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Two-dimensional Inverse Problem for an Integro-differential Equation of Hyperbolic Type

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Abstract. A multidimensional inverse problem of determining the kernel of the integral term of an integro-differential wave equation is considered. In the direct problem it is required to find the displacement function from the initial-boundary value problem. In the inverse problem it is required to determine the kernel of the integral term that depends on both the temporal and one spatial variable. Local unique solvability of the problem posed in the class of functions continuous in one of the variables and analytic in the other variable is proved with the use of the method of scales of Banach spaces of real analytic functions.

Keywords: integro-differential equation, inverse problem, delta function, integral equation, Banach theorem.

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1. Introduction. Formulation of the problem

Let us consider the integro-differential equation

$$u_{tt} = \Delta u + \int_0^t k(x,\alpha)u(x,z,t-\alpha)\,d\alpha, \qquad x \in R,\ z \in (0,l),\ t \in R, \tag{1}$$

with initial conditions

$$u|_{t<0} = 0, \tag{2}$$

and boundary conditions

$$u_z|_{z=0} = \delta'(t), \qquad u_z|_{z=l} = 0. \tag{3}$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplace operator, $\delta'(t)$ is the derivative of Dirac delta function, $l > 0$ is a finite real number.

Finding function $u(x,z,t) \in D$ (from the class of generalized functions) for known $k(x,t)$ is called the direct problem. The inverse problem consists in determination of function $k(x,t) \in C(\Pi)$ with respect to the solution of the direct problem and

$$u(x,0,t) = g(x,t), \tag{4}$$

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where $g(x,t)$ is a given smooth function, $\Pi = \{(x,t) : x \in R,\ t > 0.\}$

One-dimensional inverse problems for the differential equations were studied in [1–5]. Inverse problem (1)–(4) is a multidimensional inverse problem for differential equations. The idea to extend the method of scales of Banach spaces of analytic functions developed by L.V. Ovsyannikov [6] and L. Nirenberg [7] to multidimensional inverse problems belongs to Romanov. This method was applied, with some modifications, to study local solvability of multidimensional inverse problems [8–10]. A similar problem was studied when $z > 0$ [11]. A special feature of this work is that equation (1) is studied in a bounded domain with respect to the variable z, i.e, $z \in (0,l)$. It is proved in this paper that formulated problem is locally uniquely solvable in the class of functions analytic with respect to the variable x.

2. Study of the direct problem

First, let us consider direct problem (1)–(3), that is, we assume that function $k(x,t)$ is known. In what follows, this problem is considered in the domain $B = R \times G$, where $G = \{(z,t) :\ 0 < z < l, 0 < t < 2l - z\}$ is a combination of areas B_1 and B_2. Areas B_1 and B_2 are described as follows

$$B_1 = R \times G_1, \quad G_1 = \{(z,t) :\ 0 < z < l,\ 0 < t < z\},$$

$$B_2 = R \times G_2, \quad G_2 = \{(z,t) :\ 0 < z < l,\ z < t < 2l - z\}.$$

Lemma 2.1. *Solution of equation (1) in domain B_1 with conditions (2), (3)*

$$u(x,z,t) \equiv 0.$$

Proof. Using the d'Alembert formula, we obtain in the region $B_0 \subset B_1$ the following integral equation

$$u(x,z,t) = \frac{1}{2} \iint\limits_{\Omega(z,t)} \left[u_{xx}(x,\xi,\tau) + \int_0^\tau k(x,\alpha) u(x,\xi,\tau-\alpha) d\alpha \right] d\xi d\tau,$$

where $\Omega(z,t) = \{(z,t) : z - t + \tau \leqslant \xi \leqslant z + t - \tau,\ 0 \leqslant \tau \leqslant t\}$, $B_0 = R \times G_0$, and $G_0 = \Big\{(z,t) :\ 0 < z < l,\ 0 < t \leqslant \frac{l}{2} - |z - \frac{l}{2}|\Big\}$.

Since the obtained equation is a homogeneous equation of the Volterra type of the second kind it has only zero solution.

Therefore, $u(x,z,t) \equiv 0$ in the domain G_0. Let us take an arbitrary point $(x,z,t) \in B_1 \backslash B_0$. Let us put the term u_{zz} in equation (1) to the left side and represent the wave operator $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)$ as $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)$.

Integrating the obtained relation along the characteristic $dz/dt = 1$, from the point $(x, z-t, 0)$ to the point (x,z,t), we obtain

$$(u_t - u_z)(x,z,t) - (u_t - u_z)(x, z-t, 0) = \int_0^t \left[u_{xx}(x,z,\tau) + \int_0^\tau k(x,\tau-\alpha) u(x,\tau-t+x,\alpha) d\alpha \right] d\tau.$$

Using condition (2), we rewrite the last relation in the form

$$(u_t - u_x)|_{x=l} = \int_{\frac{t}{2}}^t \left[u_{xx}(x,z,\tau) + \int_0^\tau k(x,\tau-\alpha) u(x,\tau-t+x,\alpha) d\alpha \right] d\tau, \qquad t \in (0,l).$$

Further, using boundary condition (3) for $z = l$, we find

$$u(x,l,t) = \int_0^t \int_{\frac{\tau}{2}}^{\tau} \left[u_{xx}(x,\tau,\theta) + \int_0^{\theta} k(x,\theta-\alpha)u(x,\theta-\tau+l,\alpha)d\alpha \right] d\theta d\tau.$$

