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On the Blaschke Factors in Polydisk

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Abstract. The purpose of this work is to construct a multidimensional analogue of the Blaschke factors. The relevance of the construction of this analogue was prompted by a recent joint article by Alpay and Yger devoted to the multidimensional interpolation theory for functional spaces in special Weyl polyhedra. By such a factor we understand a set of special inner rational functions in a unit polydisk. We construct inner rational functions for the case of three complex variables, in particular, using the Lee-Yang polynomial from the theory of phase transitions in statistical mechanics.

Keywords: Blaschke product, Lee-Yang polynomial.

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Introduction

In 1915 Wilhelm Blaschke introduced a very important class of functions of one complex variable, which allowed solving important problems of interpolation theory in a unit disk.

Definition 0.1 ([1])**.** *The one-dimensional Blaschke product is a function of the form:*

$$B(z) = \prod_{k\geqslant 1} \frac{z - z_k}{1 - \bar{z}_k z}, \tag{1}$$

where $\{z_1, z_2, \dots, z_n, \dots\}$ *is a sequence of points in the unit disk* $D \subset \mathbb{C}$.

In the case of a finite number of points $\{z_k\}$ from the disk, no restrictions are imposed on them, however, when moving to a countable set of points, the so-called Blaschke condition is added for the correctness of the definition:

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty. \tag{2}$$

This concept made it possible to solve important problems of interpolation theory in a single disk. For example, Blaschke's theorem states that a sequence $\{z_k\}$ in a disk is a zero set for a holomorphic function bounded in D if and only if the series in (2) converges.

Except for the case of bounded functions, similar descriptions have been obtained for functions from Hardy classes.

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Before proceeding to an analogue of the Blaschke factors in the multidimensional case, let us take a closer look at the Blaschke factors in the one-dimensional case. Note that each factor $b_k = \dfrac{z_k - z}{1 - \bar{z}_k z}$ of the product (1) is a fractional rational function of the form:

$$b_k = \frac{p(z)}{q(z)} = z\frac{\overline{q(1/\bar{z})}}{q(z)}.$$

In the case when q has real coefficients, the functions b_k can be represented as:

$$b_k = z\frac{q(1/z)}{q(z)}.$$

The idea of our generalization of the Blaschke factors is to construct such 'elementary' factors for several complex variables. We would like to note that such a construction was carried out under the influence of the results of Alpay and Yger [2].

1. Multidimensional analogue of the Blaschke factor in the space $\mathbb{C}^3$

By the analogue of the Blaschke factor in $\mathbb{C}^3$ we shall understand the triple of special inner rational functions in the unit polydisk of $\mathbb{C}^3$. We will construct inner rational functions using the Lee-Yang polynomials (see [3]). In order to do this, we fix an arbitrary symmetric $n \times n$ matrix (a_{jk}) with real coefficients satisfying the condition $0 < |a_{jk}| < 1$. The corresponding Lee-Yang polynomial is constructed according to the given matrix as follows:

$$f(z_1, z_2, \ldots, z_n) = \sum_J \prod_{j \in J} \left(z_j \prod_{k \notin J} a_{jk} \right),$$

where J runs over the set of all subsets of $\{1, 2, \ldots, n\}$.

Let us present some important properties of this polynomial. Recall that the amoeba A_f of the polynomial f is defined as the image $\operatorname{Log} V$ of the zero set $V = \{z \in (\mathbb{C}\backslash 0)^n : f(z) = 0\}$ under the map $\operatorname{Log} : (z_1, \ldots, z_n) \to (\ln|z_1|, \ldots, \ln|z_n|)$ (see [4, 5, 6]). Taking into account the following expression:

$$f(z_1, z_2, \ldots, z_n) = z_1 z_2 \ldots z_n f(1/z_1, 1/z_2, \ldots, 1/z_n)$$

the amoeba of the polynomial f is symmetric with respect to the origin. Moreover, the following theorem is valid.

