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Irreducible Carpets of Additive Subgroups of Type G_2 Over a Field of Characteristic 0

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Abstract. It is proved that any irreducible carpet of type G_2 over a field F of characteristic 0, at least one additive subgroup of which is an R-module, where F is an algebraic extension of the field R, up to conjugation by a diagonal element defines a Chevalley group of type G_2 over an intermediate subfield between R and F.

Keywords: Chevalley group, carpet of additive subgroups, carpet subgroup.

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1. Introduction

Let Φ be a reduced indecomposable root system, $\Phi(F)$ be a Chevalley group of type Φ over the field F generated by the root subgroups

$$x_r(F) = \{x_r(t) \mid t \in F\}, \quad r \in \Phi.$$

We call *a carpet of type* Φ *of rank* l *over* F a collection of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of the field F with the condition

$$C_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js}, \quad r, s, ir+js \in \Phi,\ i,j > 0, \tag{1}$$

where $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$, and constants $C_{ij,rs}$ are equal to ± 1, ± 2 or ± 3. Inclusions (1) come from the Chevalley commutator formula

$$\big[x_s(u), x_r(t)\big] = \prod_{i,j>0} x_{ir+js}\big(C_{ij,rs}(-t)^i u^j\big), \quad r, s, ir+js \in \Phi. \tag{2}$$

Every carpet $\mathfrak{A}$ defines a *carpet subgroup* $\Phi(\mathfrak{A})$ generated by the subgroups $x_r(\mathfrak{A}_r)$, $r \in \Phi$. A carpet $\mathfrak{A}$ is called *closed* if its carpet subgroup $\Phi(\mathfrak{A})$ has no new root elements, i.e., if

$$\Phi(\mathfrak{A}) \cap x_r(F) = x_r(\mathfrak{A}_r).$$

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The definition of a carpet used here was given by V. M. Levchuk [1] (see also [2, question 7.28]), and in [3] he described irreducible carpets of rank greater than 1 over field F, at least one additive subgroup of which is an R-module, where F is an algebraic extension of the field R, under the assumption that the characteristic of the field F is different from 0 and 2 for types B_l, C_l, F_4, and for the type G_2 is different from 0, 2 and 3. It turned out that, up to conjugation by a diagonal element, all additive subgroups of the carpet coincide with one intermediate subfield between R and F. We call such carpets *constant*. A similar problem for carpets of type G_2 over a field of characteristic 2 and 3 was considered by S. K. Franchuk and she established that non-constant carpets appear in characteristic 3 [4]. We have proved that in the remaining case of characteristic 0 for the type G_2 only constant carpets are possible.

Theorem 1. *Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be an irreducible carpet of type G_2 over a field F of characteristic 0, with at least one additive subgroup $\mathfrak{A}_r$ which is an R-module, where F is an algebraic extension of the field R. Then, up to conjugation by a diagonal element, all additive subgroups $\mathfrak{A}_r$ coincide with some intermediate subfield P between the fields R and F.*

2. Preliminary results

The group $\Phi(F)$ increasing to the extended Chevalley group $\widehat{\Phi}(F)$ by all diagonal elements $h(\chi)$, where χ is a F-character integral root lattice $\mathbb{Z}\Phi$, that is, a homomorphism of the additive group $\mathbb{Z}\Phi$ into the multiplicative group F^* of the field F [5, Sec. 7.1]. Any F-character χ is uniquely defined by the values at the fundamental roots, so for any $r \in \Phi$ and $t \in F$

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t). \tag{3}$$

The next lemma states that the equality (3) fits naturally with the definition of carpet.

