myshkina*
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation

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Abstract. Examples of computing power sums of roots of systems of equations, including transcendental, are considered. Since the number of roots of such systems is, as a rule, infinite, it is necessary to study the power sums of roots in a negative degree. Formulas for finding residue integrals, their connection with power sums of roots to a negative degree, multidimensional analogues of Waring's formulas are given.

Keywords: transcendental systems of equations, power sums of roots, residue integrals.

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Introduction

Basing on the multidimensional logarithmic residue, L. A. Aizenberg has obtained formulas for power sums of roots of systems of non-linear algebraic equations in $\mathbb{C}^n$ [1, Theorem 2, Corollary 2]. These formulas enable us to find sums of values of holomorphic functions in roots without calculation of roots themselves, and to develop a new method of investigation of systems of equations in $\mathbb{C}^n$. For different types of systems, such formulas have different forms.

This was proposed by L. A. Aizenberg [1], and the development of this idea was continued in monograph [2]. The main idea of the method is to find power sums of roots of a system in positive degrees, and then use either one-dimensional or multidimensional Newton recurrent formulas to recover them. Unlike the classical elimination method, this method is less time consuming and does not increase the multiplicity of roots. The base of the method is a formula [1] obtained by using the multidimensional logarithmic residue for evaluation of sums of values of an arbitrary polynomial in roots of a given system of algebraic equations without calculation of the roots themselves.

As a rule, we cannot obtain formulas for the sums of roots of non-algebraic (transcendent) equations, because the set of the roots can be infinite, and power series of their coordinates can be divergent. However, the non-algebraic systems of equations arise, for instance, in the problems of chemical kinetics [3]. Therefore, such systems demand further investigations.

The power sums of negative degrees of roots of various transcendent systems are studied in papers [4–9]. These sums are calculated by means of residue integral over skeletons of polydisks with center at the origin. Note that this residue integral in general is not a multidimensional logarithmic residue, or the Grothendieck residue. There exist formulas of residue integrals for

*elfifenok@mail.ru

various types of homogeneous systems of lower orders, and established connections with power sums of roots of the system in negative degree.

More complicated systems are investigated in the works [7, 8]. Here the lower homogeneous parts allow expansion into product of linear factors, and the cycles of integration in the residue integrals are determined by these factors.

The subjects of the paper [9] are algebraic and transcendent systems of equations, where the lower homogeneous parts of functions form non-degenerated system of algebraic equations. Formulas were found for the residue integrals, power sums of the roots in negative degree, and multidimensional analogs of the Waring formula, i. e., the relations between the coefficients of the equations with the residue integrals. In the next section we use the results of this article.

1. Principal statements

Consider a system of equations $f(z) = 0$, where $f(z) = (f_1(z),\ f_2(z), \dots, f_n(z))$ are functions of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \tag{1}$$

where P_j is the lower homogeneous part of the function, i.e. degree of all monomials (in all variables) in Q_j is strictly greater than in P_j.

$$P_j(z) = \sum_{\|\beta^j\|=m_j} b^j_{\beta^j} z^{\beta^j},$$

and functions Q_j develop in Taylor series in a neighborhood of the origin that converge absolutely and uniformly:

$$Q_j(z) = \sum_{\|\alpha^j\|>m_j} a^j_{\alpha^j} z^{\alpha^j}.$$

For non-degenerate systems of polynomials P, i.e. such that there exists only one their common zero – the origin, one can show that ([9, Lemma 1]) the cycle

$$\Gamma_P = \{z \in \mathbb{C}^n : |P_j| = r_j,\ r_j > 0,\ j = \overline{1, n}\}$$

is a compact set that does not intersect with the coordinate axes for almost all r_j.

Denote by J_γ the residue integral

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z^{\gamma+I}} \cdot \frac{df}{f} =$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \dots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n},$$

where $\gamma = (\gamma_1, \dots \gamma_n)$.

Theorem 1.1 ([9], Theorem 1). *Under the assumtions made, for a system of equations of the form (1) we have*

$$J_\gamma = \sum_{\|\alpha\| \leqslant \|\gamma\|+n} (-1)^{\|\alpha\|} \mathfrak{M} \left[\frac{\Delta \cdot Q^\alpha}{z^\gamma \cdot P^{\alpha+I}} \right], \tag{2}$$

where Δ is the Jacobian of the system (1), $Q^\alpha = Q_1^{\alpha_1} \cdot \ldots \cdot Q_n^{\alpha_n}$, and $\mathfrak{M}$ is a linear functional that to a Laurent polynomial assigns its free term.

