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On the Integration of the Periodic Camassa-Holm Equation with a Self-Consistent Source

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Abstract. Recently, much attention has been paid to non-linear equations with a self-consistent source that have soliton solutions. Sources arise in solitary waves with a variable speed and lead to a variety of physical models. Such models are usually used to describe interactions between solitary waves. The Cauchy problem for the Camassa—Holm equation with a source in the class of periodic functions is considered in this paper. The main result of this work is a theorem on the evolution of the spectral data of the weighted Sturm—Liouville operator where potential of the operator is a solution of the periodic Camassa–Holm equation with a source. The obtained relations allow one to apply the method of the inverse spectral transform to solve the Cauchy problem for the periodic Camassa-Holm equation with a source.

Keywords: Camassa-Holm equation, self-consistent source, trace formulas, inverse spectral problem, weighted Sturm–Liouville operator.

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Introduction

The proof of complete integrability of non-linear equation

$$u_t - u_{xxt} + 2\omega\, u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1}$$

was presented [1]. The equation describes the unidirectional propagation of waves at the free surface of shallow water. Various generalizations and applications of the Camassa–Holm equation

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(CHe) were considered [2–18]. For instance, the physical significance of (1) in the context of water waves was discussed in [2–4], the existence of peaked solitons and geometry of multi-peakons was considered in [1, 5], geometric formulations and waves breaking was considered in [6, 7].

Non-linear equations with self-consistent sources that admit soliton solutions have received much attention in the recent literature. They are important integrable models in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, etc. [19–24]. Sources arise in solitary waves with a variable speed and lead to a variety of physical models. Such models are usually used to describe interactions between solitary waves. For example, the KdV equation with a self-consistent source was considered [22]. This equation describes the interaction of long and short capillary-gravity waves [23]. Another important equation with a self-consistent source is the non-linear Schrödinger equation that describes the non-linear interaction of an ion acoustic wave in the two component homogeneous plasma with the electrostatic high frequency wave [24].The N-soliton, N-cuspon, N-positon and N-negaton solutions of CHe with a self-consistent source were obtained in the class of "rapidly decreasing" functions with the use of the inverse reciprocal transformation [25]. For related issues see also [26–39] and the references therein.

We consider the periodic problem for CH equation with a self-consistent source. We will obtain a representation for the solution of problem (2)–(4) in the framework of the inverse spectral problem for Eq. (4). Namely, we find an analogue of the Dubrovin system of equations for the spectral parameters of the weighted Shturm-Liouville operator. Then the solution of the Cauchy problem for the periodic Camassa-Holm equation with a self-consistent source is obtained in the form of a uniformly converging functional series. The solvability of the Dubrovin system of equations in the case of the periodic Camassa-Holm equation without a source was studied [40–42].

The paper is organized as follows. The formulation of the problem is given in Section 1. In Section 2, some basic information on the direct and inverse spectral problems for the weighted Sturm-Liouville operator with periodic coefficient is presented. Section 3 is devoted to the evolution of the spectral data that correspond to the problem in question. ***

1. Problem statement

We consider the Camassa-Holm equation with a self-consistent source

$$\begin{gathered} u_t - u_{xxt} = uu_{xxx} + 2u_x u_{xx} - 3uu_x + \\ + \sum_{k=0}^{\infty} \alpha_k(t) s\left(\pi,\ \lambda_k, t\right) \left[q_x\left(x,t\right) \psi^2\left(x,\lambda_k,t\right) + 2q(x,t)\left(\psi^2\left(x,\lambda_k,t\right)\right)'\right] \end{gathered} \tag{2}$$

in the class of real-valued π-periodic with respect to the spatial variable x function $u = u\left(x,t\right)$. It satisfies the regularity of assumption

$$u \in C_x^3\left(t > 0\right) \cap C_t^1\left(t > 0\right) \cap C\ \ \left(t \geqslant 0\right)$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in R, \tag{3}$$

