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Theoretical Analysis for a System of Nonlinear ϕ-Hilfer Fractional Volterra-Fredholm Integro-differential Equations

Ahmed A. Hamoud*
Nedal M. Mohammed†
Department of Mathematics & Computer Science
Taiz University
Taiz-96704, Yemen
Rasool Shah‡
Department of Mathematics
Abdul Wali Khan University
Mardan-23200, Pakistan

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Abstract. We investigate the existence of solutions for a system of nonlinear ϕ-Hilfer fractional Volterra–Fredholm integro-differential equations with fractional integral conditions, by using the Krasnoselskii's fixed point theorem and Arzela–Ascoli theorem. Moreover, applying an alternative fixed point theorem due to Diaz and Margolis, we prove the Kummer stability of the system on the compact domains. An example is also presented to illustrate our results.

Keywords: ϕ-Hilfer fractional Volterra-Fredholm integro-differential equation, Kummer's stability, Arzela–Ascoli theorem, Krasnoselskii fixed point theorem.

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Introduction

Fractional order differential equations have become one of the most popular areas of research in mathematical analysis, engineering, economics, control theory, materials sciences, physics, chemistry, and biology (see [1, 2] and the references therein). Scientists have applied various mathematical approaches through diverse research-oriented aspects of fractional differential systems. For instance, existence, stability, and control theory for fractional differential equations were studied [3, 4]. For the first time, Alsina and Ger [5] studied the Hyers-Ulam stability for differential equations. Recently, mathematicians have paid more attention to the study of stability for a wide range of differential systems [6–9]. Volterra integro-differential equations which are an important class of these equations have arise widely in the mathematical modelling of many physical and biological processes, for example biological species coexisting together with increasing and decreasing rate of growth, electromagnetic theory, Wilson-Cowan model and many

*drahmed985@yahoo.com https://orcid.org/0000-0002-8877-7337
†dr.nedal.mohammed@gmail.com
‡rasoolshahawkum@gmail.com

more [10,11]. Although they have considerably been studied in science and engineering, fractional integro-differential equations with mixed fractional operators have been newly introduced [12–15].

In this article, motivated by the research going on in this direction, we study a new system for nonlinear ϕ-Hilfer fractional Volterra-Fredholm integro-differential equations with fractional integral conditions of the form

$$\begin{cases} {}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi} w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^t \mathfrak{h}(\eta, s, w(s))ds \ + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, w(s))ds, \\ \hfill \eta \in \omega := (0, \mathfrak{p}], \\ I_0^{1-\gamma;\phi} w(0) = \mathfrak{w}_0, \quad \mathfrak{w}_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}(.)$ is a ϕ-Hilfer fractional derivative of order $0 < \varsigma \leqslant 1$ and type $0 \leqslant v < 1$, and $I_0^{1-\gamma;\phi}$ is a ϕ-Riemann-Liouville fractional integral of order $1-\gamma(\gamma = \varsigma + v(1-\varsigma))$ with respect to the mapping ϕ. Furthermore, $\mathfrak{g} : \omega \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{h}, \mathfrak{k} : \omega^2 \times \mathbb{R} \to \mathbb{R}$ are given mappings, and $\mathcal{A}$ is a closed linear operator. In the following, we show the existence of solutions to equation (1) based on the Krasnoselskii fixed point theorem and Arzela–Ascoli theorem. Using Kummer's control function, we introduce a new concept of stability and further deduce that the solution of equation (1) is stable in Kummer's sense.

1. Preliminaries

In this section, we present some important definitions and mathematical concepts on the fractional calculus. For details, please see [2, 16] and the references therein. Let $[\mathrm{n}, \mathrm{m}]$ be a finite and closed interval with $0 \leqslant \mathrm{n} < \mathrm{m} < \infty$ and $\mathfrak{C}[\mathrm{n}, \mathrm{m}]$ be the space of continuous functions $\varrho : [\mathrm{n}, \mathrm{m}] \to \mathbb{R}$ equipped with the following norm

$$\|\varrho\|_{\mathfrak{C}[\mathrm{n},\mathrm{m}]} = \max_{\eta \in [\mathrm{n},\mathrm{m}]} |\varrho(\eta)|.$$

Furthermore, the weighted space $\mathfrak{C}_{\gamma,\phi}(\mathrm{n}, \mathrm{m}]$ is defined as

$$\mathfrak{C}_{1-\gamma,\phi}[\mathrm{n}, \mathrm{m}] = \left\{\varrho : (\mathrm{n}, \mathrm{m}] \to \mathbb{R}; (\phi(\eta) - \phi(\mathrm{n}))^{1-\gamma} \varrho(\eta) \in \mathfrak{C}[\mathrm{n}, \mathrm{m}]\right\} \quad \text{where} \quad 0 < \gamma < 1.$$

with norm

$$\|\varrho\|_{C_{1-\gamma,\phi}[\mathrm{n},\mathrm{m}]} = \max_{\eta \in [\mathrm{n},\mathrm{m}]} \left|(\phi(\eta) - \phi(\mathrm{n}))^{1-\gamma} \varrho(\eta)\right|,$$

where $\phi : [\mathrm{n}, \mathrm{m}] \longrightarrow \mathbb{R}$ is an arbitrary function, and $\eta \in [\mathrm{n}, \mathrm{m}]$.

Definition 1.1. *Let* $(\mathrm{n}, \mathrm{m}), -\infty \leqslant \mathrm{n} < \mathrm{m} \leqslant +\infty$ *be a finite or infinite interval of the line* $\mathbb{R}$, Γ *be the gamma function, and* $\varsigma > 0$. *Additionally, let* $\phi(\eta)$ *be a positive function defined on* $[\mathrm{n}, \mathrm{m}]$ *so that* $\phi'(\eta) \geqslant 0$ *on* $(\mathrm{n}, \mathrm{m}]$ *and* $\phi'(\eta)$ *is a continuous function on* (n, m). *The left-and right-sided fractional integrals of a function* ϱ *with respect to the function* ϕ *on* $[\mathrm{n}, \mathrm{m}]$ *are defined by*

$$I_{\mathrm{n}^+}^{\varsigma;\phi} \varrho(x) = \frac{1}{\Gamma(\varsigma)} \int_{\mathrm{n}}^x \phi'(\eta)(\phi(x) - \phi(\eta))^{\varsigma-1} \varrho(\eta) d\eta$$

and

$$I_{\mathrm{m}^-}^{\varsigma;\phi} \varrho(x) = \frac{1}{\Gamma(\varsigma)} \int_x^{\mathrm{m}} \phi'(\eta)(\phi(\eta) - \phi(x))^{\varsigma-1} \varrho(t) d\eta$$

respectively.