We shall additionally assume that the system of polynomials P does not have zeroes at the infinity in the space $\overline{\mathbb{C}}^n$ and consider the case $Q_j(z)$ are polynomials of degree s_j with the condition: for each i

$$\deg_{z_i} P_i < \deg_{z_i} Q_i, \quad \deg_{z_j} P_i \geqslant \deg_{z_j} Q_i, \quad i \neq j.$$

We make in the functions $f_j(z) = P_j(z)+Q_j(z)$ the change $z_i = \dfrac{1}{w_i}$ assuming that all $w_i \neq 0$. We get

$$f_j\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right) = \frac{1}{w_1^{m_j^1}\cdot\ldots\cdot w_j^{s_j^j}\cdot\ldots\cdot w_n^{m_j^n}}\cdot\left(\widetilde{P}_j(w)+\widetilde{Q}_j(w)\right),$$

where $\widetilde{P}_j(w)$ and $\widetilde{Q}_j(w)$ are polynomials with the property $\deg\widetilde{P}_j > \deg\widetilde{Q}_j$, and $\widetilde{f}_j(w) = \widetilde{P}_j(w)+\widetilde{Q}_j(w)$.

Denote by $\Gamma_{\widetilde{P}}$ the cycle

$$\Gamma_{\widetilde{P}} = \{w\in\mathbb{C}^n : |\widetilde{P}_j| = \varepsilon_j, \quad \varepsilon_j > 0\}.$$

Then for an arbitrary multi-index γ the integral J_γ is equal to ([9, Lemma 9])

$$J_\gamma = \frac{(-1)^n}{(2\pi\sqrt{-1})^n}\int\limits_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\widetilde{f}_1}{\widetilde{f}_1}\wedge\frac{d\widetilde{f}_2}{\widetilde{f}_2}\wedge\ldots\wedge\frac{d\widetilde{f}_n}{\widetilde{f}_n}.$$

Moreover, if $w_1,\dots,w_p$ are zeroes of the system $\widetilde{f}(w) = 0$ (counting multiplicities) where $w_k = (w_{k1}, w_{k2},\dots,w_{kn})$ then

$$J_\gamma = \sum_{j=1}^{s} w_{j1}^{\gamma_1+1}\cdot w_{j2}^{\gamma_2+1}\cdot\ldots\cdot w_{jn}^{\gamma_n+1}.$$

These zeroes are related with the zeroes $z_1,\dots,z_p$ of the original system that do not lie on the coordinate axes via $z_{km} = \dfrac{1}{w_{km}}$. Collecting obtained formulas and computing the integral in (2) using the transformation formula for the Grothendieck residue, we get the main result.

Theorem 1.2 ([9], Theorem 6)**.** *Under the assumptions made, the power sum of roots of the system (1) is equal to*

$$\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdot\ldots\cdot z_{jn}^{\gamma_n+1}} = J_\gamma =$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n}(-1)^{||\alpha||}\frac{1}{(2\pi\sqrt{-1})^n}\int\limits_{\Gamma_{\widetilde{P}}} w^{\gamma+I}\cdot\frac{\widetilde{\Delta}\cdot\widetilde{Q}^{\alpha}dw}{\widetilde{P}^{\alpha+I}} =$$

$$= \sum_{||K||\leqslant||\gamma||+n}\frac{(-1)^{||K||+n}\prod_{s=1}^{n}\left(\sum_{j=1}^{n}k_{sj}\right)!}{\prod_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\widetilde{\Delta}\cdot\det A\cdot\widetilde{Q}^{\alpha}\prod_{s,j=1}^{n}a_{sj}^{k_{sj}}}{\prod_{j=1}^{n}w_j^{\beta_j N_j+\beta_j+N_j}}\right],$$

where the summation is performed over all integer non-negative matrices $K = \|k_{sj}\|_{s,j=1}^n$ *such that the sum* $\sum_{s=1}^n k_{sj} = \alpha_j$, *and the sum* $\sum_{j=1}^n k_{js}$ *is denoted by* β_s. *The polynomial coefficients* a_{sj} *are taken from the representation*

$$w_j^{N_j+1} = \sum_{k=1}^{n} a_{jk}\widetilde{P}_k,$$

and $\det A$ *is the determinant of the matrix of coefficients.*

2. Examples

Example 1. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = 1 + a_1 z_1 + a_2 z_2 = 0, \\ f_2(z_1, z_2) = 1 + b_1 z_1 + b_2 z_2 = 0. \end{cases} \tag{3}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 + a_1 w_2 + a_2 w_1 = 0, \\ \widetilde{f}_2 = w_1 w_2 + b_1 w_2 + b_2 w_1 = 0. \end{cases}$$