Changing variables in the inner integral by the formula $\theta - \tau + l = \xi$, we rewrite the last equation as

$$u(x,l,t) = \int_0^t \int_{l-\frac{\tau}{2}}^{l} \left[u_{xx}(x,\tau-l+\xi,\tau) + \int_0^{\tau-l+\xi} k(x,\tau-l+\xi-\alpha)u(x,\xi,\alpha)d\alpha \right] d\xi d\tau. \quad (5)$$

Integrating equation (1) along the characteristic $dz/dt = 1$ from the point $(x, z-t, 0)$ to the point (x,z,t),, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) u(x,z,t) = \int_{\frac{l+z-t}{2}}^{z} \left[u_{xx}(x,\xi,t) + \int_0^{\xi+t-z} k(x,\xi+t-z-\alpha)u(x,\xi,\alpha)d\alpha \right] d\xi.$$

Further, integrating this equality along the characteristic $dz/dt = -1$ from the point (x,z,t) to the point $(x,l,t+zl)$ and using (5), we find the equation for $u(x,z,t)$ in domain $B_1 \backslash B_0$

$$u(x,z,t) = \int_0^{t+z-l} \int_{l-\frac{\tau}{2}}^{l} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi+\tau-l} k(x,\xi+\tau-l-\alpha)u(\xi,\alpha,\tau)d\alpha \right] d\xi d\tau +$$
$$+ \int_{t+z-l}^{t} \int_{\frac{l+z-t-2\tau}{2}}^{-\tau+z+t} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi+2\tau-z-t} k(x,\xi+2\tau-z-t-\alpha)u(\xi,\alpha,\tau)d\alpha \right] d\xi d\tau,$$

where $(x,z,t) \in B_1 \backslash B_0$.

This equation is also a homogeneous equation of the Volterra type. Hence,

$$u(x,z,t) \equiv 0, \qquad (x,t) \in B_1 \backslash B_0$$

.

□

Using the d'Alembert formula in B_2 area, we obtain

$$u(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right) + \frac{1}{2}\left(\delta(t+z) - \delta(t-z)\right) +$$
$$+ \frac{1}{2} \int_0^z \int_{\tau-z+t}^{-\tau+z+t} \left[u_{xx}(x,\xi,\tau) + \int_0^{\xi-\tau} k(x,\alpha)u(x,\xi,\tau-\alpha)d\alpha \right] d\xi d\tau, \quad (x,z,t) \in B_2.$$

Let us introduce the function

$$\widetilde{u}(x,z,t) = u(x,z,t) - \frac{1}{2}\left(\delta(t-z) - \delta(t+z)\right). \quad (6)$$

For function $\widetilde{u}(x,z,t)$ we have the following equation

$$\widetilde{u}(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right) +$$
$$+ \frac{1}{2} \int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\widetilde{u}_{xx}(x,\xi,\tau) + \frac{1}{2}k(x,\tau-\xi) + \int_0^{\tau-\xi} k(x,\alpha)\widetilde{u}(x,\xi,\tau-\alpha)d\alpha \right] d\tau d\xi, \quad (7)$$

where $(x,z,t) \in B_2$.

Substituting function (6) into equations (1)–(3) and equating the coefficients of the singularities as in [12–14], we obtain

$$\widetilde{u}(x,z,t)\mid_{t=z+0}=0. \tag{8}$$

Taking into account (8), we take the limit $t \to z+0$, in equation (7) and obtain

$$\frac{1}{2}\left(g(x,2z)+g(x,0)\right)=$$

$$=-\frac{1}{2}\int_0^z\int_\xi^{2z-\xi}\left[\widetilde{u}_{xx}(x,\xi,\tau)+\frac{1}{2}k(x,\tau-\xi)+\int_0^{\tau-\xi}k(x,\alpha)\widetilde{u}(x,\xi,\tau-\alpha)d\alpha\right]d\tau d\xi, \quad (x,z,t)\in B_2.$$

Let us differentiate the obtained relation with respect to z

$$g_t(x,2z)=-\int_0^z\left[\widetilde{u}_{xx}(x,\xi,2z-\xi)+\frac{1}{2}k(x,2z-2\xi)+\int_0^{2z-2\xi}k(x,\alpha)\widetilde{u}(x,z,2z-\xi-\alpha)d\alpha\right]d\xi$$

, where $(x,z,t)\in B_2$.