The fractional integrals with the above definition have a semi-group property given by

$$I_{\mathrm{n}^+}^{\varsigma;\phi} I_{\mathrm{n}^+}^{v;\phi} \varrho(x) = I_{\mathrm{n}^+}^{\varsigma+v;\phi} \varrho(x) \quad \text{and} \quad I_{\mathrm{m}^-}^{\varsigma;\phi} I_{\mathrm{m}^-}^{v;\phi} \varrho(x) = I_{\mathrm{m}^-}^{\varsigma+v;\phi} \varrho(x).$$

Additionally, for $\varsigma, \sigma > 0$, we have [17]:

(i) if $\varrho(x) = (\phi(x) - \phi(\mathrm{n}))^{\sigma-1}$ then $I_{\mathrm{n}^+}^{\varsigma;\phi} \varrho(x) = \dfrac{\Gamma(\sigma)}{\Gamma(\varsigma+\sigma)} (\phi(x) - \phi(\mathrm{n}))^{\varsigma+\sigma-1}$, and

(ii) if $\varrho(x) = (\phi(\mathrm{m}) - \phi(x))^{\sigma-1}$ then $I_{\mathrm{m}^-}^{\varsigma;\phi} \varrho(x) = \dfrac{\Gamma(\sigma)}{\Gamma(\varsigma+\delta)} (\phi(\mathrm{m}) - \phi(x))^{\varsigma+\sigma-1}$.

Definition 1.2. *Let* $(\mathrm{n}, \mathrm{m}), -\infty \leqslant \mathrm{n} < \mathrm{m} \leqslant +\infty$ *be a finite or infinite interval of the line* $\mathbb{R}$, $\phi'(\eta) \neq 0$ *for all* $\eta \in (\mathrm{n}, \mathrm{m})$, *and* $\varsigma > 0$, $n \in \mathbb{N}$. *The left-sided Riemann–Liouville derivative of a function* ϱ *with respect to* ϕ *of order* ς *correspondent to the Riemann–Liouville is defined by*

$$\begin{aligned} D^{\varsigma,\phi} \varrho(\eta) &= \left(\frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n I^{n-\varsigma,\phi} \varrho(\eta) \\ &= \frac{1}{\Gamma(n-\varsigma)} \left(\frac{1}{\phi'(\eta)} \frac{d}{dx} \right)^n \times \int_{\mathrm{n}}^{\eta} \phi'(t)(\phi(\eta) - \phi(t))^{n-\varsigma-1} \varrho(t) dt. \end{aligned}$$

Definition 1.3. *Let* $n-1 < \varsigma < n$ *with* $n \in \mathbb{N}$, $I = [\mathfrak{n}, \mathfrak{m}](-\infty \leqslant \mathfrak{n} < \mathfrak{m} \leqslant \infty)$ *and* $\varrho, \phi \in \mathfrak{C}^n([\mathfrak{n}, \mathfrak{m}], \mathbb{R})$ *be two mappings such that* $\phi'(x) > 0$ *for all* $x \in I$. *The left-and right-sided* ϕ*-Hilfer fractional derivatives* ${}^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}(.)$ *of the arbitrary function* ϱ *of order* ς *and type* $0 \leqslant v < 1$ *are defined by*

$${}^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} \varrho(x) = I_{\mathfrak{n}^+}^{v(n-\varsigma);\phi} \left(\frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{\mathfrak{n}^+}^{(1-v)(n-\varsigma);\phi} \varrho(x)$$

and

$${}^{\mathcal{H}}\mathbb{D}_{\mathfrak{m}^-}^{\varsigma,v;\phi} \varrho(x) = I_{\mathfrak{m}^-}^{v(n-\varsigma);\phi} \left(-\frac{1}{\phi'(x)} \frac{d}{dx} \right)^n I_{\mathfrak{m}^-}^{(1-v)(n-\varsigma);\phi} \varrho(x)$$

respectively.

Theorem 1.4. *If* $\varrho \in \mathfrak{C}^1[\mathfrak{n}, \mathfrak{m}]$, $\varsigma > 0$, $0 \leqslant v < 1$, *and* $\gamma = \varsigma + v(1-\varsigma)$ *then*

$${}^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} I_{\mathfrak{n}^+}^{\varsigma;\phi} \varrho(x) = \varrho(x) \quad \textit{and} \quad {}^{\mathcal{H}}\mathbb{D}_{\mathfrak{m}^-}^{\varsigma,v;\phi} I_{\mathfrak{m}^-}^{\varsigma;\phi} \varrho(x) = \varrho(x).$$

Additionally, we have

$$I_{\mathfrak{n}^+}^{\varsigma;\phi}\ {}^{\mathcal{H}}\mathbb{D}_{\mathfrak{n}^+}^{\varsigma,v;\phi} \varrho(x) = \varrho(x) - \frac{(\phi(x) - \phi(\mathfrak{n}))^{\gamma-1}}{\Gamma(\gamma)} I_{\mathfrak{n}^+}^{(1-v)(1-\varsigma);\phi} \varrho(\mathfrak{n})$$

and

$$I_{\mathfrak{m}^-}^{\varsigma;\phi}\ {}^{\mathcal{H}}\mathbb{D}_{\mathfrak{m}^-}^{\varsigma,v;\phi} \varrho(x) = \varrho(x) - \frac{(\phi(\mathfrak{m}) - \phi(x))^{\gamma-1}}{\Gamma(\gamma)} I_{\mathfrak{m}^-}^{(1-v)(1-\varsigma);\phi} \varrho(\mathfrak{m}).$$

Proof. Ref. [17].