Theorem 1.1 (M. Passare, A. Tsikh [7]). *Let A be the amoeba of the Lee-Yang polynomial, then the closed positive and negative orthants $\pm\mathbb{R}^n_+$ intersect the amoeba A only at the origin:*

$$\mathbb{R}^n_+ \cap A = -\mathbb{R}^n_+ \cap A = \{0\}$$

Consider the Lee–Yang polynomial in three variables associated with the matrix

$$(a_{jk}) = \begin{pmatrix} a_{11} & a & b \\ a & a_{22} & c \\ b & c & a_{33} \end{pmatrix},$$

where $\{a_{11}, a_{22}, a_{33}, a, b, c\} \in (-1, 1)\backslash\{0\}$. The corresponding Lee–Yang polynomial is

$$f = (z_1 z_2 z_3 + bc z_1 z_2 + ab z_2 z_3 + ac z_1 z_3) + (ab z_1 + ac z_2 + bc z_3 + 1).$$

We introduce the following

Notation. *Split the polynomial into two parts and denote*

- $f_1 = z_1z_2z_3 + bcz_1z_2 + abz_2z_3 + acz_1z_3$,
- $f_2 = abz_1 + acz_2 + bcz_3 + 1$.

Next, we fix an arbitrary (z_1^0, z_2^0, z_3^0) from the distinguished boundary

$$\Delta = \{|z_j| = 1, j = 1, 2, 3\}$$

of the polydisk $D^3 \subset \mathbb{C}^3$ and consider the following sequence of functions:

$$p_1 = f_1(z_1^0, z_2, z_3), \quad p_2 = f_1(z_1, z_2^0, z_3), \quad p_3 = f_1(z_1, z_2, z_3^0),$$
$$q_1 = f_2(z_1^0, z_2, z_3), \quad q_2 = f_2(z_1, z_2^0, z_3), \quad q_3 = f_2(z_1, z_2, z_3^0).$$

Definition 1.1. *We call the map* $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ *a three-dimensional analogue of the Blaschke factor if the zeros of the polynomial* $f_2 = abz_1 + acz_2 + bcz_3 + 1$ *do not intersect the open unit polydisk* D^3 *(in this case,* q_i *will be called valid).*

Recall an important definition.

Definition 1.2 (see [8]). *A function* $g \in H^\infty(D^n)$ *is called inner if its radial boundary values* $g^*(s)$ *satisfy the condition* $|g^*(w)| = 1$ *almost everywhere on* T^n*.*

In fact, the Blaschke product is an inner function. Our theorem below shows that the definition we introduced corresponds to this property.

Theorem 1.2. *The functions* p_j/q_j *in Definition 1.1 of the Blaschke factors are inner functions in the polydisk* D^3*.*

Proof. Since $q_1 = f_2(z_1^0, z_2, z_3)$, the zeros of the denominator q_1 on the unit distinguished boundary are also zeros of the polynomial f_2. It can be noted that for $|ab|+|ac|+|bc| < 1$, the polynomial f_2 has no zeros in the closure of the unit polydisk $\bar{D}^3$, but then the denominators q_j have no zeros in the same closure. For $|ab| + |ac| + |bc| = 1$, the polynomial f_2 has a single zero on the distinguished boundary $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ and has no zeros inside the polydisk. In this case, the denominator q_i has a single zero on the distinguished boundary if $\hat{z}_i = z_i^0$; otherwise, q_i does not vanish in the closure of a single polydisk. If the inequality $|ab| + |ac| + |bc| > 1$ is satisfied, then the polynomial f_2 has zeros inside the polydisk, so the corresponding denominators q_j are not valid. Thus, the permissible denominators vanish at no more than one point from the distinguished boundary. Therefore, almost everywhere on T^3 we have:

$$p_1 = f_1(z_1^0, z_2, z_3) = z_1^0z_2z_3 + bcz_1^0z_2 + abz_2z_3 + acz_1^0z_3 =$$
$$= z_1^0z_2z_3\left(\frac{bc}{z_3} + \frac{ab}{z_1^0} + \frac{ac}{z_2} + 1\right) = z_1^0z_2z_3f_2\left(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3}\right),$$

