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On Centralizers of the Graph Automorphisms of Niltriangular Subalgebras of Chevalley Algebras

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Abstract. Graph automorphisms of a Chevalley group correspond to each type of reduced indecomposable root system Φ, which Coxeter graph has a non-trivial symmetry. It is well-known, that a Chevalley algebra and its niltriangular subalgebra N has a graph automorphism θ exaclty when Φ is of type A_n, D_n or E_6. We note connections with homomorphisms of root systems introduced in 1982.

The main theorem on the centralizers in N of the automorphism θ gives new representations of niltriangular subalgebras, using also the unique series of unreduced indecomposable root system of type BC_n.

Keywords: Chevalley algebra, niltriangular subalgebra, homomorphisms of root systems.

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1. Preliminary notes

The semisimple complex Lie algebras are classified in Cartan–Killing theory parallel with the root systems of a Euclidean space V.

For any indecomposable root system Φ in V and for any field K we have corresponding Chevalley algebra $L_\Phi(K)$. The elements e_r $(r \in \Phi)$ and the basis of the suitable Cartan subalgebra H with the condition $He_r \subseteq Ke_r$ [1, Sec. 4.2] form a basis of $L_\Phi(K)$. The elements e_r $(r \in \Phi^+)$ for the positive root system Φ^+ in Φ form a basis of niltriangle subalgebra $N\Phi(K)$.

The system of fundamental roots Π, which is a basis in Φ^+, is unique. The Cartan numbers $A_{rs} = 2(r,s)/(r,s)$ $(r,s \in \Pi)$ are integer, they form the Cartan matrix. We call a graph with nodes, one associated with each fundamental root, such that the ith node is joined to the jth node by a bond of strength $A_{rs}A_{sr}$, a Coxeter graph of the root system Φ, by the terminology of J.-P. Serre [2] (see also remark in [3, Sec. 1]). If we mark each node by a number (r,r) we obtain the Dynkin diagram.

The classification of simple complex Lie algebras, to within isomorphism, is connected with the classification of indecomposable root systems, to within equivalence. There exist 9 series of reduced root systems [4, Tables I– IX] – the classical types A_n, B_n, C_n, D_n, and the exceptional types G_2, F_4, E_6, E_7 and E_8.

Co-roots $h_r = 2r/(r,r)$ $(r \in \Phi)$ give also *a dual root system* Φ^* with base Π^* and, also, the systems Φ, $(\Phi^*)^*$ are equivalence. Note that the systems of type B_n, C_n are dual.

On the other hand, all indecomposable unreduced root systems, to within equivalence, are exhausted by the systems of type BC_n. For $n = 1$ we have

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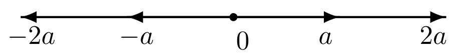


Type BC_1.

In 1982 in [5] homomorphisms of root systems were introduced. Denote by $L_0(\Phi)$ the additive subgroup in V generated by the roots of Φ (the lattice of roots).

Definition 1. *We call a mapping $\Phi \to \Phi'$ a homomorphism of Φ into Φ' if this mapping may be extended to a homomorphism of the group $L_0(\Phi)$ into the group $L_0(\Phi')$.*

We consider homomorphisms of the root system Φ, which are not isomorphisms. Such homomorphisms exist only if the Coxeter graph of the system Φ has a non-trivial symmetry and all roots in Φ have an equal length [5].

The symmetry may be linearly extended to a permutation $\bar{}$ on the root system Φ. Under the same restrictions on Φ the Chevalley algebra $L_\Phi(K)$ has a graph automorphism

$$\theta: \ e_r \to e_{\bar{r}} \quad (r \in \Pi),$$

according to the proof of Proposition 12.2.3 in [1]. By [6], the same conclusion is true for the induced automorphism of subalgebra $N\Phi(K)$.

By [7] and [8], an arbitrary (not necessary associative) algebra A is said to be an exact enveloping algebra for a Lie algebra L if L is isomorphic to the associated algebra $A^{(-)}$. The both algebras L and A may be defined by structure constants in the same basis, in contrast to the universal associative enveloping algebra.

The existance and the structure of exact enveloping algebras for the Lie algebras $N\Phi(K)$ were considered in [7]. Centralizers of graph automorphisms of Lie algebras $N\Phi(K)$ play an important role in the investigation of uniqueness (problem of I. P. Shestakov, 2017).

For the classical types on this way special exact enveloping algebras

$$RA_{n-1}(K) = NT(n, K), \quad RB_n(K), \quad RC_n(K), \quad RD_n(K).$$

were discovered (the associative algebra $NT(n, K)$ of all (lower) niltriangular matrices of degree n over K exists only for the type A_{n-1}):

The problem on non-trivial homomorphismd of root systems and embeddings of the centralizers is studied for Φ of type D_n and $\theta^2 = 1$, [9]. This problem was represented at the student conference in SFU (2022) by students of IMFI (M. V. Dektyarev, D. R. Khismatulin, A. D. Pakhomova and I. V. Salmina).