The solution of a hypergeometric differential equation is called a confluent hypergeometric function [18]. There exist different standard forms of confluent hypergeometric functions, such as Kummer's functions, Tricomi's functions, Whittaker's functions, and Coulomb's wave functions. In this paper, we apply the following Kummer (confluent hypergeometric) function to study our stability:

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_2)}{\Gamma(\mathfrak{P}_1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_2 + k)} \frac{\mathfrak{z}^k}{k!} \tag{2}$$

which is the solution of the differential equation

$$\mathfrak{z}\frac{d^2\mathfrak{u}}{dz} + (\mathfrak{P}_2 - \mathfrak{z})\frac{d\mathfrak{u}}{d\mathfrak{z}} - \mathfrak{P}_1\mathfrak{u}(\mathfrak{z}) = 0,$$

where $\mathfrak{z}, \mathfrak{P}_1 \in \mathbb{C}$ and $\mathfrak{P}_2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Kummer's function was introduced by Kummer in 1837. The series (2) is also known as the confluent hyper-geometric function of the first kind, and is convergent for any $\mathfrak{z} \in \mathbb{C}$. In this article, we apply it on the real line $\mathbb{R}$ as our control function. Clearly, for $\mathfrak{P}_1 = \mathfrak{P}_2$, we have

$$\Phi(\mathfrak{P}_1, \mathfrak{P}_2; \mathfrak{z}) = {}_1F_1(\mathfrak{P}_1, \mathfrak{P}_1; \mathfrak{z}) = \frac{\Gamma(\mathfrak{P}_1)}{\Gamma(\mathfrak{P}_1)}\sum_{k=0}^{\infty}\frac{\Gamma(\mathfrak{P}_1 + k)}{\Gamma(\mathfrak{P}_1 + k)}\frac{\mathfrak{z}^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{k!} = e^{\mathfrak{z}}.$$

Letting $\varsigma, v \in \omega$, we consider the following inequality for $\epsilon > 0$

$$\begin{aligned} &\left| {}^{\mathcal{H}}\mathbb{D}_{0+}^{\varsigma,v;\phi}v(\eta) - \mathcal{A}(v(\eta)) + \mathfrak{g}(\eta, v(\eta)) - \int_0^t \mathfrak{h}(\eta, s, v(s))ds - \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, v(s))ds \right| \leqslant \\ &\leqslant \epsilon\Phi(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}) \end{aligned} \tag{3}$$

where Φ is the Kummer's function (see [18]), to define a new stability concept called Kummer's stability.

Definition 1.5. *For a positive constant $\mathcal{C}_\Phi$, for all $\epsilon > 0$, and every solution $v \in (\mathfrak{C}[0, \mathfrak{p}], \mathbb{R})$ to inequality* (3)*, if we can find a solution $w \in (\mathfrak{C}[0, \mathfrak{p}], \mathbb{R})$ to Equation* (1)*, with the following property:*

$$|w(\eta) - v(\eta)| \leqslant C_\Phi \in \Phi(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}) \text{ for all } \eta \in [0, \mathfrak{p}]$$

then we say that equation (1) *has Kummer's stability with respect to $\Phi(\varsigma, v, (\phi(\eta) - \phi(0))^{\varsigma})$.*

Our approach is motivated by the fact that inversion of a perturbed differential operator may result from the sum of a compact operator and a contraction mapping (see [19–21] and the references therein). We begin by stating the following Krasnoselskii fixed point theorem, which has many applications in studying the existence of solutions to differential equations:

Theorem 1.6 (Krasnoselskii fixed point theorem)**.** *Let X be a Banach space and $\mathfrak{M} \subseteq X$ be a closed, convex, and non-empty set. Additionally, let $\mathfrak{T}, \mathfrak{S}$ be mappings so that:*

- $\mathfrak{T}u + \mathfrak{S}v \in \mathfrak{M}$ *whenever* $u, v \in \mathfrak{M}$,
- *The operator $\mathfrak{T}$ is continuous and compact, and*
- *Mapping $\mathfrak{T}$ is a contraction.*
  Then, there exists a $w \in \mathfrak{M}$ so that $w = \mathfrak{T}w + \mathfrak{S}w$.

In addition, we mention an alternative fixed point theorem presented by Diaz and Margolis in 1967, and it plays a crucial role in proving our stability result [22].

Theorem 1.7. *Consider the generalized complete metric space $(\mathfrak{X}, \mathrm{Y})$ and let Θ be a self-map operator which is a strictly contraction mapping with the Lipschitz constant $\kappa < 1$. Then, we have two options:*
(i) either for every $\mathrm{n} \in \mathbb{N}, \mathrm{Y}(\Theta^{\mathrm{n}+1}\mathfrak{z}, \Theta^{\mathrm{n}}\mathfrak{z}) = +\infty$; or
(ii) if there exists $\mathrm{n} \in \mathbb{N}$ so that the operator Θ satisfies $Y(\Theta^{n+1}\mathfrak{z}, \Theta^{n}\mathfrak{z}) < \infty$ for some $\mathfrak{z} \in \mathfrak{X}$, then the sequence $\{\Theta^n\mathfrak{z}\}$ tends to a unique fixed point $\mathfrak{z}^*$ of Θ in the set $\mathfrak{X}^* = \{\mathfrak{v} \in \mathfrak{X} : Y(\Theta^n\mathfrak{v}, \Theta^n\mathfrak{v}) < \infty\}$. Furthermore, for all $\hat{z} \in \mathfrak{X}$

$$Y(\mathfrak{z}, \mathfrak{z}^*) \leqslant \frac{1}{1-\kappa}Y(\mathfrak{z}, \Theta_{\mathfrak{z}}).$$

Now, we are ready to prove that equation (1) is equivalent to an integral equation. Then, by the above theorem, we infer that a fixed point exists for the integral equation, so equation (1) has at least one solution.

Proposition 1.8. *Assume that $\mathfrak{g} : \omega \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{h} : \omega^2 \times \mathbb{R} \to \mathbb{R}$ are real-valued continuous mappings, and $\mathcal{A}$ is a closed operator, then the following integral equation is equivalent to equation* (1)*:*

$$\begin{aligned} w(\eta) \quad = \quad & \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi}\Big[\mathfrak{g}(\eta, w(\eta), w(\eta)) + \int_0^s \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \\ & + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,u(\tau))d\tau + \mathcal{A}(w(\eta))\Big] \end{aligned} \tag{4}$$

where $\gamma \geqslant 0$ and we obtain from $\gamma = \varsigma + v(1-\varsigma)$ for $0 < \varsigma \leqslant 1$ and $0 \leqslant v < 1$ in (1)*.*

Proof. Using the properties of the ϕ-Hilfer fractional derivative outlined in the preliminaries, we have where $\gamma = \varsigma + v(1-\varsigma)$. So, by the above equality, we have

$$^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\boldsymbol{w}(\eta) = I^{v(n-\varsigma);\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta) = I^{\gamma-\varsigma;\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta),$$

where $\gamma = \varsigma + v(1-\varsigma)$. So, by the above equality, we have

$$I^{\gamma-\varsigma;\phi}D^{\gamma;\phi}\boldsymbol{w}(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta, w(\eta)) + \int_0^\eta \mathfrak{h}(\eta,\mathrm{s},w(s))ds + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\mathrm{s},w(s))ds.$$