$$p_2 = f_1(z_1, z_2^0, z_3) = z_1z_2^0z_3 + bcz_1z_2^0 + abz_2^0z_3 + acz_1z_3 =$$
$$= z_1z_2^0z_3\left(\frac{bc}{z_3} + \frac{ab}{z_1} + \frac{ac}{z_2^0} + 1\right) = z_1z_2^0z_3f_2\left(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3}\right),$$

$$p_3 = f_1(z_1, z_2, z_3^0) = z_1z_2z_3^0 + bcz_1z_2 + abz_2z_3^0 + acz_1z_3^0 =$$
$$= z_1z_2z_3^0\left(\frac{bc}{z_3^0} + \frac{ab}{z_1} + \frac{ac}{z_2} + 1\right) = z_1z_2z_3^0f_2\left(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0}\right).$$

From these equalities we obtain the following chains of equalities for modules of functions $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$, which are valid almost everywhere on the distinguished boundary:

$$\left|\frac{p_1}{q_1}\right| = \left|\frac{z_1^0 z_2 z_3 f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = |z_1^0 z_2 z_3| \left|\frac{f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1^0}, \frac{1}{z_2}, \frac{1}{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = \left|\frac{f_2(\bar{z_1^0}, \bar{z_2}, \bar{z_3})}{f_2(z_1^0, z_2, z_3)}\right| = \left|\frac{\bar{f_2}(z_1^0, z_2, z_3)}{f_2(z_1^0, z_2, z_3)}\right| = 1,$$

$$\left|\frac{p_2}{q_2}\right| = \left|\frac{z_1 z_2^0 z_3 f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = |z_1 z_2^0 z_3| \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2^0}, \frac{1}{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = \left|\frac{f_2(\bar{z_1}, \bar{z_2^0}, \bar{z_3})}{f_2(z_1, z_2^0, z_3)}\right| = \left|\frac{\bar{f_2}(z_1, z_2^0, z_3)}{f_2(z_1, z_2^0, z_3)}\right| = 1,$$

$$\left|\frac{p_3}{q_3}\right| = \left|\frac{z_1 z_2 z_3^0 f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = |z_1 z_2 z_3^0| \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| =$$
$$= \left|\frac{f_2(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = \left|\frac{f_2(\bar{z_1}, \bar{z_2}, \bar{z_3^0})}{f_2(z_1, z_2, z_3^0)}\right| = \left|\frac{\bar{f_2}(z_1, z_2, z_3^0)}{f_2(z_1, z_2, z_3^0)}\right| = 1.$$

That is, moduli of the radial boundary values are almost everywhere on the distinguished boundary equal to 1, so functions
$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$$
are inner by definition. □

To describe the valid denominators of q_i, we need the following

Definition 1.3 (see [9], Sec. 14, p. 125)**.** *Let C be a nonempty convex set. Then the closed convex set*
$$\hat{C} = \{x | \forall\, x^* \in C, \langle x, x^* \rangle \leqslant 1\},$$
is called the polar of the set C.

If the set C itself is closed and contains the origin, then it coincides with the polar of its polar set
$$\hat{\hat{C}} = C.$$
For more information about convex sets and other properties of the polar, see [9].

Let us find the polar of the cube $K = [-1, 1]^3$. For this we need to find all points that satisfy the equation
$$xx^* + yy^* + zz^* \leqslant 1 \quad \forall\, x^* \in [-1, 1],\ y^* \in [-1, 1],\ z^* \in [-1, 1].$$
Since we compare everything with one in this equation, there is no point in checking the fulfillment of this inequality for intermediate values. Because if the inequality holds for boundary values, then it holds automatically for values inside the segment. Therefore, we have the system:
$$xx^* + yy^* + zz^* \leqslant 1 \quad \forall\, x^*, y^*, z^* \in \{-1, 1\},$$
which can be written as a single inequality $|x| + |y| + |z| \leqslant 1$ in the conjugate space.