Subtract the second equation from the first one and pass to the system of the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 + a_1 w_2 + a_2 w_1 = 0, \\ \widetilde{f}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2 = 0. \end{cases} \tag{4}$$

The Jacobian $\widetilde{\Delta}$ of the system (4) is equal to

$$\widetilde{\Delta} = \begin{vmatrix} w_2 + a_2 & w_1 + a_1 \\ a_2 - b_2 & a_1 - b_1 \end{vmatrix} = (-a_2 + b_2) w_1 + (a_1 - b_1) w_2 + (a_1 b_2 - a_2 b_1).$$

Note that

$$\begin{cases} \widetilde{Q}_1 = a_1 w_2 + a_2 w_1, \\ \widetilde{Q}_2 = 0. \end{cases}$$

$$\begin{cases} \widetilde{P}_1 = w_1 w_2, \\ \widetilde{P}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2. \end{cases}$$

Let us calculate $\det A$. Since

$$w_1^2 = a_{11} \widetilde{P}_1 + a_{12} \widetilde{P}_2,$$
$$w_2^2 = a_{21} \widetilde{P}_1 + a_{22} \widetilde{P}_2,$$

where $\widetilde{P}_1 = w_1 w_2$, $\widetilde{P}_2 = (a_2 - b_2) w_1 + (a_1 - b_1) w_2$. Therefore, the elements of a_{ii} are equal

$$a_{11} = -\frac{a_1 - b_1}{a_2 - b_2}, \quad a_{12} = \frac{w_1}{a_2 - b_2},$$

$$a_{21} = -\frac{a_2 - b_2}{a_1 - b_1}, \quad a_{22} = \frac{w_2}{a_1 - b_1}.$$

Therefore,

$$\det A = -\frac{w_2}{a_2 - b_2} + \frac{w_1}{a_1 - b_1} = \frac{(a_2 - b_2) w_1 - (a_1 - b_1) w_2}{(a_1 - b_1)(a_2 - b_2)}.$$

By Theorem 1.1

$$J_{(0,0)} = \sum_{\|K\| = k_{11} + k_{12} + k_{21} + k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11} + k_{12})! \cdot (k_{21} + k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times\mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A\cdot\widetilde{Q}_1^{k_{11}+k_{21}}\cdot\widetilde{Q}_2^{k_{12}+k_{22}}\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{2(k_{11}+k_{12})}\cdot w_2^{2(k_{21}+k_{22})}}\right],$$

$$J_{(0,0)}=\sum_{\|K\|=k_{11}+k_{12}+k_{21}+k_{22}\leqslant 2}\frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times$$

$$\times\mathfrak{M}\left[\frac{(-1)^{k_{11}+k_{21}}((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))}{(a_1-b_1)^{1+k_{21}+k_{22}-k_{11}}\cdot(a_2-b_2)^{1+k_{11}+k_{12}-k_{21}}}\right]\times$$

$$\times\left[\frac{(a_1w_2+a_2w_1)^{k_{11}+k_{21}}\cdot 0^{k_{12}+k_{22}}}{w_1^{2k_{11}+k_{12}}\cdot w_2^{2k_{21}+k_{22}}}\right].$$

Calculate the values of the sums using the fact that $\widetilde{Q}_2=0$.