Let us differentiate this equation with respect to z once more. Making a preliminary change of variables in the second integral $2z-2\xi=\eta$ and solving the resulting equation for function $k(x,z)$, we find

$$k(x,z)=-4g_{tt}(x,z)-4\int_0^{\frac{z}{2}}\left[\widetilde{u}_{xxt}(x,\xi,z-\xi)+2\int_0^{z-2\xi}k(x,\alpha)\widetilde{u}(x,z,z-\xi-\alpha)d\alpha\right]d\xi, \tag{9}$$

where $(x,z,t)\in B_2$. To obtain the equation for $\widetilde{u}_t(x,z,t)$ we differentiate equation (7) with respect to t

$$\begin{aligned}\widetilde{u}_t(x,z,t)&=\frac{1}{2}\left(g_t(x,t+z)+g_t(x,t-z)\right)-\frac{1}{2}k(x,t-z)z+\\&+\frac{1}{2}\int_0^z\left[\widetilde{u}_{xx}(x,\xi,t+z-\xi)-\widetilde{u}_{xx}(x,\xi,t-z+\xi)+\frac{1}{2}k(x,t+z-2\xi)+\right.\\&\left.+\int_0^{t+z-2\xi}k(x,\alpha)\widetilde{u}(x,\xi,t+z-\xi-\alpha)d\alpha-\int_0^{t-z}k(x,\alpha)\widetilde{u}(x,\xi,t-z+\xi-\alpha)d\alpha\right]d\xi.\end{aligned} \tag{10}$$

□

3. Theorem on solvability of the inverse problem

Let us introduce the Banach space $A_s(r), s>0$ of analytic functions $h(x), x\in R$ for which the norm is finite

$$\|h\|_s(r)=\sup_{|x|<r}\sum_{|\alpha|=0}^{\infty}\frac{s^{|\alpha|}}{\alpha!}\left\|\frac{\partial^\alpha}{\partial x^\alpha}h(x)\right\|<\infty$$

.

Here $r>0, s>0$, α is a non-negative integer.

In what follows, parameter r is fixed while parameter s is treated as a variable parameter. Further, parameter r is omitted for simplicity in the notation for the norms of the space A_s. When parameter s is changed the scale of Banach spaces A_s appears. The following property is obvious: if $h(x)\in A_s$ then $h(x)\in A_{s'}$ for all $s'\in(0,s)$. Hence $A_s\subset A_{s'}$ if $s'<s$ and the following inequality holds

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha}h(x)\right\|_{s'}\leqslant \alpha^\alpha\frac{\|h\|_s}{(s-s')^\alpha}\quad \forall\alpha\quad 0<s'<s\leqslant s_0. \tag{11}$$

In what follows, parameter r is fixed. The norm of function $f(x,z,t)$ in A_{s_0} for fixed z and t is denoted by $\|f\|_{s_0}(z,t)$. This norm in $C_{(z,t)}(G_2, A_{s_0})$ is defined by the equality

$$\|f\|_{C_{(z,t)}(G_2,A_{s_0})} = \sup_{(z,t)\in G_2} \|f\|_{s_0}(z,t),$$

where $C_{(z,t)}(G_2, A_{s_0})$ denotes the class of functions continuous with respect to variables z and t in domain G_2 with values in A_{s_0}.

Theorem 3.1. *Let*

$$(g(x,+0), g_t(x,+0)) \in A_{s_0}, \quad (g(x,z), g_t(x,z), g_{tt}(x,z)) \in (C_l[0,2l], A_{s_0})$$

$$\max\{\|g\|_{s_0}(t), \|g_t\|_{s_0}(t), \|g_{tt}\|_{s_0}(t)\} \leqslant R, \quad t \in [0,2l].$$

where $R > 0$ is given number.

Then there is $a \in (0,l)$ such that for any $s \in (0,s_0)$ in domain $\Gamma_{sl} = B_2 \cap \{(x,z,t) : 0 \leqslant z \leqslant a(s_0 - s)\}$ there is a unique solution of the system of equations (7), (9), (10) for which

$$(\widetilde{u}(x,z,t), \widetilde{u}_t(x,z,t)) \in C(A_{s_0}, F), \quad k(x,t) \in C(A_{s_0}, [0, a(s_0 - s)])$$

$$F = \{(z,t,s) : (z,t) \in D_{0l}, \quad 0 < z < a(s_0 - s)\}$$

. Moreover,

$$\|\widetilde{u} - \widetilde{u}_0\|_s(z,t) \leqslant R; \quad \|\widetilde{u}_t - \widetilde{u}_{0t}\|_s(z,t) \leqslant \frac{R}{s_0 - s}; \quad \|k - k_0\|_s(z) \leqslant \frac{R}{(s_0 - s)^2}.$$

Proof. It is convenient to introduce the notation

$$\varphi_1(x,z,t) = \widetilde{u}(x,z,t), \quad \varphi_2(x,z,t) = \widetilde{u}_t(x,z,t) + \frac{1}{2}k(x,t-z)z, \quad \varphi_3(x,z) = k(x,z)$$

$$\varphi_1^0(x,z,t) = \frac{1}{2}\left(g(x,t+z) + g(x,t-z)\right), \quad \varphi_2^0(x,z,t) = \frac{1}{2}\left(g_t(x,t+z) + g_t(x,t-z)\right),$$

$$\varphi_3^0(x,z) = -4g_{zz}(x,z).$$

Then we obtain from equations (6), (9), (10) that

$$\varphi_1(x,z,t) = \varphi_1^0(x,z,t) +$$

$$+\frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}(x,\xi,\tau) + \frac{1}{2}\varphi_3(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3(x,\alpha)\varphi_1(x,\xi,\tau-\alpha)d\alpha\right] d\tau d\xi, \quad (12)$$

$$\varphi_2(x,z,t) = \varphi_2^0(x,z,t) + \frac{1}{2}\int_0^z \Bigg[\varphi_{1xx}(x,\xi,t+z-\xi) - \varphi_{1xx}(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3(x,t+z-2\xi) +$$

$$+ \int_0^{t+z-2\xi} \varphi_3(x,\alpha)\varphi_1(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3(x,\alpha)\varphi_1(x,\xi,t-z+\xi-\alpha)d\alpha\Bigg] d\xi, \quad (13)$$

$$\varphi_3(x,z) = \varphi_3^0(x,z) -$$

$$-4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}(x,\xi,z-\xi) - \frac{1}{2}\varphi_3(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3(x,\alpha)\varphi_1(x,z,z-\xi-\alpha)d\alpha\right] d\xi. \quad (14)$$