We need the constructed polar to describe the valid denominators, namely

Theorem 1.3. *The denominators q_i are valid if and only if the pairwise products $(ab, ac, bc) = (x, y, z)$ lie in the polar and satisfy the system of inequalities:*

$$\begin{cases} 1 > \dfrac{yz}{x} > 0 \\ 1 > \dfrac{xz}{y} > 0 \\ 1 > \dfrac{xy}{z} > 0 \end{cases}.$$

Proof. If the denominators q_j are valid, then the polynomial $f_2 = abz_1 + acz_2 + bcz_3 + 1$ has no zeros inside the unit polydisk. And this is possible, as we have shown above, if and only if $|ab| + |ac| + |bc| \leqslant 1$, that is, when the pairwise products of (ab, ac, bc) lie in the polar. The system of inequalities arises from the following reasoning:

$$\begin{cases} a \in (-1,1)\backslash\{0\} \\ b \in (-1,1)\backslash\{0\} \\ c \in (-1,1)\backslash\{0\} \end{cases} \sim \begin{cases} 0 < a^2 < 1 \\ 0 < b^2 < 1 \\ 0 < c^2 < 1 \end{cases} \sim \begin{cases} 0 < \frac{bc \cdot a^2}{bc} < 1 \\ 0 < \frac{ac \cdot b^2}{ac} < 1 \\ 0 < \frac{ab \cdot c^2}{ab} < 1 \end{cases} \sim$$

$$\sim \begin{cases} 0 < \frac{ab \cdot ac}{bc} < 1 \\ 0 < \frac{ab \cdot bc}{ac} < 1 \\ 0 < \frac{ac \cdot bc}{ab} < 1 \end{cases} \sim \begin{cases} 0 < \frac{xy}{z} < 1 \\ 0 < \frac{xz}{y} < 1 \\ 0 < \frac{yz}{x} < 1 \end{cases}.$$

□

Let us visualize the resulting set. In Fig. 1, 2, we can see the points from the boundary of the polar satisfying the resulting system of inequalities. In the first figure, the edges of the polar set of the cube (of the regular octahedron) are highlighted in red, and the intersection is green.

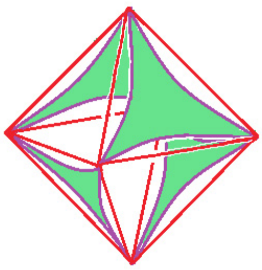

Fig. 1. Visual representation of the intersection of solutions of the system of inequalities and the boundary of the polar

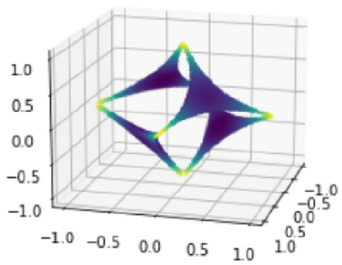

Fig. 2. Computer representation of the intersection of solutions of the system of inequalities and the boundary of the polar

2. The second approach to constructing an analogue of the Blaschke factors

In this section, when constructing a generalization of the Blaschke factors, we start from the form of this factor in the one-dimensional case. As we noted earlier, each factor $\dfrac{z_k - z}{1-\bar{z}_k z}$ of the product (1) is a rational function of the form:

$$b_k = \frac{p(z)}{q(z)} = z\frac{\overline{q(1/\bar{z})}}{q(z)},$$

where $q(z) = 1-\bar{z}_k z$ is a polynomial of the first degree that has no zeros inside the unit disk D. In this case, when switching to the multidimensional case, we can take as such a polynomial the following one

$$q(z_1,\ldots,z_n) = 1+\zeta_1 z_1+\ldots+\zeta_n z_n,$$

where $(\zeta_1,\ldots,\zeta_n) \in \{(z_1,\ldots,z_n) \in D^n : |z_1|+\ldots+|z_n| \leqslant 1\}$. With such restrictions on the coefficients of a given linear polynomial, it will not have zeros inside the unit polydisk D^n. Note that these constraints are consistent with the one-dimensional case. As a numerator $p(z_1,\ldots,z_n)$ we can take the following polynomial, which also agrees with the case of one complex variable:

$$p(z_1,\ldots,z_n) = z_1\cdot\ldots\cdot z_n\cdot\overline{q(1/\bar{z}_1,\ldots,1/\bar{z}_n)}.$$