Lemma 1 ([6], Lemma 1)**.** *Conjugating the carpet subgroup $\Phi(\mathfrak{A})$ with the diagonal element $h(\chi)$, we obtain the carpet subgroup*

$$h(\chi)\Phi(\mathfrak{A})h(\chi)^{-1} = \Phi(\mathfrak{A}'),$$

defined by the carpet

$$\mathfrak{A}' = \{\mathfrak{A}'_r \mid r \in \Phi\}, \text{ where } \mathfrak{A}'_r = \chi(r)\mathfrak{A}_r.$$

It is natural to call the carpet $\mathfrak{A}'$ from Lemma 1 *conjugate* to the original carpet $\mathfrak{A}$, and we can talk about conjugate carpets without relating them to carpet subgroups. Therefore, such statements are permissible. "Up to conjugation by a diagonal element, the carpet $\mathfrak{A}$ coincides with the carpet $\mathfrak{A}'$."

For a root system of type A_2 (see Fig. 1), there is one kind of commutator formula

$$[x_a(t), x_b(u)] = x_{a+b}(\pm tu).$$

Therefore, the carpet conditions have only one form $\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}$.

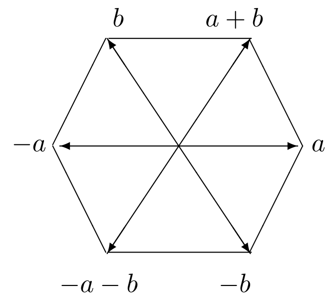

Fig. 1

For a root system of type G_2 (see Fig. 2), there are four kinds of commutator formulas

$$[x_a(t), x_b(u)] = x_{a+b}(\pm tu)x_{2a+b}(\pm t^2u)x_{3a+b}(\pm t^3u)x_{3a+2b}(\pm t^3u^2), \tag{4}$$

$$[x_a(t), x_{a+b}(u)] = x_{2a+b}(\pm 2tu)x_{3a+b}(\pm 3t^2u)x_{3a+2b}(\pm 3tu^2), \tag{5}$$

$$[x_a(t), x_{2a+b}(u)] = x_{3a+b}(\pm 3tu), \tag{6}$$

$$[x_b(t), x_{3a+b}(u)] = x_{3a+2b}(\pm tu). \tag{7}$$

So that, in this case, the carpet conditions look more impressive than for other types of root systems, and the formulas (4), (5), (6), (7) provide, respectively, the following forms

$$\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}, \quad \mathfrak{A}_a^2\mathfrak{A}_b \subseteq \mathfrak{A}_{2a+b}, \quad \mathfrak{A}_a^3\mathfrak{A}_b \subseteq \mathfrak{A}_{3a+b}, \quad \mathfrak{A}_a^3\mathfrak{A}_b^2 \subseteq \mathfrak{A}_{3a+2b},$$

$$2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}, \quad 3\mathfrak{A}_a^2\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{3a+b}, \quad 3\mathfrak{A}_a\mathfrak{A}_{a+b}^2 \subseteq \mathfrak{A}_{3a+2b},$$

$$3\mathfrak{A}_a\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{3a+b},$$

$$\mathfrak{A}_b\mathfrak{A}_{3a+b} \subseteq \mathfrak{A}_{3a+2b}.$$

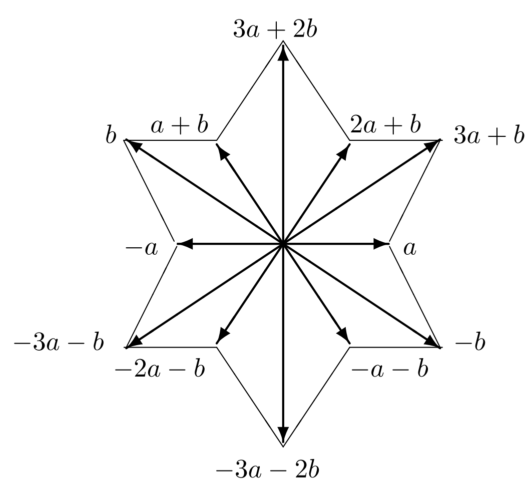

Fig. 2

The proof of the following lemma is elementary, so we omit it.