where $u_0\left(x\right) \in C^3(R)$ is the given real-valued π -periodic function. In equation (2) $q(x,t) = u(x,t) - u_{xx}\left(x,t\right)$, and $\psi\left(x,\lambda_k,t\right)$ is the Floquet solution (normalized by the condition $\psi\left(0,\lambda_k,t\right) = 1$) of the weighted Sturm–Liouville equation

$$y'' = \frac{1}{4}y + \lambda q(x,t)y, \quad x \in R. \tag{4}$$

Here λ_k is a zero of the function $\Delta^2(\lambda) - 4$, where $\Delta(\lambda) = c(\pi, \lambda, t) + s'(\pi, \lambda, t)$. The solutions of equation (4) are denoted by $c(x, \lambda, t)$ and $s(x, \lambda, t)$. They satisfy the initial conditions $c(0, \lambda, t) = 1$, $c'(0, \lambda, t) = 0$ and $s(0, \lambda, t) = 0$, $s'(0, \lambda, t) = 1$, respectively. In system (2) functions $\alpha_k(t)$, $k \in Z$ can be prescribed arbitrary within the class of real-valued continuous functions that have uniform asymptotic decay $\alpha_k = O\left(\frac{1}{k^2}\right)$, $k \to +\infty$. Using the expression for the Floquet solutions, one can derive the identity

$$s(\pi, \lambda_k, t)\psi^2(\tau, \lambda_k, t) = s(\pi, \lambda_k, t, \tau), \tag{5}$$

where $s(x, \lambda, t, \tau)$ is the solution of the Eq. (4) with coefficient $q(x + \tau, t)$ that satisfies the initial conditions $s(0, \lambda, t, \tau) = 0$, $s'(0, \lambda, t, \tau) = 1$. Equality (5), the uniform decay condition $\alpha_k(t) = O\left(\frac{1}{k^2}\right)$ and asymptotic formulas ([42])

$$s(\pi, \lambda, t, \tau) = O(\lambda^{-\frac{1}{2}}), \quad \frac{\partial s(\pi, \lambda, t, \tau)}{\partial \tau} = O(\lambda^{-\frac{1}{2}})$$

provide uniform convergence of the series in equation (1). In (2) and elsewhere the prime "$\prime$" means the derivative with respect to the variable x.

The aim of this work is to provide a procedure for constructing the solution $u(x,t)$, $\psi(x, \lambda_k, t)$ of problem (2)–(4) using the inverse spectral theory for the weighted Sturm–Liouville equation (4)

2. Preliminaries

For the sake of completeness some facts from the inverse spectral theory of the weighted Sturm–Liouville equation (4) is summarized in this section (see [1, 2, 41–43]).

The spectrum of the weighted Sturm–Liouville operator (4) with $q(x, 0) < 0$, $x \in R$ is absolutely continuous and coincides with the set

$$E = \{\lambda \in R \colon -2 \leqslant \triangle(\lambda) \leqslant 2\} = [\lambda_0, \ \lambda_1] \cup [\lambda_2, \ \lambda_3] \cup \dots \cup [\lambda_{2n}, \ \lambda_{2n+1}] \dots .$$

The intervals $(-\infty, \lambda_0)$, $(\lambda_{2n-1}, \lambda_{2n})$, $n \geqslant 1$ are called the gaps or lacunae.

The numbers ξ_n, $n \geqslant 1$ with the signs $\sigma_n = sign\{s'(\pi, \xi_n) - c(\pi, \xi_n)\}$, $n \geqslant 1$ are called the spectral parameters of the weighted Sturm–Liouville equation (4) with $q(x, 0) < 0$, $x \in R$. Let us notice that ξ_n coincides with the eigenvalues of the Dirichlet problem for equation (4). Moreover, the inclusions $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$ and the equality

$$s(\pi, \lambda) = 2sh\frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\xi_n}\right) \tag{6}$$

are fulfilled.