Now, applying $I^{\gamma;\phi}$ to both sides of the above equation and using Theorem 1.4 , we obtain

$$I^{\varsigma,\phi}I^{\gamma-\varsigma;\phi}D^{\Upsilon\phi}w(\eta) = I^{\varsigma;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^s \mathfrak{h}(\eta,\tau,w(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right)$$

and

$$I^{\Upsilon\phi}D^{\gamma\phi}w(\eta) = I^{\varsigma;\phi}\left(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^s h(\eta,\tau,w(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right).$$

Then,

$$w(\eta) = \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I^{\varsigma;\phi}\Big(\mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^{\mathrm{s}} \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\Big).$$

Conversely, assuming that $w \in \mathcal{C}[0,p]$ satisfies equation (4), we claim that the fractional integro-differential equation (1) holds. We apply $^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}$ to the equation (4) and imply by Theorem 1.4 that

$$\begin{aligned} & ^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}w(\eta) = \\ & = {}^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\Big(\frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I^{\varsigma;\phi}\mathcal{A}(w(\eta)) + I^{\varsigma;\phi}\mathfrak{g}(\eta,w(\eta)) + I^{\varsigma;\phi}\int_0^{\mathrm{s}} \mathfrak{h}(\eta,\tau,u(\tau))d\tau + \\ & \qquad + I^{\varsigma;\phi}\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\Big). \end{aligned}$$

From $^{\mathcal{H}}\mathbb{D}^{\varsigma,v;\phi}\mathfrak{w}_0 = 0$, we obtain

$$^{\mathcal{H}}\mathbb{D}_{0^+}^{\varsigma,v;\phi}w(\eta) = \mathcal{A}(w(\eta)) + \mathfrak{g}(\eta,w(\eta)) + \int_0^\eta \mathfrak{h}(t,s,u(s))ds + \int_0^{\mathfrak{p}} \mathfrak{k}(t,s,u(s))ds.$$

This completes the proof. □

Remark 1. Let $\mathfrak{w} \in \mathfrak{C}(\omega, \mathbb{R})$ satisfy inequality (3). Then the following integral inequality holds

$$\begin{aligned}
&\Big|\mathfrak{w}(t) - \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 - \\
&-I_{0+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta))\Big) - I_{0+}^{\varsigma;\phi}\left[\int_0^s \mathfrak{h}(\eta,\tau,w(\tau))d\tau\right] - I_{0+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right] - I_{0+}^{\varsigma;\phi}(\mathcal{A}(w(\eta)))\Big| \\
&\leqslant \frac{\epsilon}{\Gamma(\varsigma)}\int_0^\eta \phi'(t)(\phi(x)-\phi(\eta))^{\varsigma-1}\Phi\left(\varsigma, v; (\phi(\eta)-\phi(0))^\varsigma\right) ds \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\int_0^\eta \phi'(\eta)(\phi(x)-\phi(\eta))^{\varsigma-1}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{(\phi(\eta)-\phi(0))^{t\varsigma}}{t!}ds \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^\eta \phi'(t)(\phi(x)-\phi(\eta))^{\varsigma-1}(\phi(\eta)-\phi(0))^{t\varsigma}ds \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^\eta (\phi(x)-\phi(\eta))^{\varsigma-1}(\phi(\eta)-\phi(0))^{t\varsigma}d\phi(s) \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}\int_0^{\phi(\eta)-\phi(0)} (\phi(\eta)-\phi(0)-w)^{\varsigma-1}w^{t\varsigma}dw \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma-1}\int_0^{\phi(\eta)-\phi(0)} \left(1-\frac{w}{\phi(\eta)-\phi(0)}\right)^{\varsigma-1} w^{t\varsigma}dw \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma(t+1)}\int_0^1 (1-v)^{\varsigma-1}v^{t\varsigma}dv \\
&= \frac{\epsilon}{\Gamma(\varsigma)}\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{1}{t!}(\phi(\eta)-\phi(0))^{\varsigma(t+1)}\frac{\Gamma(t\varsigma+1)\Gamma(\varsigma)}{\Gamma((t+1)\varsigma+1)} \\
&\leqslant \epsilon\frac{\Gamma(v)}{\Gamma(\varsigma)}\sum_{t=0}^\infty \frac{\Gamma(\varsigma+t)}{\Gamma(v+t)}\frac{(\phi(\eta)-\phi(0))^{\varsigma(t+1)}}{t!} \\
&\leqslant \epsilon\Phi\left(\varsigma, v; (\phi(\eta)-\phi(0))^\varsigma\right).
\end{aligned}$$

2. Existence results

In this section, we study equation (1) under the following hypotheses:

- (H1). $\mathfrak{g} \in \mathbb{C}(\omega \times \mathbb{R}, \mathbb{R})$. Moreover, there exists $\mathfrak{q}_1$ such that

$$|\mathfrak{g}(\eta, w)| \leqslant \mathfrak{q}_1\mathfrak{M}_1$$

  where $\eta \in \omega, w \in \mathbb{C}([0,\mathfrak{p}], \mathbb{R})$ and $\mathfrak{M}_1 = \|w\|_{\mathfrak{C}[0,p]}$.

- (H2). There exist $\mathfrak{q}_2^h, \mathfrak{q}_2^k > 0$ such that $|\mathfrak{h}(\eta, s, w)| \leqslant \mathfrak{q}_2^h|w(\eta)|$, $|\mathfrak{k}(\eta, s, w)| \leqslant \mathfrak{q}_2^k|w(\eta)|$ for all $\eta \in \omega$ and $w \in \mathfrak{C}([0,p], \mathbb{R})$.

- (H3). The operator A is bounded and $\|\mathcal{A}\| < \dfrac{\Gamma(\varsigma+1)}{\Gamma(\gamma)(\phi(p)-\phi(0))^\gamma}$.

- (H4). The function $\phi(\eta)$ is uniformly continuous for all $\eta \in \omega$.

