Journal of Siberian Federal University. Mathematics & Physics 2022, 15(5), 555-558

DOI: 10.17516/1997-1397-2022-15-5-555-558
УДК 517.9

Uniqueness of an Interpolating Entire Function with Some Properties

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Received 19.04.2022, received in revised form 23.05.2022, accepted 20.06.2022

Abstract. We consider the problem of power series analytic continuation by coefficients interpolation by entire or meromorphic functions. We prove uniqueness of an interpolating function with some properties. Also, under assumptions of Polya‘s theorem about extendability of the sum of power series to the whole complex plane, except, possibly, some boundary arc, we find at least one singular point location.

Keywords: power series, analytic continuation, indicator function.

Citation: A.J. Mkrtchyan, Uniqueness of an Interpolating Entire Function with Some Properties, J. Sib. Fed. Univ. Math. Phys., 2022, 15(5), 555–558. DOI: 10.17516/1997-1397-2022-15-5-555-558.

Consider a power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

whose domain of convergence is the unit disk $D_1 := \{z \in \mathbb{C} : |z| < 1\}$. One possible approach to treat analytic continuation problem is to interpolate the coefficients of this power series.

The function φ *interpolates* the coefficients f_n of the power series (1), if

$$\varphi(n) = f_n \ \text{ for all } \ n \in \mathbb{N}. \tag{2}$$

Following this approach many results were obtained in the case when the coefficients of the series are interpolated by an analytic function (see for example [1, 2]).

In particular, Lindelöf and Le Roy gave the conditions for which the series extends analytically into a sectorial domain:

Theorem 1 (Le Roy, Lindelöf [3, 4])**.** *The sum of the series* (1) *extends analytically to the open sector* $\mathbb{C} \setminus \Delta_\sigma$ *if there is an entire function* $\varphi(\zeta)$ *of exponential type interpolating the coefficients* f_n *whose indicator function* $h_\varphi(\theta)$ *satisfies the condition*

$$h_\varphi(\theta) \leqslant \sigma |\sin\theta| \ \text{ for } \ |\theta| \leqslant \frac{\pi}{2}.$$

The *indicator* function $h_\varphi(\theta)$ for an entire function φ is defined as the upper limit [5]

$$h_\varphi(\theta) = \varlimsup_{r\to\infty} \frac{\ln|\varphi(re^{i\theta})|}{r}, \ \ \theta \in \mathbb{R}.$$

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Δ_σ is the sector $\{z = re^{i\theta} \in \mathbb{C} : |\theta| \leqslant \sigma\}$, $\sigma \in [0, \pi)$.

Pólya found conditions for analytic continuability of a series to the whole complex plane except for some boundary arc :

Theorem 2 (Polya [6]). *The series* (1) *extends analytically to* $\mathbb{C}$, *except possibly the arc* $\partial D_1 \cap \Delta_\sigma$, *if and only if there exists an entire function of exponential type* $\varphi(\zeta)$ *interpolating the coefficients* f_n *such that*

$$h_\varphi(\theta) \leqslant \sigma|\sin\theta| \ \textit{for} \ |\theta| \leqslant \pi.$$

Note that despite the fact that an entire function interpolating the coefficients always exists, it is sometimes easier to construct and work with meromorphic interpolating functions of a special form than with entire functions (see [7]).
Let

$$\psi(\zeta) = \phi(\zeta)\frac{\prod_{j=1}^{p}\Gamma(a_j\zeta + b_j)}{\prod_{k=1}^{q}\Gamma(c_k\zeta + d_k)}, \tag{3}$$

where $\phi(\zeta)$ is an entire function, $a_j \geqslant 0$, $j = 1, \ldots, p$, and

$$\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} c_k = 0. \tag{4}$$

$$l = \sum_{k=1}^{q} |c_k| - \sum_{j=1}^{p} a_j.$$

Theorem 3. *The series* (1) *extends analytically to the open sector* $\mathbb{C}\setminus\Delta_\sigma$, *if there exists a meromorphic function* $\psi(\zeta)$ *of the form* (3) *interpolating the coefficients* f_n, *such that the indicator of the associated with* $\psi(\zeta)$ *entire function*

$$\varphi(\zeta) := \phi(\zeta)\frac{\prod_{j=1}^{p} {a_j}^{a_j\zeta}}{\prod_{k+1}^{q} |c_k|^{c_k\zeta}}$$

satisfies the conditions:

$$1)\ h_\varphi(0) = 0, \qquad 2)\ \max\left\{h_\varphi\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}l, h_\varphi\left(\frac{\pi}{2}\right) + \frac{\pi}{2}l\right\} \leqslant \sigma.$$

Also note that all these results do not say anything about uniqueness of the interpolating function. We can see that if the function $\varphi(z)$ is of exponential type and interpolates the coefficients of a power series, then any function of the form $\varphi(z) + A\sin\pi z$ is also of exponential type and also interpolates the same coefficients. We will show that the properties of the indicator of interpolating functions in these theorems ensure uniqueness of the interpolating functions.