Let us assume that numbers $a_1, a_2, \dots, a_n$ are determined by the recurrence relation

$$a_{n+1} = a_n \frac{(n+1)^2}{(n+1)^2+1}$$

. They form a decreasing numerical sequence and a is the limit of this sequence:

$$a = \lim_{n\to\infty} a_n = a_0 \prod_{n=1}^{\infty} \frac{(n+1)^2}{(n+1)^2+1}.$$

The positive number $a_0 < \dfrac{l}{s_0}$ will be selected later. Let us construct successive approximations as follows

$$\varphi_1^{n+1}(x,z,t) = \varphi_1^0(x,z,t)+$$
$$+\frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}^n(x,\xi,\tau) + \frac{1}{2}\varphi_3^n(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,\tau-\alpha)d\alpha\right] d\tau d\xi, \tag{15}$$

$$\varphi_2^{n+1}(x,z,t) = \varphi_2^0(x,z,t)+$$
$$+\frac{1}{2}\int_0^z \left[\varphi_{1xx}^n(x,\xi,t+z-\xi) - \varphi_{1xx}^n(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3^n(x,t+z-2\xi)+\right. \tag{16}$$
$$\left.+\int_0^{t+z-2\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3^n(x,\alpha)\varphi_1^n(x,\xi,t-z+\xi-\alpha)d\alpha\right] d\xi,$$

$$\varphi_3^{n+1}(x,z) = \varphi_3^0(x,z)-$$
$$-4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}^n(x,\xi,z-\xi) - \frac{1}{2}\varphi_3^n(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3^n(x,\alpha)\varphi_1^n(x,z,z-\xi-\alpha)d\alpha\right] d\xi. \tag{17}$$

We define function $s'(z)$ by the formula

$$s'(z) = \frac{s+\nu^n(z)}{2}, \qquad \nu^n(z) = s_0 - \frac{z}{a_n}. \tag{18}$$

Let us introduce the notation $\psi_i^n = \varphi_i^{n+1} - \varphi_i^n$, $i = 1,2,3$. For $n = 0$ the following relations hold

$$\psi_1^0(x,z,t) =$$
$$= \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\varphi_{1xx}^0(x,\xi,\tau) + \frac{1}{2}\varphi_3^0(x,\tau-\xi) + \int_0^{\tau-\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,\tau-\alpha)d\alpha\right] d\tau d\xi,$$

$$\psi_2^0(x,z,t) =$$
$$= \frac{1}{2}\int_0^z \left[\varphi_{1xx}^0(x,\xi,t+z-\xi) - \varphi_{1xx}^0(x,\xi,t-z+\xi) + \frac{1}{2}\varphi_3^0(x,t+z-2\xi)+\right.$$
$$\left.+\int_0^{t+z-2\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \varphi_3^0(x,\alpha)\varphi_1^0(x,\xi,t-z+\xi-\alpha)d\alpha\right] d\xi,$$

$$\psi_3^0(x,z) =$$
$$= 4\int_0^{\frac{z}{2}} \left[\varphi_{2xx}^0(x,\xi,z-\xi) - \frac{1}{2}\varphi_3^0(x,t-2\xi) + 2\int_0^{z-2\xi} \varphi_3^0(x,\alpha)\varphi_1^0(x,z,z-\xi-\alpha)d\alpha\right] d\xi$$

.

For $n = 1$ we have

$$\psi_1^1(x,z,t) = \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\psi_{1xx}^0(x,\xi,\tau) + \frac{1}{2}\psi_3^0(x,\tau-\xi)+$$

$$+\int_0^{\tau-\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,\tau-\alpha+\varphi_3^0(x,\alpha)\psi_1^0(x,\xi,\tau-\alpha)\big)\, d\alpha\Big] d\tau d\xi,$$

$$\psi_2^1(x,z,t) = \frac{1}{2}\int_0^z \Big[\psi_{1xx}^0(x,\xi,t+z-\xi) - \psi_{1xx}^0(x,\xi,t-z+\xi) + \frac{1}{2}\psi_3^0(x,t+z-2\xi)+$$

$$+\int_0^{t+z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,t+z-\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,t+z-\xi-\alpha)\big)\, d\alpha-$$

$$-\int_0^{t-z} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,t-z+\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,t-z+\xi-\alpha)\big)\, d\alpha\Big] d\xi,$$

$$\psi_3^1(x,z) = -4\int_0^{\frac{z}{2}} \Big[\psi_{2xx}^0(x,\xi,z-\xi) - \frac{1}{2}\psi_3^0(x,t-2\xi)+$$

$$+2\int_0^{z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^1(x,\xi,z+\xi-\alpha) + \varphi_3^0(x,\alpha)\psi_1^0(x,\xi,z+\xi-\alpha)\big)\, d\alpha\Big] d\xi.$$