The boundaries λ_n of the spectrum and the spectral parameters ξ_n, σ_n are called the spectral data of the weighted Sturm–Liouville equation (4). The determination of spectral data of (4) is called the direct spectral problem and conversely, the restoration of coefficient q of (4) by spectral data is called the inverse spectral problem.

The spectrum of the weighted Sturm–Liouville operator (4) with coefficient $q(x + \tau) < 0$ does not depend on the real parameter τ but the spectral parameters do. The spectral parameters satisfy the system of Dubrovin differential equations

$$\frac{d\xi_n(\tau)}{d\tau}=\frac{\xi_n(\tau)}{sh\left(\frac{\pi}{2}\right)}\ \frac{\sqrt{\Delta^2\left(\xi_n(\tau)\right)-4}}{\prod_{m\neq n}\left(1-\frac{\xi_n(\tau)}{\xi_m(\tau)}\right)},\ \xi_n\left(0\right)=\xi_n, \tag{7}$$

where the radical $\sqrt{\Delta^2\left(\xi_n\right)-4}$ is given by

$$\sqrt{\Delta^2\left(\xi_n\right)-4}=-\triangle\left(\xi_n\right)-\frac{1}{\frac{\partial s(\pi,\xi_n)}{\partial\lambda}}\int_0^\pi q\left(x\right)s^2\left(x,\xi_n\right)dx,\quad n\geqslant 1.$$

System of Dubrovin equations (7) and the following trace formula

$$q\left(\tau\right)=\frac{1}{2}\sum_{n\geqslant 1}\left[\frac{1}{\xi_n\left(\tau\right)}-\left(\frac{1}{\xi_n\left(\tau\right)}\right)''\right]-\frac{1}{4}\sum_{k\geqslant 0}\frac{1}{\lambda_k} \tag{8}$$

provide the method for solving the inverse problem.

3. Main result

The main result of the paper is stated in the following theorem.

Theorem 3.1. *Let us assume that $u(x,t)$ and $\psi(x,\lambda_k,t)$ are solution of problem (2)–(4). Then the spectrum of problem (4) does not depend on t, and spectral parameters $\xi_n=\xi_n\left(t\right),\ \sigma_n=\sigma_n\left(t\right),\ n\geqslant 1$ satisfy the analogue of the system of Dubrovin equations*

$$\dot{\xi}_n=\left\{\frac{1}{2\xi_n}-\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\xi_j}+\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}+\sum_{k=0}^{\infty}\frac{\xi_n\alpha_k(t)s\left(\pi,\ \lambda_k,\mathrm{t}\right)}{\xi_n-\lambda_k}\right\}h_n\left(\xi\right), \tag{9}$$

where

$$h_n\left(\xi\right)=-\frac{\sigma_n\xi_n\sqrt{\left(1-\frac{\xi_n}{\lambda_0}\right)\ \prod_{i=1}^{\infty}\left(1-\frac{\xi_n}{\lambda_{2i-1}}\right)\left(1-\frac{\xi_n}{\lambda_{2i}}\right)}}{\prod\limits_{j\neq n,\,j=1}^{\infty}\left(1-\frac{\xi_n}{\xi_j}\right)}.$$

The sign $\sigma_n(t)=\pm 1$ changes at each collision of the point $\xi_n(t)$ with the boundaries of its gap $[\lambda_{2n-1},\lambda_{2n}]$. Moreover, the following initial conditions are fulfilled

$$\xi_n(t)|_{t=0}=\xi_n^0,\qquad \sigma_n(t)|_{t=0}=\sigma_n^0,\ n\geqslant 1, \tag{10}$$

where $\xi_n^0,\ \sigma_n^0,\ n\geqslant 1$ are spectral parameters of the weighted Sturm–Liouville equations (4) that correspond to the coefficient $q_0\left(x\right)=u\left(x,0\right)-u_{xx}\left(x,0\right)<0$.