Proposition 1. *In each of Theorems 1, 2, 3 above, the interpolating function with the corresponding property is unique.*

Proof. We shall treat here the case of Theorem 2. (proofs for Theorems 1 and 3 are similar). Let $\varphi(z)$ and $g(z)$ be two entire functions of exponential type that interpolate the coefficients f_n and both satisfing the condition

$$h_g(\theta) \leqslant \sigma|\sin\theta| \ \textit{for} \ |\theta| \leqslant \pi,$$

Consider the function $F(z) = \varphi(z) - g(z)$, which is an analytic function of exponential type for $Re\ z \geqslant 0$ and $F(n) = 0, n = 0, 1, \ldots$ according to the Carlson theorem [8]:

Theorem (Carlson). *If the function is analytic and of finite exponential type σ for $Re\ z \geqslant 0$ and $F(n) = 0, n = 0, 1, \dots,$ then either $F(n) \equiv 0$, or its exponential type $\sigma \geqslant \pi$ for $Re\ z \geqslant 0$.*

So either $F(z) \equiv 0$ from which it follows that $g(z) \equiv \varphi(z)$, or of exponential type $\sigma \geqslant \pi$ for $Re\ z \geqslant 0$, that is, there is $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $h_F(\theta) \geqslant \pi$. According to the property of the indicator

$$h_F(\theta) \leqslant \max\{h_g(\theta), h_\varphi(\theta)\}$$

and we obtain

$$\max\{h_g(\theta), h_\varphi(\theta)\} \geqslant \pi,$$

which contradicts the hypothesis of Theorem 2. Therefore, our assumption about the existence of another interpolating function with properties that satisfy the conditions of the theorem is not true. □

Proposition 2. *Under the assumptions of Theorem 2, at least one of the boundary points $e^{ih_\varphi(\frac{\pi}{2})}$, $e^{-ih_\varphi(-\frac{\pi}{2})}$ is a singular point.*

Proof. Note that Theorem 2 is satisfied when $\sigma \geqslant \max\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\}$. The arc $\partial D_1 \cap \Delta_\sigma$, where the series may not continue, is minimal for $\sigma = \max\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\}$. Assume that both points $e^{ih_\varphi(\frac{\pi}{2})}$ and $e^{-ih_\varphi(-\frac{\pi}{2})}$ are not singular for the sum of the series. This means that, there is an open arc on the boundary $\partial D_1 \setminus \Delta_{\sigma_1}$, where the sum of the series is regular and which includes the arc $\partial D_1 \setminus \Delta_\sigma$. Therefore, according to Theorem 2, there must exist an interpolating function ϕ of exponential type for which the condition

$$h_\phi(\theta) \leqslant \sigma_1 |\sin\theta| \ \ for \ \ |\theta| \leqslant \pi$$

is satisfied and, therefore, for which

$$\max\left\{h_\phi(\frac{\pi}{2}), h_\phi(-\frac{\pi}{2})\right\} < \max\left\{h_\varphi(\frac{\pi}{2}), h_\varphi(-\frac{\pi}{2})\right\}$$

that is, another interpolating function must exist, which contradicts Proposition 1. □

The research is supported by a grant of the Russian Science Foundation (project no. 20-11-20117)

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Единственность интерполирующей целой функции с определенными свойствами

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Аннотация. Рассматривается вопрос аналитической продолжимости степенных рядов путем интерполяции коэффициентов целыми или мероморфными функциями. Доказывается единственность интерполирующей функции с определенными свойствами. Также в теореме Полиа о продолжимости суммы ряда на всю комплексную плоскость кроме быть может некоторой граничной дуги находится местоположение по крайней мере одной особой точки.

Ключевые слова: степенные ряды, аналитичиское продолжение, индикатор функция.