Thus, for any n we obtain

$$\psi_1^n(x,z,t) = \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\psi_{1xx}^{n-1}(x,\xi,\tau) + \frac{1}{2}\psi_3^{n-1}(x,\tau-\xi)+$$

$$+\int_0^{\tau-\xi} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,\tau-\alpha+\varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,\tau-\alpha)\big)\, d\alpha\Big] d\tau d\xi,$$

$$\psi_2^n(x,z,t) = \frac{1}{2}\int_0^z \Big[\psi_{1xx}^{n-1}(x,\xi,t+z-\xi) - \psi_{1xx}^{n-1}(x,\xi,t-z+\xi) + \frac{1}{2}\psi_3^{n-1}(x,t+z-2\xi)+$$

$$+\int_0^{t+z-2\xi} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,t+z-\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,t+z-\xi-\alpha)\big)\, d\alpha-$$

$$-\int_0^{t-z} \big(\psi_3^{n-1}(x,\alpha)\varphi_1^n(x,\xi,t-z+\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,t-z+\xi-\alpha)\big)\, d\alpha\Big] d\xi,$$

$$\psi_3^n(x,z) = -4\int_0^{\frac{z}{2}} \Big[\psi_{2xx}^{n-1}(x,\xi,z-\xi) - \frac{1}{2}\psi_3^{n-1}(x,t-2\xi)+$$

$$+2\int_0^{z-2\xi} \big(\psi_3^0(x,\alpha)\varphi_1^n(x,\xi,z+\xi-\alpha) + \varphi_3^{n-1}(x,\alpha)\psi_1^{n-1}(x,\xi,z+\xi-\alpha)\big)\, d\alpha\Big] d\xi.$$

Next we show that if $a \in (0,l)$ is chosen in a suitable way then for any $n = 1,2,\ldots$ the following inequalities hold

$$\lambda_n = \max\Big\{ \sup_{(z,t,s)\in F_n} \Big[\|\psi_1^n\|_s(z,t)\frac{\nu^n(z)-s}{z}\Big], \sup_{(z,t,s)\in F_n} \Big[\|\psi_2^n\|_s(z,t)\frac{(\nu^n(z)-s)^2}{z}\Big],$$
$$\sup_{(z,t,s)\in F_n} \Big[\|\psi_3^n\|_s(z)\frac{(\nu^n(z)-s)^3}{z}\Big]\Big\} < \infty \tag{19}$$

$$\left\|\varphi_i^{n+1} - \varphi_0^{n+1}\right\|_s(z,t) \leqslant \frac{R_0}{(s_0-s)^{i-1}},\ i=1,2, \quad \left\|\varphi_3^{n+1} - \varphi_0^{n+1}\right\|_s(z) \leqslant \frac{R_0}{(s_0-s)^2}. \tag{20}$$

where

$$F_n = \big\{(z,t,s) : (z,t) \in G_l,\ 0 < z < a_n(s_0 - s),\ 0 < s < s_0\big\}.$$

Let $n = 0$. Then, taking into account (11), we obtain

$$\|\psi_1^0\|_s(z,t) \leqslant$$

$$\frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\|\varphi_{1xx}^0\|_s(\xi,\tau) + \frac{1}{2}\|\varphi_3^0\|_s(\tau-\xi) + \int_0^{\tau-\xi} \|\varphi_3^0\|_s(\alpha)\|\varphi_1^0\|_s(\xi,\tau-\alpha)d\alpha\Big]d\tau d\xi \leqslant$$

$$\leqslant \frac{1}{2}\int_0^z \int_{\xi-z+t}^{-\xi+z+t} \Big[\frac{4R}{(s(\xi)-s')^2} + \frac{R}{2} + R^2 t\Big]d\tau d\xi.$$

Let us use formula (18) for $n = 0$:

$$\|\psi_1^0\|_s(z,t) \leqslant \int_0^z (z-\xi)\Big[\frac{16R}{(\nu^0(z)-s)^2} + \frac{R}{2} + R^2 t\Big]d\xi \leqslant$$

$$\leqslant \Big[16 + s_0^2(0,5+2Rl)\Big]\int_0^z \frac{(z-\xi)d\xi}{(\nu^0(z)-s)^2} \leqslant a_0 R\Big[16 + s_0^2(0,5+2Rl)\Big]\frac{z}{\nu^0(z)-s}, \quad (z,t,s) \in F_0.$$

Let us estimate other components in a similar way:

$$\|\psi_2^0\|_s(z,t) \leqslant \frac{1}{2}\int_0^z \Big[\|\varphi_{1xx}^0\|(x,\xi,t+z-\xi) + \|\varphi_{1xx}^0\|(x,\xi,t-z+\xi) + \frac{1}{2}\|\varphi_3^0\|(x,t+z-2\xi) +$$

$$+ \int_0^{t+z-2\xi} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,\xi,t-z+\xi-\alpha)d\alpha\Big]d\xi \leqslant$$

$$\leqslant \frac{a_0}{2}R\Big[32 + s_0^2(0,5+4Rl)\Big]\frac{z}{(\nu^0(z)-s)^2}, \quad (z,t,s) \in F_0,$$