Lemma 2.1. *Let the operator* $\mathcal{T} : \mathbb{C}[0, \mathfrak{p}] \to \mathcal{C}[0, \mathfrak{p}]$ *given as*

$$(\mathcal{T}w)(\eta) = \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi} \mathfrak{g}(\eta, w(\eta)) + \\ + I_{0^+}^{\varsigma,\phi} \left[\int_0^s \mathrm{h}(\eta, \tau, w(\tau)) d\tau \right] + I_{0^+}^{\varsigma,\phi} \left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta, \tau, w(\tau)) d\tau \right] + I_{0^+}^{\varsigma;\phi} (\mathcal{A}(w(\eta)))$$

and assume that the hypotheses (H1)–(H3) *are satisfied. Then, the operator* $\mathcal{T}$ *maps from the closed ball* $\mathfrak{B}_\tau = \{w \in \mathcal{C}([0, \mathfrak{p}] : \|w\| \leqslant \tau\}$ *into itself, if*

$$\tau \geqslant \frac{\Gamma(\varsigma + \gamma) |r_0|}{\Gamma(\varsigma + \gamma) - \frac{\Gamma(\gamma)\mathfrak{C}_\phi^\varsigma}{\Gamma(\varsigma+\gamma)} \left[\mathfrak{q}_2^h \mathfrak{p} + \mathfrak{q}_2^k \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 + \frac{\Gamma(\varsigma+1)}{\Gamma(\gamma)\mathfrak{C}_\phi^\gamma} \right]} \tag{5}$$

where $\mathfrak{C}_\phi := (\phi(\mathfrak{p}) - \phi(0))$.

Proof. Clearly, we need to prove that if $w(\eta) \in \mathfrak{B}_\tau$ then $(\mathcal{T}w)(\eta) \in \mathfrak{B}_\tau$. For all $\eta \in [0, \mathfrak{p}]$, we have

$$\begin{aligned}
|(\mathcal{T}w)(\eta)| \leqslant\ & |\mathfrak{w}_0| + \frac{1}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(s) - \phi(0))^{1-\varsigma}} \max_{s \in [0,\eta]} | (\phi(s) - \phi(0))^{1-\gamma} \mathfrak{g}(s, w(s) \mid ds \\
& + \frac{1}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\varsigma}} \max_{s \in [0,\eta]} (\phi(s) - \phi(0))^{1-\gamma} \\
& \times \left[\int_0^s |h(\eta, \tau, w(\tau))| d\tau + \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, \tau, w(\tau)) d\tau \right] ds \\
& + \frac{\|\mathcal{A}\|}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1} \max_{s\in[0,\eta]} \left| (\phi(s) - \phi(0))^{1-\gamma} w(s) \right| ds}{(\phi(\eta) - \phi(s))^{1-\varsigma}} \\
& \leqslant |\mathfrak{w}_0| + \frac{\mathfrak{q}_1 \mathfrak{M}_1 \tau}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\varsigma}} ds + \frac{\mathfrak{q}_2^h \tau \mathfrak{p}}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\varsigma}} ds \\
& + \frac{\mathfrak{q}_2^k \tau \mathfrak{p}}{\Gamma(\varsigma)} \int_0^{\mathfrak{p}} \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\varsigma}} ds + \frac{\|\mathcal{A}\| \tau}{\Gamma(\varsigma)} \int_0^\eta \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1} ds}{(\phi(\eta) - \phi(s))^{1-\varsigma}} \\
& \leqslant |\mathfrak{w}_0| + \frac{1}{\Gamma(\zeta)} \left[\|\mathcal{A}\| \tau + \mathfrak{q}_2^h \tau \mathfrak{p} + \mathfrak{q}_2^k \tau \mathfrak{p} + \mathfrak{q}_1 \mathfrak{N}_1 r \right] \int_0^t \frac{\phi'(s)(\phi(s) - \phi(0))^{\gamma-1}}{(\phi(\eta) - \phi(s))^{1-\varsigma}} ds \\
& \leqslant |\mathfrak{w}_0| + \frac{(\phi(\eta) - \phi(0))^{\varsigma+\gamma-1}}{\Gamma(\varsigma)} \left[\|\mathcal{A}\|^\tau + \mathfrak{q}_2^h \tau \mathfrak{p} + \mathfrak{q}_2^k \tau \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau \right] B(\varsigma, \gamma),
\end{aligned}$$

where B is the beta function. From the formula

$$B(\varsigma, \gamma) = \frac{\Gamma(\varsigma)\Gamma(\gamma)}{\Gamma(\varsigma + \gamma)}$$

we have

$$|(\mathcal{T}w)(\eta)| \leqslant |\mathfrak{w}_0| + \frac{\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^\varsigma}{\Gamma(\varsigma + \gamma)} \left[\|\mathcal{A}\| \tau + \mathfrak{q}_2^h \tau \mathfrak{p} + \mathfrak{q}_2^k \tau \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau \right].$$

Applying $H3$ and condition (5), we have

$$|(\mathcal{T}w)(\eta)| \leqslant |\mathfrak{w}_0| + \frac{\Gamma(\gamma)\mathfrak{C}_\phi^\varsigma}{\Gamma(\varsigma + \gamma)} \left[\mathfrak{q}_2^h \tau \mathfrak{p} + \mathfrak{q}_2^k \tau \mathfrak{p} + \mathfrak{q}_1 \mathfrak{M}_1 \tau + \frac{\Gamma(\varsigma + 1)\tau}{\Gamma(\gamma)\mathfrak{C}_\phi^\gamma} \right] \leqslant \tau.$$

This completes the proof. □

The following theorem shows the existence of solutions to the fractional differential equation (1) using Krasnoselskii's fixed point theorem listed above.

Theorem 2.2. *Assume that hypotheses (H1)–(H4) are satisfied. Then, equation* (1) *has a solution.*

Proof. Define $\Im : \mathcal{C}[0,\mathfrak{p}] \to \mathcal{C}[0,\mathfrak{p}]$ as

$$(\Im w)(\eta) = (\Im_1 w)(\eta) + (\Im_2 w)(\eta)$$

where

$$\begin{aligned}(\Im_1 w)(\eta) \quad := \quad & \frac{(\phi(\eta)-\phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + I_{0^+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta)) + I_{0^+}^{\varsigma,\phi}\left[\int_0^s \mathrm{h}(\eta,\tau,w(\tau))d\tau\right] + \\ & + I_{0^+}^{\varsigma,\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta,\tau,w(\tau))d\tau\right]\end{aligned}$$

and

$$(\Im_2 w)(\eta) := I_{0^+}^{\varsigma;\phi}(\mathcal{A}(w(\eta))).$$

From Proposition 1.8, solving Equation (1) is equivalent to finding a fixed point for the operator $\Im$ defined on the space $\mathfrak{C}[0,\mathfrak{p}]$.