Proof. Let us rewrite (2) as follows

$$q_t=uu_{xxx}+2u_xu_{xx}-3uu_x+G\left(x,t\right), \tag{11}$$

$$y_n\left(x,t\right)=\frac{1}{\alpha_n\left(t\right)}s\left(x,\xi_n,t\right), \tag{12}$$

$$\alpha_n\left(t\right)=\sqrt{\int_0^\pi s^2\left(x,\xi_n,t\right)q\left(x,t\right)dx}.$$

Differentiating the identity

$$y_n'' - \frac{1}{4}y_n = \xi_n q y_n \tag{13}$$

with respect to t, we obtain

$$\dot{y}_n'' - \frac{1}{4}\dot{y}_n = \dot{\xi}_n q y_n + \xi_n \dot{q}_t y_n + \xi_n q \dot{y}_n. \tag{14}$$

Multiplying identity (14) by y_n and integrating it over x from 0 to π, we obtain

$$\dot{\xi}_n \int_0^\pi q y_n^2 dx = \int_0^\pi \left(\dot{y}_n' y_n - y_n{}' \dot{y}_n\right)' dx - \int_0^\pi \xi_n q_t y_n^2 dx.$$

Hence, taking into account Dirichlet boundary condition and normalization

$$\int_0^\pi q\,(x,t) y_n^2 dx = 1,$$

we have

$$\dot{\xi}_n = -\int_0^\pi \xi_n q_t y_n^2 dx. \tag{15}$$

Substituting Eq. (11) into (15), we obtain

$$\dot{\xi}_n = -\xi_n \int_0^\pi \left(u u_{xxx} + 2u_x u_{xx} - 3u u_x\right) y_n^2 dx - \int_0^\pi \xi_n G\,(x,t)\, y_n^2 dx. \tag{16}$$

We seek the antiderivative of the first integrand in (16) as a quadratic form with respect to y_n and y_n', that is,

$$\left(a y_n^2 + b y_n y_n' + c y_n'^2\right)' = -\xi_n \left(u u_{xxx} + 2u_x u_{xx} - 3u u_x\right) y_n^2 \tag{17}$$

where $a = a\,(x,t,\xi_n)$, $b= b\,(x,t,\xi_n)$ and $c = c\,(x,t,\xi_n)$ are independent of y_n and y_n'. Using equality (13), we obtain from (17) that

$$\begin{gathered} y_n^2(a' + \frac{1}{4}b + \xi_n bq) + y_n y_n'(2a + b' + \frac{1}{2}c + 2\xi_n cq) + (y_n')^2(b + c') = \\ = -\xi_n \left(u u_{xxx} + 2u_x u_{xx} - 3u u_x\right) y_n^2. \end{gathered} \tag{18}$$

Comparing the left and right hand sides of (18) we find that $b = -c'$, $a = \frac{1}{2}c'' - \frac{1}{4}c - c\xi_n q$ and

$$\frac{1}{2}c''' - \frac{1}{2}c' - 2c'\xi_n q - c\xi_n\,(t)\, q' = -\xi_n \left(u u_{xxx} + 2u_x u_{xx} - 3u u_x\right). \tag{19}$$

It is easy to check that function $c = \frac{1}{2\xi_n} - u\,(x,t)$ satisfies equality (19). Substituting obtained values of $a\,(x,t,\xi_n)$, $b\,(x,t,\xi_n)$ and $c\,(x,t,\xi_n)$ into (17) we arrive at

$$\begin{aligned} -\int_0^\pi \xi_n \left(u u_{xxx} + 2u_x u_{xx} - 3u u_x\right) y_n^2 dx &= \int_0^\pi \left(a y_n^2 + b y_n y_n' + c y_n'^2\right)' dx = \\ &= \left(a y_n^2 + b y_n y_n'\right)\Big|_0^\pi + \left(\frac{1}{2\xi_n} - u\,(x,t)\right) y_n'^2\Big|_0^\pi = \\ &= \left(\frac{1}{2\xi_n} - u\,(0,t)\right)\left[y_n'^2\,(\pi,t) - y_n'^2\,(0,t)\right] \end{aligned} \tag{20}$$