$$\|\psi_3^0\|_s(z) \leqslant 4\int_0^{\frac{z}{2}} \Big[\|\varphi_{2xx}^0\|(x,\xi,z-\xi) + \frac{1}{2}\|\varphi_3^0\|(x,t-2\xi) +$$

$$+ 2\int_0^{z-2\xi} \|\varphi_3^0\|(x,\alpha)\|\varphi_1^0\|(x,z,z-\xi-\alpha)d\alpha\Big]d\xi \leqslant a_0 R\Big[402 + s_0^2(2+16Rl)\Big]\frac{z}{(\nu^0(z)-s)^3},$$

$$(z,t,s) \in F_0.$$

To obtain these estimates the following inequalities are used

$$\frac{1}{\nu^0(\xi)-s} \leqslant \frac{1}{\nu^0(z)-s}, \qquad \nu^0(z) - s < s_0$$

. They are true for $\xi \in (0,z)$, $s \in (0,s_0)$, $(z,t,s) \in F_0$. The obtained estimates show the validity of inequality (19) for $n = 0$. Further, for $(z,t,s) \in F_1$ we find that

$$\left\|\varphi_i^1 - \varphi_0^0\right\|_s(z,t) = \|\psi_i^0\|_s(z,t) \leqslant \frac{a_0\lambda_0 z}{(\nu^0(z)-s)^i} \leqslant \frac{2^{i-1}a_0\lambda_0}{(s_0-s)^{i-1}}, \ \ i = 1,2.$$

$$\left\|\varphi_3^1 - \varphi_0^0\right\|_s(z) = \|\psi_3^0\|_s(z) \leqslant \frac{a_0\lambda_0 z}{(\nu^0(z)-s)^3} \leqslant \frac{4a_0\lambda_0}{(s_0-s)^2}$$

. Thus, if a_0 is chosen so that $4a_0\lambda_0 \leqslant R$ then inequalities (20) are true $n = 0$.

Let us show by the method of mathematical induction that inequalities (19), (20) also hold for other n if a_0 is chosen appropriately. Let us assume that inequalities (19), (20) are true for $n = 1, 2, \dots j$. Then for $(z, t, s) \in F_{j+1}$ we have

$$\|\psi_1^{j+1}\|(z,t) = \frac{1}{2}\int_0^z \int\limits_{\xi-z+t}^{-\xi+z+t} \Big[\|\psi_{1xx}^j\|(x,\xi,\tau) + \frac{1}{2}\|\psi_3^j\|(x,\tau-\xi)+$$

$$+\int\limits_0^{\tau-\xi} \Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,\tau-\alpha + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,\tau-\alpha)\Big)\,d\alpha\Big]d\tau d\xi \leqslant$$

$$\leqslant \frac{1}{2}\int\limits_0^z \int\limits_{\xi-z+t}^{-\xi+z+t} \Big[\frac{4a_0\lambda_j\xi}{(s(\xi)-s')^2(\nu^j(\xi)-s)} + \frac{a_0\lambda_j\xi}{(\nu^j(\xi)-s)^3} + \frac{a_0\lambda_j\xi Rt}{(\nu^j(\xi)-s)^3} + \frac{a_0\lambda_j\xi Rt(1+s_0^2)}{(\nu^j(\xi)-s)(s_0-s)^2}\Big]d\tau d\xi \leqslant$$

$$\leqslant \lambda_j a_0^2\Big[17 + 6Rl + 2Rls_0^2\Big]\frac{z}{\nu^{j+1}(z)-s} =: \lambda_j a_0\eta_1(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)}, \quad (z,t,s)\in F_0.$$

Here function $s'(\xi)$ is taken in form (18) with $n = j$ and inequalities

$$\|\varphi_1^{j+1}\|_s(z,t) \leqslant 2R, \quad \|\varphi_2^j\|_s(z) \leqslant R\frac{1+s_0^2}{(s_0-s)}, \quad \|\varphi_3^j\|_s(z) \leqslant R\frac{1+s_0^2}{(s_0-s)^2},$$

are valid according to the inductive hypothesis as well as the obvious inequalities $a_j \leqslant a_0$, $\nu^{j+1}(z) < \nu^j(z)$.

Similar reasoning for ψ_2^{j+1}, ψ_3^{j+1} leads to the inequalities

$$\|\psi_2^{j+1}\|_s(z,t) \leqslant \frac{1}{2}\int_0^z \big[\|\psi_{1xx}^j\|(x,\xi,t+z-\xi) + \|\psi_{1xx}^j\|(x,\xi,t-z+\xi) + \frac{1}{2}\|\psi_3^j\|(x,t+z-2\xi)+$$

$$+\int_0^{t+z-2\xi}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,t+z-\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,t+z-\xi-\alpha)\Big)\,d\alpha+$$

$$+\int_0^{t-z}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,t-z+\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,t-z+\xi-\alpha)\Big)\,d\alpha\big]d\xi \leqslant$$

$$\leqslant \lambda_j\frac{a_0}{2}\big[33 + 6Rl + 2Rls_0^2\big]\frac{z}{(\nu^{j+1}(z)-s)^2} =: \lambda_j a_0\eta_2(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)^2}, \quad (z,t,s)\in F_0,$$