Suppose that τ satisfies condition (5) and $\mathfrak{B}_\tau = \{w \in \mathfrak{C}([0,p] : \|w\| \leqslant \tau\}$. Due to Lemma 2.1, the operator $\Im$ maps $\mathfrak{B}_\tau$ into itself. Now, we use Krasnoselskii fixed point theorem to show that $\Im$ has a fixed point. □

Claim 1. The operator $\Im_1$ is continuous on $\mathfrak{B}_\tau$. Let $\{w_n\}$ be a sequence in $\mathfrak{B}_\tau$ that converges to w. We need to prove that $\Im_1 w_n \longrightarrow \Im_1 w$. For each $\eta \in [0,\mathfrak{p}]$, we have

$$\begin{aligned}& \left|(\Im_1 w_n)(\eta) - (\Im_1 w)(\eta)\right| \leqslant \\ \leqslant & \frac{1}{\Gamma(\varsigma)}\int_0^{\eta\eta} \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(0))^{1-\varsigma}}\Big| \max_{s\in[0,\eta]}(\phi(s)-\phi(0))^{1-\gamma} g(s, w_n(s), (w_n(s))) - \\ & - g(s, w(s), (w(s))\Big| ds + \frac{1}{\Gamma(\varsigma)}\int_0^{\eta} \frac{\phi'(s)(\phi(s)-\phi(0))^{\gamma-1}}{(\phi(\eta)-\phi(0))^{1-\varsigma}} \max_{s\in[0,\eta]}(\phi(s)-\phi(0))^{1-\gamma}\times \\ & \times\left[\int_0^s \Big|\mathfrak{h}(\eta,\tau,w_n(\tau)) - h(\eta,\tau,w(\tau))\Big| d\tau + \int_0^{\mathfrak{p}}\Big|\mathfrak{k}(\eta,\tau,w_n(\tau)) - \mathfrak{k}(\eta,\tau,w(\tau))\Big| d\tau\right] ds.\end{aligned}$$

Since $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{k}$ are continuous, and $w_n \to w$ as $n \to +\infty$ in $\mathfrak{B}_\tau$, we can conclude that $|(\Im_1 w_n)(\eta) - (\Im_1 w)(\eta)| \to 0$ as $n \to +\infty$ by Lebesgue dominated conoergence theorem.

Claim 2. $\Im_1$ is an equicontinuous operator. To prove our second claim, we let $\eta_1, \eta_2 \in \mathtt{w}$ with $\eta_2 < \eta_1$ and $w \in \mathfrak{B}_\tau$,

$$\begin{aligned}& |(\Im_1 w)(\eta_1) - (\Im_1 w)(\eta_2)| \leqslant \\ & \leqslant \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\gamma}(\phi(s)-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}|\mathfrak{w}_0| \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_1\mathfrak{M}_1\mathfrak{r}}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_2^h\mathfrak{r}p}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds \\ & + \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}\mathfrak{q}_2^k\mathfrak{r}p}{\Gamma(\varsigma)}\int_{\eta_2}^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds\end{aligned}$$

$$\leqslant \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\gamma}(\phi(s)-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}|\mathfrak{w}_0|$$
$$+\frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma}}{\Gamma(\varsigma)}\left(\mathfrak{q}_1\mathfrak{M}_1\mathfrak{r}+\mathfrak{q}_2^h\mathfrak{r}+\mathfrak{q}_2^k\mathfrak{r}\right)\int_0^{\eta_1}\phi'(s)(\phi(s)-\phi(0))^{1-\gamma}ds$$
$$\leqslant \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\gamma}(\phi(\mathfrak{p})-\phi(0))^{1-\gamma}}{\Gamma(\gamma)}|r_0|$$
$$+\frac{\Gamma(\gamma)(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma+\gamma}}{\Gamma(\varsigma+1)}\left(\mathfrak{q}_2^h\mathfrak{r}+\mathfrak{q}_2^k\mathfrak{r}+\mathfrak{q}_1\mathfrak{M}_1\mathfrak{r}\right).$$

Hence, we have

$$|(\Im_1 w)(\eta_1)-(\Im_1 w)(\eta_2)|\leqslant$$
$$\leqslant \frac{(\phi(\eta_1)-\phi(\eta_2))^{1-\gamma}}{\Gamma(\gamma)}|\mathfrak{w}_0|+\frac{\Gamma(\gamma)(\phi(\eta_1)-\phi(\eta_2))^{1-\varsigma+\gamma}}{\Gamma(\varsigma+1)(\phi(b)-\phi(0))^{1-\gamma}}\left(\mathfrak{q}_2^h\mathfrak{r}+\mathfrak{q}_2^k\mathfrak{r}+\mathfrak{q}_1\mathfrak{M}_1\mathfrak{r}\right)$$

regarding (H4), the right-hand side of the above inequality tends to zero whenever $\eta_1\to\eta_2$ so it clearly claims that $\Im_1$ is equicontinuous. Furthemnore, using the previous lemma, it is uniformly bounded. Therefore, by Arzela-Ascoli Theorem, $\Im_1$ is compact on $\mathfrak{B}_{\mathrm{r}}$.

Claim 3. The operator $\Im_2$ is a contraction. Let $w_1,w_2\in\mathbb{C}_{1-\gamma,\varsigma}([0,\mathfrak{p}])$, then, we have

$$|(\Im_2 w_1)(t)-(\Im_2 w_2)(t)|\leqslant$$
$$\leqslant \frac{\|A\|}{\Gamma(\varsigma)}\int_0^{\eta}\frac{\phi'(s)}{(\phi(\eta)-\phi(s))^{1-\varsigma}}|w_1(s)-w_2(s)|\,ds$$
$$\leqslant \frac{\|A\|\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^{\gamma}}{\Gamma(\varsigma+1)}|w_1(\eta)-w_2(\eta)|.$$

By (H3), we infer that $\|A\|\Gamma(\gamma)(\phi(\mathfrak{p})-\phi(0))^{\gamma}<\Gamma(\varsigma+1)$. Thus, $\Im_2$ is a contraction mapping. By Theorem 1.6, the mapping $\mathcal{T}$ has at least a fixed point, which directly implies that equation (1) has a solution. This completes the proof.