Let us find the antiderivative of the second integrand in (16)

$$\int_0^\pi \xi_n G(x,t)\, y_n^2 dx = \sum_{k=0}^{\infty} \alpha_k(t)\xi_n s(\pi,\ \lambda_k,t) \int_0^\pi \left[q_x \psi^2(x,\lambda_k,t) + 2q\left(\psi^2(x,\lambda_k,t)\right)'\right] y_n^2 dx. \quad (21)$$

It is easy to see that

$$\begin{aligned} I &= \int_0^\pi \left[q_x \psi^2(x,\lambda_k,t) + 2q\left(\psi^2(x,\lambda_k,t)\right)'\right] y_n^2 dx = \\ &= \int_0^\pi \left[2\left(q\psi^2(x,\lambda_k,t)\right)' - q_x\psi^2(x,\lambda_k,t)\right] y_n^2 dx = \\ &= \int_0^\pi \left(q\psi^2(x,\lambda_k,t)\right)' y_n^2 dx - \int_0^\pi q_x \psi^2(x,\lambda_k,t) y_n^2 dx + \\ &\quad + q\psi^2(x,\lambda_k,t)\, y_n^2\big|_0^\pi - \int_0^\pi q\psi^2(x,\lambda_k,t)\left(y_n^2\right)' dx = \\ &= \int_0^\pi q\left(\psi^2(x,\lambda_k,t)\right)' y_n^2 dx - \int_0^\pi 2q\psi^2(x,\lambda_k,t) y_n y_n' dx = \\ &= \int_0^\pi 2q\psi(x,\lambda_k,t)\, y_n\left[\psi'(x,\lambda_k,t)\, y_n - \psi(x,\lambda_k,t)\, y_n'\right] dx = \\ &= -\int_0^\pi 2q\psi(x,\lambda_k,t)\, y_n W\left\{\psi(x,\lambda_k,t),\ y_n\right\} dx. \end{aligned} \quad (22)$$

Using the equality

$$q\psi(x,\lambda_k,t)\, y_n = \frac{\left(\psi(x,\lambda_k,t)\, y_n' - \psi'(x,\lambda_k,t)\, y_n\right)'}{\xi_n - \lambda_k}$$

and conditions

$$\psi(\pi,\lambda_k,t) = \psi(0,\lambda_k,t) = 1$$

we obtain

$$\begin{aligned} I &= \int_0^\pi \frac{2}{\lambda_k - \xi_n} W\left\{\psi(x,\lambda_k,t),\ y_n\right\} W'\left\{\psi(x,\lambda_k,t),\ y_n\right\} dx = \\ &= \frac{1}{\lambda_k - \xi_n} W^2\left\{\psi(x,\lambda_k,t), y_n\right\}\big|_0^\pi = \\ &= \frac{1}{\lambda_k - \xi_n}\left(y_n'^2(\pi)\,\psi^2(\pi,\lambda_k,t) - y_n'^2(0)\,\psi^2(0,\lambda_k,t)\right) = \\ &= \frac{1}{\lambda_k - \xi_n}\left(y_n'^2(\pi) - y_n'^2(0)\right). \end{aligned} \quad (23)$$

Substituting (23) into (21), we have

$$\int_0^\pi \xi_n G(x,t)\, y_n^2 dx = \sum_{k=0}^{\infty} \frac{\xi_n \alpha_k(t) s(\pi,\ \lambda_k,t)}{\lambda_k - \xi_n}\left(y_n'^2(\pi) - y_n'^2(0)\right). \quad (24)$$