$(0,0,0,0):$

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))}{(a_1-b_1)\cdot(a_2-b_2)}\right]=0,$$

$(1,0,0,0):$

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)}{(a_2-b_2)^2\cdot w_1^2}\right]=$$

$$=\frac{a_2(a_1b_2-a_2b_1)}{a_2-b_2},$$

$(0,0,1,0):$

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)}{(a_1-b_1)^2\cdot w_2^2}\right]=$$

$$=\frac{-a_1(a_1b_2-a_2b_1)}{a_1-b_1},$$

$(2,0,0,0):$

$$\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2\cdot(a_1-b_1)}{(a_2-b_2)^3\cdot w_1^4}\right]=$$

$$=\frac{-a_2^2(a_1-b_1)}{a_2-b_2},$$

$(0,0,2,0):$

$$-\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2\cdot(a_2-b_2)}{(a_1-b_1)^3\cdot w_2^4}\right]=$$

$$=\frac{-a_1^2(a_2-b_2)}{a_1-b_1},$$

$(1,0,1,0):$

$$-\mathfrak{M}\left[\frac{((a_2-b_2)w_1-(a_1-b_1)w_2)\cdot((a_1-b_1)w_2-(a_2-b_2)w_1+(a_1b_2+a_2b_1))\cdot(a_1w_2+a_2w_1)^2}{(a_1-b_1)\cdot(a_2-b_2)\cdot w_1^2\cdot w_2^2}\right]=$$

$$=\frac{-a_1^2(a_2-b_2)^2}{(a_1-b_1)(a_2-b_2)}+\frac{-a_2^2(a_1-b_1)^2}{(a_1-b_1)(a_2-b_2)}+4a_1a_2.$$

Therefore,

$$J_{(0,0)}=4a_1a_2+\frac{a_2(a_1b_2-a_2b_1)}{a_2-b_2}-\frac{a_1(a_1b_2-a_2b_1)}{a_1-b_1}-\frac{2a_1^2(a_2-b_2)}{a_1-b_1}-\frac{2a_2^2(a_1-b_1)}{a_2-b_2}.$$

Calculate the power sum of the root system (3) directly. We multiply the first equation of the system by b_2, the second by a_2 and subtract one from another. Thus

$$z_1 = \frac{a_2 - b_2}{a_1 b_2 - a_2 b_1},$$

$$z_2 = -\frac{a_1 - b_1}{a_1 b_2 - a_2 b_1}.$$

By Theorem 1.2

$$J_{(0,0)} = \sum_{j=1}^{p} \frac{1}{z_{j1} \cdot z_{j2}} = -\frac{(a_1 b_2 - a_2 b_1)^2}{(a_1 - b_1)(a_2 - b_2)},$$

which coincides with the value found above.

Example 2. We shall use the result of the previous example in the case of a non-algebraic system. Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2\pi^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k+1)!},$$

which uniformly and absolutely converge on the complex plane and have the order of growth equal to 1/2.

Consider the system of equations

$$\begin{cases} f_1(z_1, z_2) = \dfrac{\sin\sqrt{a_1 z_1 + a_2 z_2}}{\sqrt{a_1 z_1 + a_2 z_2}} = \prod\limits_{m=1}^{\infty}\left(1 - \dfrac{a_1 z_1 + a_2 z_2}{m^2\pi^2}\right) = 0, \\ f_2(z_1, z_2) = \dfrac{\sin\sqrt{b_1 z_1 + b_2 z_2}}{\sqrt{b_1 z_1 + b_2 z_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{b_1 z_1 + b_2 z_2}{s^2\pi^2}\right) = 0. \end{cases}$$

Using the formula obtained above in Example 1 and the well-known expansion of the series, we obtain that the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = \sum_{m,s=1}^{\infty} \frac{(a_1 b_2 - a_2 b_1)^2}{\pi^4 (s^2 a_1 - m^2 b_1)(m^2 b_2 - s^2 a_2)}.$$

$$J_{(0,0)} = \sum_{m=1}^{\infty} \frac{4 a_1 a_2}{\pi^4 m^2} + \sum_{m,s=1}^{\infty} \frac{a_1 b_2 - a_2 b_1}{\pi^4 m^2}\left[\frac{a_1}{m^2 b_1 - s^2 a_1} - \frac{a_2}{m^2 b_2 - s^2 a_2}\right] -$$
$$- \sum_{m,s=1}^{\infty} \frac{2 a_2^2 (m^2 b_1 - s^2 a_1)}{\pi^4 m^4 (m^2 b_2 - s^2 a_2)} - \sum_{m,s=1}^{\infty} \frac{2 a_1^2 (m^2 b_2 - s^2 a_2)}{\pi^4 m^4 (m^2 b_1 - s^2 a_1)}.$$

The value of this series is found in the articles [5] and [7].

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Примеры вычисления степенных сумм корней систем уравнений

Евгения К. Мышкина
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. Рассмотрены примеры вычисления степенных сумм корней систем уравнений, в том числе трансцендентных. Так как число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Приведены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.