Now fix an arbitrary point $(z_1^0,\ldots,z_n^0)$ from the distinguished boundary

$$\Delta = \{|z_j| = 1, j = 1,\ldots,n\}$$

of the polydisk D^n and consider the following set of functions:

$$\begin{aligned} p_1 &= p(z_1^0,\ldots,z_n), \quad \ldots, \quad p_n = p(z_1,\ldots,z_n^0),\\ q_1 &= q(z_1^0,\ldots,z_n), \quad \ldots, \quad q_n = q(z_1,\ldots,z_n^0). \end{aligned}$$

Definition 2.1. *The map* $\left(\dfrac{p_1}{q_1},\ldots,\dfrac{p_n}{q_n}\right)$ *is called the multidimensional analogue of the Blaschke factor.*

Theorem 2.1. *The functions p_j/q_j in the definition of the analogue of the Blaschke factor are inner rational functions in the polydisk D^n.*

Proof. Since $q_1 = q(z_1^0,\ldots,z_n)$, the zeros of the denominator q_1 on the unit distinguished boundary are also zeros of the polynomial q. It can be noted that for $|\zeta_1|+\ldots+|\zeta_n| < 1$ the polynomial q has no zeros in the closure of the unit polydisk $\bar{D^n}$, but then the denominators q_j have no zeros in the same closure. For $|\zeta_1|+\ldots+|\zeta_n| = 1$, the polynomial q has a single zero on the distinguished boundary $(\hat{z}_1,\hat{z}_2,\hat{z}_3)$ and has no zeros inside the polydisk. In this case, the denominator q_i has a single zero on the distinguished boundary if $\hat{z}_i = z_i^0$; otherwise, q_i does not vanish in the closure of a single polydisk. Thus, q_i vanish at no more than one point from the distinguished boundary. Therefore, almost everywhere on T^n we have

$$p_1 = p(z_1^0,z_2,\ldots,z_n) = z_1^0 z_2\cdot\ldots\cdot z_n + \bar{\zeta}_1 z_2\ldots z_n+\ldots+\bar{\zeta}_n z_1^0 z_2\ldots z_{n-1} =$$
$$= z_1^0\cdot\ldots\cdot z_n\left(\frac{\bar{\zeta}_1}{z_1^0}+\ldots+\frac{\bar{\zeta}_n}{z_n}+1\right) = z_1^0\cdot\ldots\cdot z_n\cdot\bar{q}\left(\frac{1}{\bar{z}_1^0},\ldots,\frac{1}{\bar{z}_n}\right).$$

From this equality we obtain the following chain of equalities for the module of the function $\dfrac{p_1}{q_1}$, which holds almost everywhere on the distinguished boundary:

$$\left|\frac{p_1}{q_1}\right| = \left|\frac{z_1^0 \cdot \ldots \cdot z_n \cdot \bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| = |z_1^0 \cdot \ldots \cdot z_n| \left|\frac{\bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| =$$

$$= \left|\frac{\bar{q}(\frac{1}{\bar{z}_1^0}, \ldots, \frac{1}{\bar{z}_n})}{q(z_1^0, \ldots, z_n)}\right| = \left|\frac{\bar{q}(z_1^0, \ldots, z_n)}{q(z_1^0, \ldots, z_n)}\right| = 1.$$

That is, the module of the radial boundary values are almost everywhere on the distinguished boundary equal to 1, so the function $\dfrac{p_1}{q_1}$ is inner by definition. Having carried out similar reasoning for the remaining rational functions

$$\frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n},$$

we obtain the statement of the theorem. □

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О множителях Бляшке в полидиске

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Аннотация. Цель настоящей работы состоит в построении многомерного аналога множителя Бляшке. На актуальность построения данного аналога нас натолкнула недавняя совместная статья Алпая и Ижера, которая посвящена многомерной интерполяционной теории для функциональных пространств в специальных полиэдрах Вейля. Под таким множителем мы будем понимать набор специальных внутренних рациональных функций в единичном поликруге. Построение внутренних рациональных функций для случая трех комплексных переменных произведем, в частности, с помощью многочлена Ли-Янга из теории фазовых переходов статистической механики.

Ключевые слова: произведение Бляшке, многочлен Ли-Янга.