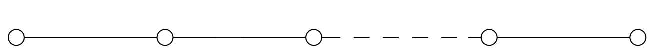
2. Centralizers of graph automorphisms

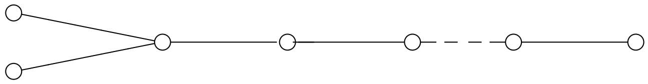
It is well-known that the number

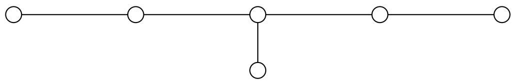
$$p(\Phi) := \max\{(r, r)/(s, s) \mid r, s \in \Phi\}$$

is equal to 1 or 2 or Φ of type G_2 and $p(\Phi) = 3$.

The root systems of the same length (i.e. with $p(\Phi)$=1), having a non-trivial symmetry of order 2, exhausted by the types A_n, D_n and E_6 with Coxeter graphs, correspondingly,

A_n :  (n nodes, $n \geqslant 1$)

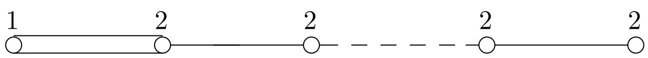
D_n :  (n nodes, $n \geqslant 4$)

E_6 : 

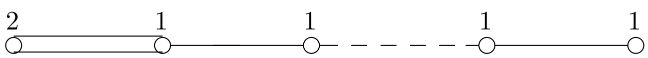
In this cases the graph automorphism θ is defined for the Chevalley algebra and for its subalgebra $N\Phi(K)$. In addition, either θ is of order 2, or Φ of type D_4 and $\theta^3 = 1$.

By [2, Ch. V, Sec. 15], a Coxeter graph gives *Dynkin diagram*, if we mark each node r by a number (r, r) (we assume that short roots have a lengh 1).

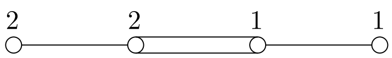
The root systems of type B_n are C_n dual and they have the same Coxeter graph. However, Dynkin diagrams for these types coinsides when $n = 2$. In this case the root systems are equivalent and the Coxeter graph have a non-trivial symmetry of order 2, as for types F_4 and G_2:

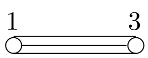
B_n : 

($n \geqslant 2$)

C_n : 

($n \geqslant 2$)

F_4 : 

G_2 : 

Note that Chevalley algebras (in contrast to Chevalley groups) of types F_4, G_2 and $B_2 = C_2$ doesn't have a graph automorphism [10].

We study the centralizer $C(\theta)$ of the graph automorphism θ of the Lie algebra $N\Phi(K)$, i. e. the subalgebra of all θ-stationary elements. Note, that the root system of type A_{2n}, by [5, Lemma 7], has a homomorphism to the unreduced root system of type BC_n.

The main result of the article is

Theorem 1. *Let θ be a graph automorphism of a Lie algebra $N\Phi(K)$. Then one of the following statements is valid.*

(a) $\theta^3 = 1$, Φ of type D_4 and $C(\theta) \simeq NG_2(K)$;

(b) $\theta^2 = 1$, Φ type D_n ($n \geqslant 4$) and $C(\theta) \simeq NB_{n-1}(K)$;

(c) $\theta^2 = 1$, Φ of type A_{2n-1} ($n \geqslant 3$) and $C(\theta) \simeq NC_n(K)$;

(d) $\theta^2 = 1$, Φ of type E_6 and $C(\theta) \simeq NF_4(K)$;

(e) $\theta^2 = 1$, Φ of type A_{2n} ($n \geqslant 2$) and the centralizer $C(\theta)$ in $NA_{2n}(K)$ is a subalgebra, associated with the unreduced root system of type BC_n.

A special case was considered in [9, Lemma 3.6].

Lemma 1. *The algebra $RB_n(K)$ is represented in the algebra $RD_{n+1}(K)$ by a centralizre of the graph automorphism θ of order 2 of the Lie algebra $RD_{n+1}(K)^{(-)}$.*

The authors showed that, using this lemma, an increasing sequence with uniquely defined isomorphic embeddings of algebras

$$RB_{n-1}(K) \subset RD_n(K) \subset RB_n(K) \subset RD_{n+1}(K) \subset \cdots, \quad n = 3, 4, 5, \ldots .$$

may be obtained.

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О централизаторах графовых автоморфизмов нильтреугольных подалгебр алгебр Шевалле

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Аннотация. Для группы Шевалле графовые автоморфизмы связывают с каждым типом ассоциированной приведенной неразложимой системы корней Φ, граф Кокстера которой допускает нетривиальную симметрию. Известно, что алгебра Шевалле и, аналогично, ее нильтреугольная подалгебра N обладает графовым автоморфизмом θ точно когда Φ — типа A_n, D_n или E_6. Мы отмечаем связь с введенными в 1982 году гомоморфизмами систем корней.

Основная теорема о централизаторах в N автоморфизма θ приводит к новым представлениям нильтреугольных подалгебр, использующим и единственную серию неприведенных неразложимых систем корней типа BC_n.

Ключевые слова: алгебра Шевалле, нильтреугольная субалгебра, гомоморфизмы систем корней.