Substituting (20) and (24) into (16), we obtain

$$\dot{\xi}_n = \left\{\frac{1}{2\xi_n} - u(0,t) + \sum_{k=0}^{\infty} \frac{\xi_n \alpha_k(t) s(\pi,\ \lambda_k,t)}{\xi_n - \lambda_k}\right\}\left[y_n'^2(\pi,t) - y_n'^2(0,t)\right]. \quad (25)$$

Using to (12) and identity

$$\alpha_n^2(t) = \int_0^\pi s^2(x,\xi_n,t)\,q(x,t)\,dx = s'(\pi,\xi_n,t)\,\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\,|_{\lambda=\xi_n}, \tag{26}$$

we derive the following equality

$$\begin{gathered} y_n'^2(\pi,t) - y_n'^2(0,t) = \frac{1}{\alpha_n^2(t)}\left[s^2(\pi,\xi_n,t) - s'^2(0,\xi_n,t)\right] = \\ = \frac{1}{s'(\pi,\xi_n,t)\,\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\,|_{\lambda=\xi_n}}\left[s'^2(\pi,\xi_n,t) - 1\right] = \\ = \frac{1}{\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\,|_{\lambda=\xi_n}}\left[s'(\pi,\xi_n,t) - \frac{1}{s'(\pi,\xi_n,t)}\right]. \end{gathered} \tag{27}$$

Now, substituting the values $x=\pi$ and $\lambda=\xi_n(t)$ into relation

$$c(x,\lambda,t)\,s'(x,\lambda,t) - c'(x,\lambda,t)\,s(x,\lambda,t) = 1,$$

we find that

$$c(\pi,\xi_n,t) = \frac{1}{s'(\pi,\xi_n,t)}. \tag{28}$$

Using (28) and identity

$$[c(\pi,\lambda,t) - s'(\pi,\lambda,t)]^2 = (\Delta^2(\lambda) - 4) - 4c'(\pi,\lambda,t)s(\pi,\lambda,t))$$

we obtain the following equality

$$s'(\pi,\xi_n,t) - \frac{1}{s'(\pi,\xi_n,t)} = \sigma_n(t)\sqrt{\Delta^2(\xi_n) - 4}, \tag{29}$$

where

$$\sigma_n(t) = sign\left\{s'(\pi,\xi_n,t) - c(\pi,\xi_n,t)\right\},\ n \geqslant 1.$$

Using (6) and expansions

$$\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\,|_{\lambda=\xi_n} = -2\xi_n^{-1} sh\frac{\pi}{2}\prod_{\substack{j\neq n\\ j=1}}^{\infty}\left(1 - \frac{\xi_n}{\xi_j}\right), \tag{30}$$

$$\Delta^2(\xi_n) - 4 = 4sh^2\frac{\pi}{2}\left(1 - \frac{\xi_n}{\lambda_0}\right)\prod_{i=1}^{\infty}\left(1 - \frac{\xi_n}{\lambda_{2i-1}}\right)\left(1 - \frac{\xi_n}{\lambda_{2i}}\right), \tag{31}$$

we deduce

$$\frac{s'(\pi,\xi_n,t) - \frac{1}{s'(\pi,\xi_n,t)}}{\frac{\partial s(\pi,\lambda,t)}{\partial\lambda}\,|_{\lambda=\xi_n}} = \frac{\sigma_n\sqrt{\Delta^2(\xi_n) - 4}}{\frac{\partial s(\pi,\xi_n,t)}{\partial\lambda}} = -\frac{\sigma_n\sqrt{\left(1 - \frac{\xi_n}{\lambda_0}\right)\prod_{i=1}^{\infty}\left(1 - \frac{\xi_n}{\lambda_{2i-1}}\right)\left(1 - \frac{\xi_n}{\lambda_{2i}}\right)}}{\xi_n^{-1}\prod_{\substack{j\neq n,\\ j=1}}^{\infty}\left(1 - \frac{\xi_n}{\xi_j}\right)}. \tag{32}$$

From (25), (32) and trace formulas

$$u(0,t) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\xi_k} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}$$

we arrive at (9).