$$\|\psi_3^{j+1}\|_s(z) \leqslant 4\int_0^{\frac{z}{2}}\Big[\|\psi_{2xx}^j\|(x,\xi,z-\xi) + \frac{1}{2}\|\psi_3^j\|(x,t-2\xi)+$$

$$+2\int_0^{z-2\xi}\Big(\|\psi_3^j\|(x,\alpha)\|\varphi_1^{j+1}\|(x,\xi,z+\xi-\alpha) + \|\varphi_3^j\|(x,\alpha)\|\psi_1^j\|(x,\xi,z+\xi-\alpha)\Big)\,d\alpha\Big]d\xi \leqslant$$

$$\leqslant \lambda_j a_0\big[402 + 48Rl + 12Rls_0^2\big]\frac{z}{(\nu^{j+1}(z)-s)^3} =: \lambda_j a_0\eta_3(R,l,s_0)\frac{z}{(\nu^{j+1}(z)-s)^3}, \quad (z,t,s)\in F_0,$$

It follows from the obtained estimates that

$$\lambda_{j+1} \leqslant \lambda_j\rho, \quad \lambda_{j+1} < \infty, \quad \rho := \max a_0\{\eta_1,\eta_2,\eta_3\}. \tag{21}$$

At the same time for $(x,t,s)\in F_{j+2}$ we have

$$\|\varphi_i^{j+2} - \varphi_i^0\|_s(z,t) \leqslant \sum_{n=0}^{j+1}\|\varphi_i^{n+1} - \varphi_i^n\|_s(z,t) = \sum_{n=0}^{j+1}\|\psi_i^n\|_s(z,t) \leqslant \sum_{n=0}^{j+1}\frac{\lambda_n z}{(\nu^n(z)-s)^i} \leqslant$$

$$\leqslant \frac{1}{(s_0-s)^{i-1}}\sum_{n=0}^{j+1}\frac{\lambda_n a_n^i a_{j+2}}{(a_n-a_{j+2})^i} \leqslant \frac{\lambda_0 a_0}{(s_0-s)^{i-1}}\sum_{n=0}^{j+1}\rho^n(n+1)^{2i}, \quad i=1,2,3.$$

Let us choose $a_0 \in (0,l)$ in such a way that

$$\rho < 1, \quad \lambda_0 a_0 \sum_{n=0}^{\infty} \rho^n (n+1)^6 \leqslant R.$$

Then

$$\|\varphi_i^{j+2} - \varphi_i^0\|_s(z,t) \leqslant \frac{R}{(s_0-s)^{i-1}}, \ \ (x,t,s) \in F_{j+2}, \ \ i=1,2,3.$$

Since the choice does not depend on the number of approximations the successive approximations φ_i^n, $i=1,2,3$, belong to

$$C(F,A_s), \quad F = \bigcap_{n=0}^{\infty} F_n$$

and the following inequalities

$$\|\varphi_i^n - \varphi_i^0\|_s(z,t) \leqslant \frac{R}{(s_0-s)^{i-1}}, \ \ (x,t,s) \in F, \ \ i=1,2,3.$$

are true. For $s \in (0,s_0)$ the series

$$\sum_{n=0}^{\infty} (\varphi_i^n - \varphi_i^{n-1})$$

converge uniformly in the norm of space $C(F,A_s)$ therefore $\psi_i^n \to \psi_i$. The limit functions are elements of $C(F,A_s)$ and satisfy equations (12), (13), (14).

Let us now prove the uniqueness of the found solution. Let us assume that $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$ are any two solutions that satisfy the inequalities

$$\|\varphi_i^{(k)} - \varphi_i^0\|_s(z,t) \leqslant R, \ \ i=1,2,3, \ \ ,k=1,2, \ \ (x,t,s) \in F.$$

Let us introduce $\widetilde{\varphi}_i = \varphi_i^{(1)} - \varphi_i^{(2)}$ $i=1,2,3$ and

$$\lambda = \max_{1\leqslant i\leqslant 3} \left\{ \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_1\|_s(z,t) \frac{\nu(z)-s}{z} \right], \quad \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_2\|_s(z,t) \frac{(\nu(z)-s)^2}{z} \right], \right.$$

$$\left. \sup_{(z,t,s)\in F} \left[\|\widetilde{\varphi}_3\|_s(z) \frac{(\nu(z)-s)^3}{z} \right] \right\} < \infty,$$

where

$$\nu(z) = s_0 - \frac{z}{a}, \quad a = a_0 \prod_{n=0}^{\infty} \frac{(n+1)^2}{(n+1)^2+1}.$$

Then, from equations (15), (16), (17) one can obtain the following relations for functions $\widetilde{\varphi}_i$

$$\widetilde{\varphi}_1(x,z,t) =$$

$$= \frac{1}{2} \int_0^z \int_{\xi-z+t}^{-\xi+z+t} \left[\widetilde{\varphi}_{1xx}(x,\xi,\tau) + \frac{1}{2}\widetilde{\varphi}_3(x,\tau-\xi) + \int_0^{\tau-\xi} \widetilde{\varphi}_3(x,\alpha)\widetilde{\varphi}_1(x,\xi,\tau-\alpha)d\alpha \right] d\tau d\xi,$$