Lemma 2. *Let F be an algebraic extension of the field R and A is a subring of the field F which is an R-module. Then A is the field between R and F.*

Lemma 3. *Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be an irreducible carpet of type A_2 over a field F, $\{a, b\}$ is the fundamental system for Φ and let $1 \in \mathfrak{A}_{-a} \cap \mathfrak{A}_{-b}$ and the additive subgroup $\mathfrak{A}_{a+b}$ is an R-module, where F is an algebraic extension of the field R. Then all $\mathfrak{A}_r$ coincide with some fixed subfield of the field F.*

Proof. By [3, Lemma 3] all $\mathfrak{A}_r$ coincide with some fixed subring of the field F, and by Lemma 2 this subring is a field. The lemma is proved. □

3. Proof of Theorem 1

Up to conjugation, diagonal elements can be assumed to be $1 \in \mathfrak{A}_{-a} \cap \mathfrak{A}_{-b}$. Then, by virtue of the carpet conditions, from the commutator formula (4) we obtain $1 \in \mathfrak{A}_r$ for all $r \in \Phi^-$. Without loss of generality, we can assume that $\mathfrak{A}_{2a+b}$ or $\mathfrak{A}_{3a+2b}$ is an R-module. Since the field R has characteristic 0, then for any non-zero integer n we use the equality $n\mathfrak{A}_r = \mathfrak{A}_r$ without mentioning in case when the additive subgroup $\mathfrak{A}_r$ is an R-module.

Let $\mathfrak{A}_{2a+b}$ be an R-module. Due to the carpet conditions $2\mathfrak{A}_{-a-b}\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_a$ and $2\mathfrak{A}_{-a}\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{a+b}$ we get the inclusions $\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_a$ and $\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{a+b}$ respectively. Hence, due to the carpet condition $2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}$ it follows that $\mathfrak{A}_{2a+b}$ is a ring, and by virtue of Lemma 2 it is a field. In particular, $1 \in \mathfrak{A}_{2a+b}$. Therefore, due to the carpet conditions from the commutator formula (4), replacing the pair of roots (a, b) with the pairs $(2a + b, -3a - b)$ and $(2a+b, -3a-2b)$ we obtain $1 \in \mathfrak{A}_r$ for all $r \in \Phi$. Let $\mathfrak{A}_{2a+b} = P$. From the six carpet conditions of type $2\mathfrak{A}_a\mathfrak{A}_{a+b} \subseteq \mathfrak{A}_{2a+b}$ we obtain the equalities $\mathfrak{A}_r = P$ for all short roots of r. By Lemma 3, all additive subgroups $\mathfrak{A}_r$ indexed by long roots r coincide with some fixed field Q. Now, from the carpet conditions $\mathfrak{A}_a\mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}$ and $\mathfrak{A}_a\mathfrak{A}_{2a+b} \subseteq \mathfrak{A}_{3a+b}$ we obtain the inclusions $Q \subseteq P$ and $P \subseteq Q$ respectively. Thus, in this case we have established that all additive subgroups of the carpet coincide with the field P.

Let $\mathfrak{A}_{3a+2b}$ be an R-module. By Lemma 3, all additive subgroups $\mathfrak{A}_r$ indexed by long roots r coincide with some fixed field P. In particular, $1 \in \mathfrak{A}_{3a+2b}$. Therefore, due to the carpet conditions from the commutator formula (4), when the pair of roots (a, b) is replaced by the pairs $(-2a - b, 3a + 2b)$ and $(-a - b, 3a + 2b)$ we get $1 \in \mathfrak{A}_r$ for all $r \in \Phi$. Further, just as in the previous case, we obtain that all additive subgroups of the carpet coincide with the field P.

The theorem is proved.

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Неприводимые ковры аддитивных подгрупп типа G_2 над полем характеристики 0

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Аннотация. Доказано, что любой неприводимый ковер типа G_2 над полем F характеристики 0, хотя бы одна аддитивная подгруппа которого является R-модулем, где F — алгебраическое расширение поля R, с точностью до сопряжения диагональным элементом определяет группу Шевалле типа G_2 над промежуточным подполем между R и F.

Ключевые слова: группа Шевалле, ковер аддитивных подгрупп, ковровая подгруппа.