3. Stability analysis

In this section, we present the Kummer stability with respect to $\Phi(\varsigma,v;(\phi(\eta)-\phi(0))^{\varsigma})$ for equation (1) based on Theorem 1.7. We begin by assuming the following hypotheses:
(K1) $\mathfrak{g}\in\mathbb{C}(\omega\times\mathbb{R},\mathbb{R})$. Moreover, there exists $\mathcal{L}_{\mathfrak{g}}>0$ such that

$$|\mathfrak{g}(\eta,\mathfrak{w}_1)-\mathfrak{g}(\eta,\mathfrak{w}_2)|\leqslant\mathcal{L}_{\mathfrak{g}}|\mathfrak{w}_1-\mathfrak{w}_2| \tag{6}$$

for all $\eta\in[0,\mathfrak{p}]$.
(K2) $\mathfrak{h},\mathfrak{k}:\omega^2\times\mathbb{R}\to\mathbb{R}$ are continuous functions which satisfies a Lipschitz condition in the third argument, i.e., there exist $\mathcal{L}_{\mathfrak{h}},\mathcal{L}_{\mathfrak{k}}>0$ such that

$$|\mathfrak{h}(\eta,s,\mathfrak{w})-\mathfrak{h}(\eta,s,\mathfrak{v})|\leqslant\mathcal{L}_{\mathfrak{h}}|\mathfrak{w}-\mathfrak{v}|, \tag{7}$$

$$|\mathfrak{k}(\eta,s,\mathfrak{w})-\mathfrak{k}(\eta,s,\mathfrak{v})|\leqslant\mathcal{L}_{\mathfrak{k}}|\mathfrak{w}-\mathfrak{v}| \tag{8}$$

for all $s,\eta\in\omega$ and $\mathfrak{w},\mathfrak{v}\in\mathbb{R}$.

Theorem 3.1. *Suppose that $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{k}$ satisfy $K1$ and $K2$. Additionally, let*

$$\|\mathcal{A}\| < \frac{\Gamma(\varsigma+1) - (2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}})\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}{\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}. \tag{9}$$

If a continuously differentiable function $w : \omega \to \mathbb{R}$ for $\epsilon \geqslant 0$ satisfies

$$\left| {}^{\mathcal{H}}\mathbb{D}_{0+}^{\varsigma,v;\phi} w(\eta) - \mathcal{A}(w(\eta)) - \mathfrak{g}(\eta, w(\eta)) - \int_0^{\eta} \mathfrak{h}(\eta, s, w(s))ds - \int_0^{\mathfrak{p}} \mathfrak{k}(\eta, s, w(s))ds \right| \leqslant$$
$$\leqslant \epsilon \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right)$$

for all $\eta \in \omega$, then there exists a unique continuous function $v_0 : \omega \to \mathbb{R}$ that satisfies equation (1) *and*

$$|w(\eta) - v_0(\eta)| \leqslant$$
$$\leqslant \frac{\Gamma(\varsigma+1)\epsilon}{\Gamma(\varsigma+1) - (2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathfrak{L}_{\mathfrak{h}} + \mathfrak{p}\mathfrak{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}} \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \tag{10}$$

for all $\eta \in \omega$.

Proof. Let $\mathfrak{Y} := \mathcal{C}_{1-\gamma,\phi}(0, \mathfrak{p}]$ be endowed with the following generalized metric, defined by

$$d^*(\mathfrak{w}, \mathfrak{v}) = \inf\left\{\mathcal{C} \geqslant 0 : |\mathfrak{w}(\eta) - \mathfrak{v}(\eta)| \leqslant \mathcal{C}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \quad \text{for all} \quad \eta \in \omega\right\} \tag{11}$$

for all $\mathfrak{w}, \mathfrak{v} \in \mathfrak{Y}$. It is not difficult to see that $(\mathfrak{Y}, d^*)$ is a complete generalized metric space [6]. Define the operator $\mathcal{S} : \mathfrak{Y} \to \mathfrak{Y}$ by

$$(\mathcal{S}w)(\eta) := \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)} + \mathfrak{w}_0 + I_{0+}^{\varsigma;\phi}(\mathcal{A}(w(\eta))) + I_{0+}^{\varsigma;\phi}\mathfrak{g}(\eta, w(\eta)) +$$
$$+ I_{0+}^{\varsigma;\phi}\left[\int_0^s \mathfrak{h}(\eta, \tau, w(\tau))d\tau\right] + I_{0+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}(\eta, \tau, w(\tau))d\tau\right]$$

for all $\eta \in \omega$ and $w \in \mathfrak{Y}$. For any $\mathfrak{w}, \mathfrak{v} \in \mathfrak{Y}$), choose a constant $\mathcal{K}$ so that $d^*(\mathfrak{w}, \mathfrak{v}) \leqslant \mathcal{K}$, i.e,

$$|\mathfrak{w}(t) - \mathfrak{v}(t)| \leqslant \mathcal{K}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right) \tag{12}$$

for all $\eta \in \omega$. So, using Remark 1, we have

$$|(\mathcal{S}\mathfrak{w})(\eta) - (\mathcal{S}\mathfrak{v})(\eta)| \leqslant$$
$$\leqslant \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} |\mathfrak{g}(\eta, \mathfrak{w}(\eta)) - \mathfrak{g}(t, \mathfrak{v}(\eta))| ds +$$
$$+ \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \left[\int_0^s |\mathfrak{h}(\eta, \tau, \mathfrak{w}(\tau)) - \mathfrak{h}(\eta, \tau, v(\tau))| d\tau\right] ds +$$
$$+ \frac{1}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \left[\int_0^{\mathfrak{p}} |\mathfrak{k}(\eta, \tau, \mathfrak{w}(\tau)) - \mathfrak{k}(\eta, \tau, \mathfrak{v}(\tau))| d\tau\right] ds +$$
$$+ \frac{\|\mathcal{A}\|}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} |\mathfrak{w}(\eta) - \mathfrak{v}(\eta)|$$
$$\leqslant \frac{(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\mathcal{K}\epsilon}{\Gamma(\varsigma)} \int_0^{\eta} \frac{\phi'(s)}{(\phi(\eta) - \phi(s))^{\varsigma}} \Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right)$$
$$\leqslant \frac{(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}{\Gamma(\zeta+1)} \mathcal{K}\epsilon\Phi\left(\varsigma, v; (\phi(\eta) - \phi(0))^{\varsigma}\right),$$

which indicates that

$$d^*((\mathcal{S}\mathfrak{w}), (\mathcal{S}\mathfrak{v})) \leqslant \frac{(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}{\Gamma(\varsigma+1)} d^*(\mathfrak{w}, \mathfrak{v})$$

for all $\mathfrak{w}, \mathfrak{v} \in (\mathfrak{Y}, d^*)$. From (8), we have $(2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma} < \Gamma(\varsigma+1)$. Hence, the operator $\mathcal{S}$ is a strict contraction. Moreover, for element $\mathfrak{v}_0 \in (\mathfrak{Y}, d^*)$, we have