$$\widetilde{\varphi}_2(x,z,t) = \frac{1}{2}\int_0^z \left[\widetilde{\varphi}_{1xx}(x,\xi,t+z-\xi) - \widetilde{\varphi}_{1xx}(x,\xi,t-z+\xi) + \frac{1}{2}\widetilde{\varphi}_3(x,t+z-2\xi) + \right.$$

$$\left. + \int_0^{t+z-2\xi} \widetilde{\varphi}_3(x,\alpha)\widetilde{\varphi}_1(x,\xi,t+z-\xi-\alpha)d\alpha - \int_0^{t-z} \widetilde{\varphi}_3(x,\alpha)\widetilde{\varphi}_1(x,\xi,t-z+\xi-\alpha)d\alpha \right] d\xi,$$

$$\widetilde{\varphi}_3(x,z) = -4\int_0^{\frac{z}{2}} \left[\widetilde{\varphi}_{2xx}(x,\xi,z-\xi) - \frac{1}{2}\widetilde{\varphi}_3(x,t-2\xi) + 2\int_0^{z-2\xi} \widetilde{\varphi}_3(x,\alpha)\widetilde{\varphi}_1(x,z,z-\xi-\alpha)d\alpha \right] d\xi$$

which are similar to equalities for ψ_i^n, $i=1,2,3$.

Applying the estimates given above, we find an analogue of inequality (21) in the form

$$\lambda < \lambda\rho',$$

where $\rho' := \max a\{\eta_1, \eta_2, \eta_3\}$. Since $a < a_0$ then $\rho' < \rho < 1$. Therefore $\lambda = 0$ and $\varphi_i^{(1)} = \varphi_i^{(2)}$, $i=1,2,3$. The theorem is proved.

References

[1] D.K.Durdiev, Zh.D.Totieva, The problem of determining the one-dimensional core of the equation of viscoelasticity, *Siberian Journal of Industrial Mathematics*, **16**(2013), no. 2, 72–82 (Russian).

[2] D.K.Durdiev, Zh.ShSafarov, The inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded region. *Mat. Notes*, **97**(2015), no. 6, 855–867. DOI: http://dx.doi.org/10.4213/mzm10659

[3] J.Sh.Safarov One-dimensional inverse problem for the equation of viscoelasticity in a bounded region. *Journal SVMO*, **17**(2015), no. 3, 44–55 (in Russian).

[4] J.Sh.Safarov, D.K.Durdiev, Inverse problem for the integro-differential equation of acoustics, *Differ. equat.*, **54**(2018), no. 1, 136–144. DOI: 10.1134/S0374064118010119

[5] J.Sh.Safarov, Global solvability of the one-dimensional inverse problem for the integro-differential equation of acoustics, *J. Sib. Fed. Univ. Math Phys.*, **11**(2018), no. 6, 753–763. DOI: https://doi.org/10.17516/1997-1397-2018-11-6-753-763

[6] L.Nirenberg, Topics in nolinear Functional Analysis, *Courant institutie Math. Sci.*, 1974.

[7] L.B.Ovsyannikov, Singular operator in the scale of Banach spaces, *Dokl. Akad. Nauk SSSR*, **163**(1963), no. 4, 819–822.

[8] V.G.Romanov, On the local solvability of some multidimensional inverse problems for equations of hyperbolic type, *Differ. equat.*, **25**(1989), no. 2, 275–284 (in Russian).

[9] V.G.Romanov, On the solvability of inverse problems for hyperbolic equations in the class of functions analytic with respect to some of the variables, *Dokl. Akad. Nauk SSSR*, **304**(1989), no. 4, 807–811 (in Russian).

[10] V.G.Romanov, A two-dimensional inverse problem for the viscoelasticity equation, *Siberian Math. J.*, **53**(2017), no. 6, 1128–1138. DOI:10.1134/S0037446612060171

[11] D.K.Durdiev, A multidimensional inverse problem for an equation with memory,*Siberian Math. J.*, **35**(1994), no. 3, 514–521.

[12] D.K.Durdiev, A.A.Rahmonov Inverse problem for the system integro-differential equation SH waves in a visco-elastic porous medium: global solubility, *Theoretical and Math Phys.*, **195**(2018), no. 3, 925–940.

[13] D.K.Durdiev, J.Sh.Safarov, Local solvability of the problem of determining the spatial part of a multidimensional kernel in an integro-differential equation of hyperbolic type, *Samara State Technical University Bulletin*, **4**(2012), no. 29, 37–47 (in Russian).

[14] D.K.Durdiev, Zh.D.Totieva, The problem of determining the one-dimensional matrix kernel of the system of visco-elasticity equation, *Math. Met. Appl. Sci.*, **41**(2018) no. 17, 8019–8032.

Двумерная обратная задача для интегро-дифференциального уравнения гиперболического типа

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Аннотация. Рассматривается двумерная обратная задача определения ядра интегрального члена в интегро дифференциальном уравнении гиперболического типа. В прямой задаче требуется найти функцию смещения из начально-краевой задачи.В обратной задаче требуется определение ядра интегрального члена зависящего как от временной, так и от одной пространственной переменной. Доказывается, локальная, однозначная разрешимость поставленной задачи в классе функций непрерывных по одной из переменных и аналитический по другой переменной, на основе метода шкал банаховых пространств вещественных аналитических функций.

Ключевые слова: интегро-дифференциальное уравнение, обратная задача, дельта функция, интегральное уравнение, теорема Банаха.