It was noticed that if instead of Dirichlet boundary conditions periodic or anti-periodic boundary conditions are assumed then equation (25) remains valid, and one can deduce $\dot{\lambda}_n(t) = 0$ for all $n \in Z$. Hence, the spectrum of problem (13) does not depend on parameter t, and the theorem is proved.

Remark 1. Theorem 4.1 provides the method for solving problem (2)–(4). The method is as follows.

(i) Solving the direct spectral problem (4) with $q\,(x+\tau)$, we obtain spectral data λ_n, $n \geqslant 0$, $\xi_n^0(\tau)$, $\sigma_n^0\,(\tau)$, $n \geqslant 1$.

(ii) Using the result of Theorem 4.1, we find the solution of the Cauchy problem $\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau)$, $\sigma_n\,(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \geqslant 1$ for the Dubrovin-type system (9).

(iii) Finally, using trace formulas

$$u\,(\tau,t) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\xi_k(\tau,t)} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}$$

we obtain the expressions for $u\,(\tau,t)$. After that the Floquet solutions $\psi\,(x,\lambda_k,t)$ of equation (4) can be determined. The uniform convergence of the last series follows from the asymptotic behaviour of the eigenvalues $\xi_k(\tau,t)$ and λ_k.

Corollary 1. The theorem was proved [43] which states that length of the gaps of the weighted Sturm–Liouville equations (4) with π-periodic real-valued coefficient decreases exponentially if and only if the coefficient is analytic. From this theorem we conclude that if $q_0(x)$ is real analytical function then length of the gaps corresponding to this coefficient decreases exponentially. The same gaps correspond to the coefficient $q(x,t)$. Thus, solution $u(x,t)$ of problem (2)–(4) is real analytical function with respect to x for all moments of time.

Corollary 2. The theorem was proved [43] which states that potential is of period $\dfrac{\pi}{k}$ if and only if all intervals $[\lambda_{2n-1},\lambda_{2n}]$ collapse whenever n is not a multiple of k. From this theorem we conclude that if function $q_0(x)$ have the period $\dfrac{\pi}{k}$ then the same eigenvalues with the same multiplicities correspond to $q(x,t)$. Thus, solution $q(x,t)$ of problem (2)–(4) are the $\frac{\pi}{k}$-periodic functions with respect to x for all moments of time.

Conclusion

A simple method for constructing a source for the Camassa-Holm equation is presented. It is shown that inverse spectral method of the weighted Sturm-Liouville operator with periodic coefficient is applicable to solving the Cauchy problem for the Camassa-Holm equation with a source in the class of periodic functions. An analogue of the Dubrovin system of differential equations is derived, and then the CH equation with a source is solved in the class of periodic functions. Solution is represented as uniformly converging functional series.

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Интегрирование периодического уравнения Камасса-Холма с самосогласованным источником

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Аннотация. В последнее время большой интерес вызывают нелинейные эволюционные уравнения с самосогласованными источниками. Физически источники возникают в уединенных волнах с переменной скоростью и приводят к разнообразию динамики физических моделей. Что касается их приложений, такие системы обычно используются для описания взаимодействий между различными уединенными волнами. В данной статье мы рассматриваем задачу Коши для уравнения Камасса–Холма с источником в классе периодических функций. Основной результат настоящей работы представляет собой теорему об эволюции спектральных данных оператора Штурма–Лиувилля с весом потенциал которого является решением уравнения Камасса–Холма с источником. Полученные равенства позволяют применить метод обратной задачи для решения задачи Коши для уравнения Камасса–Холма с источником в классе периодических функций.

Ключевые слова: уравнение Камасса–Холма, самосогласованный источник, формулы следов, обратная спектральная задача, оператор Штурма–Лиувилля с весом.