$$\begin{aligned}
&|(\mathcal{S}\mathfrak{v}_0)(\eta) - \mathfrak{v}_0(\eta)| \leqslant \\
\leqslant &\left|\mathfrak{v}_0(\eta) - \frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{v}_0 - I_{0^+}^{S\phi\phi}\mathfrak{g}\left(\eta, \mathfrak{v}_0(\eta)\right) - I_{0^+}^{S/\phi}\left[\int_0^{\varsigma} \mathfrak{h}\left(\eta, \tau, v_0(\tau)\right) d\tau\right] - I_{0^+}^{S_0}\left(\mathcal{A}\left(\mathfrak{v}_0(\eta)\right)\right)\right| \\
\leqslant &\epsilon\Phi\left(\alpha, \beta; (\phi(\eta) - \phi(0))^{\alpha}\right)
\end{aligned}$$

for all $\eta \in \omega$. In summary, $d^*\left(\mathcal{S}\mathfrak{v}_0, \mathfrak{v}_0\right) \leqslant 1$ and $d^*\left(\mathcal{S}^{\mathrm{n}}\mathfrak{v}_0, \mathcal{S}^{\mathrm{n}+1}\mathfrak{v}_0\right) < +\infty$ for all $\mathrm{n} \in \mathbb{N}$. According to Theorem 1.7, there exists a unique continuous function $\mathfrak{w} : \omega \to \mathbb{R}$ such that $\mathcal{S}\mathfrak{w} = \mathfrak{w}, \mathfrak{w}$ satisfies Equation (1) for all $\eta \in \omega$ and

$$\begin{aligned}
\mathfrak{w}(\eta) = &\frac{(\phi(\eta) - \phi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{w}_0 + \\
+ &I_{0^+}^{\varsigma;\phi}\mathfrak{g}\left(\eta, \mathfrak{v}_0(\eta)\right) + I_{0^+}^{\varsigma;\phi}\left[\int_0^{s} \mathfrak{h}\left(\eta, \tau, \mathfrak{v}_0(\tau)\right) d\tau\right] + I_{0^+}^{\varsigma;\phi}\left[\int_0^{\mathfrak{p}} \mathfrak{k}\left(\eta, \tau, \mathfrak{v}_0(\tau)\right) d\tau\right] + I_{0^+}^{\varsigma;\phi}\left(\mathcal{A}\left(\mathfrak{v}_0(\eta)\right)\right)
\end{aligned}$$

for every $\eta \in \omega$. In addition, it follows from the above calculations that

$$d^*\left(\mathfrak{w}, \mathfrak{v}_0\right) \leqslant \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+1) - (2\mathcal{L}_{\mathfrak{g}} + \mathfrak{p}\mathcal{L}_{\mathfrak{h}} + \mathfrak{p}\mathcal{L}_{\mathfrak{k}} + \|\mathcal{A}\|)\,\Gamma(\gamma)(\phi(\mathfrak{p}) - \phi(0))^{\varsigma}}$$

which justifies inequality (10). □

4. Application

This section presents an example which illustrate the validity and applicability of our main results.

Example 1. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and $w(\eta)$ be a continuous function on $[0,1]$ so that $|K(\eta,\lambda)w(\eta)| < \dfrac{\Gamma(1/3)}{3\Gamma(2/9)}(e-1)^{-\frac{2}{9}}$. Consider the following fractional Volterra-Fredholm integro-differential system

$$\begin{cases}
{}^{\mathcal{H}}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{2}{3};e^{\eta}} w(\eta) = \displaystyle\int_0^1 K(\eta,\lambda)w(\eta)d\eta + \frac{1}{5}\sin(w(\eta)) + \int_0^{\eta} \sin\left(\frac{3}{5}\eta w(s)\right) ds + \\
\qquad\qquad + \displaystyle\int_0^1 \cos\left(\frac{2}{5}\eta w(s)\right) ds \\
4I_0^{\frac{2}{9};\phi} w(0) = \mathfrak{w}_0, \quad \mathfrak{w}_0 \in \mathbb{R}
\end{cases} \tag{13}$$

for all $\eta \in [0,1]$ and continuous real-valued functions w on $[0,1]$, we have $\left|\dfrac{1}{5}\sin(w(\eta))\right| \leqslant \dfrac{1}{5}|w(\eta)|$. Moreover,

$$\left|\int_0^t \sin\left(\frac{3}{5}\eta w(s)\right) ds\right| \leqslant \frac{3}{5}|(w(\eta))|, \quad \left|\int_0^1 \cos\left(\frac{2}{5}\eta w(s)\right) ds\right| \leqslant \frac{2}{5}|(w(\eta))|$$

for all $\eta \in [0,1]$. Furthermore, by assumption, the operator $3\int_0^1 K(\eta,\lambda)w(\eta)d\eta$ is bounded and we have $\left|\int_0^1 K(\eta,\lambda)w(\eta)d\eta\right| \leqslant \dfrac{\Gamma(4/3)}{\Gamma(2/9)(e-1)^{2/9}}$, for all $\eta,\lambda \in [0,1]$ and continuous functions w. So (H1)–(H3) are satisfied for $\mathfrak{q}_1 = 1/5$ and $\mathfrak{q}_2^h = 3/5$, $\mathfrak{q}_2^k = 2/5$. Therefore, Theorem 2.2 proved that equation (13) has at least one solution.

Conclusions

In this paper, we considered a class of fractional Volterra–Fredholm integro-differential equations including a closed linear operator. Next, we used the Krasnoselskii fixed-point theorem to investigate the existing result under some mild conditions. Moreover, we introduced and then proved the Kummer stability of ϕ-Hilfer fractional Volterra–Fredholm integro-differential equations on the compact domains.

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Теоретический анализ системы нелинейных ϕ-хильферовских дробных интегро-дифференциальных уравнений Вольтерра-Фредгольма

Ахмед А. Хамуд
Недал М. Мохаммед
Кафедра математики & Информатика
Университет Таиз
Таиз-96704, Йемен
Расул Сах
Кафедра математики
Университет Абдул Вали Хана
Мардан-23200, Пакистан

Аннотация. Существование решений системы нелинейных ϕ-хильферовских дробных интегро-дифференциальных уравнений Вольтерра–Фредгольма с дробно-интегральными условиями исследуется с помощью теорем Красносельского о неподвижной точке и теорем Арцела–Асколи. Более того, применяя альтернативную теорему о неподвижной точке Диаса и Марголиса, мы доказываем куммеровскую устойчивость системы на компактных областях. Также представлен пример, иллюстрирующий наши результаты.

Ключевые слова: ϕ-хильферовское дробное интегро-дифференциальное уравнение Вольтера-Фредгольма, устойчивость Куммера, теорема Арцела–Асколи, теорема Красносельского о фиксированной точке.