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Tutorial on Rational Rotation C^*-Algebras

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Abstract. The rotation algebra $\mathcal{A}_\theta$ is the universal C^*-algebra generated by unitary operators U, V satisfying the commutation relation $UV = \omega VU$ where $\omega = e^{2\pi i\theta}$. They are rational if $\theta = p/q$ with $1 \leqslant p \leqslant q-1$, othewise irrational. Operators in these algebras relate to the quantum Hall effect [2,26,30], kicked quantum systems [22,34], and the spectacular solution of the Ten Martini problem [1]. Brabanter [4] and Yin [38] classified rational rotation C^*-algebras up to $*$-isomorphism. Stacey [31] constructed their automorphism groups. They used methods known to experts: cocycles, crossed products, Dixmier-Douady classes, ergodic actions, K–theory, and Morita equivalence. This expository paper defines $\mathcal{A}_{p/q}$ as a C^*-algebra generated by two operators on a Hilbert space and uses linear algebra, Fourier series and the Gelfand–Naimark–Segal construction [16] to prove its universality. It then represents it as the algebra of sections of a matrix algebra bundle over a torus to compute its isomorphism class. The remarks section relates these concepts to general operator algebra theory. We write for mathematicians who are not C^*-algebra experts.

Keywords: bundle topology, Gelfand–Naimark–Segal construction, irreducible representation, spectral decomposition.

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1. Uniqueness of universal rational rotation C^*-algebras

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T} \subset \mathbb{C}$ denote the sets of positive integer, integer, rational, real, complex and unit circle numbers. For a Hilbert space H let $\mathcal{B}(H)$ be the C^*-algebra of bounded operators on H. All homomorphisms are assumed to be continuous. We assume famliarity with the material in Section 4.

Fix $p, q \in \mathbb{N}$ with $p \leqslant q-1$ and $\gcd(p,q) = 1$, define $\sigma := e^{2\pi i/q}$ and $\omega := \sigma^p$, and let $\mathfrak{C}_{p/q}$ be the set of all C^*-algebras generated by a set $\{U, V\} \subset \mathcal{B}(H)$ satisfying $UV = \omega VU$. Since $\{U, V\} = \{V, U\}$, $\mathfrak{C}_{(q-p)/q} = \mathfrak{C}_{(q-p)/q}$. M_q and the circle subalgebra of $L^2(\mathbb{T})$ generated by $(Uf)(z) := zf(z)$ and $(Vf)(z) := f(\omega z)$ belong to $\mathfrak{C}_{(q-p)/q}$. The circle algebra is isomorpic to the tensor product $C(\mathbb{T}) \otimes M_q$.

Definition 1. *$\mathcal{A} \in \mathfrak{C}_{p/q}$ generated by $\{U, V\} \subset \mathcal{B}(H)$ satisfying $UV = \omega VU$ is called universal if for every $\mathcal{A}_1 \in \mathfrak{C}_{p/q}$ generated by $\{U_1, V_1\} \subset \mathcal{B}(H_1)$ satisfying $U_1V_1 = \omega V_1U_1$, there exists a $*$-homomorphism $\Psi : \mathcal{A} \mapsto \mathcal{A}_1$ satisfying $\Psi(U) = U_1$ and $\Psi(V) = V_1$.*

Lemma 1. *If $\mathcal{A}, \mathcal{A}_1 \in \mathfrak{C}_{p/q}$ are both universal, then they are isomorphic.*

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Proof. Let U, V, U_1, V_1 be as in Definition 1. There exists $*$-homomorphisms $\Psi : \mathcal{A} \mapsto \mathcal{A}_1$ and $\Psi_1 : \mathcal{A}_1 \mapsto \mathcal{A}$ with $\Psi_1 \circ \Psi(U) = U,\ \Psi_1 \circ \Psi(V) = V,\ \Psi \circ \Psi_1(U_1) = U_1,\ \Psi \circ \Psi_1(V_1) = V_1$. Since $\{U, V\}$ generates $\mathcal{A}$, $\Psi_1 \circ \Psi$ is the identity map on $\mathcal{A}$. Similarly, $\Psi \circ \Psi_1$ is the identity map on $\mathcal{A}_1$. Therefore Ψ is a $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$ and A is $*$-isomorphic to A_1. □

2. Construction of universal rational rotation C^*-algebras

Define the Hilbert space $H_q := L^2(\mathbb{R}^2, \mathbb{C}^q)$ consisting of Lebesgue measurable $v : \mathbb{R}^2 \mapsto \mathbb{C}^q$ satisfying $\int_{\mathbb{R}^2} v^* v < \infty$, equipped with the scalar product $< v, w > := \int_{\mathbb{R}^2} w^* v$. Define $\mathcal{P}_q$ to be the subset of continuous $a : \mathbb{R}^2 \mapsto \mathcal{M}_q$ satisfying

$$a(x_1, x_2) = a(x_1 + q, x_2) = a(x_1, x_2 + q),\ (x_1, x_2) \in \mathbb{R}^2 \tag{1}$$

and regarded as a C^*-subalgebra of $\mathcal{B}(H_q)$ acting by $(av)(x) := a(x)v(x),\quad a \in \mathcal{P}_q,\ v \in H_q$. The operator norm of $a \in \mathcal{P}_q$ satisfies

$$||a|| = \max_{x \in [0,q]^2} ||a(x)||. \tag{2}$$

Define $U, V \in \mathcal{P}_q$ by

$$U(x_1, x_2) := e^{2\pi i x_1/q} U_0, \quad V(x_1, x_2) := e^{2\pi i x_2/q} V_0, \tag{3}$$

where $U_0, V_0 \in \mathcal{M}_q$ are defined by (7), and define $\mathcal{A}_{p/q}$ to be the C^*-subalgebra of $\mathcal{P}_q$ generated by $\{U, V\}$. Choose $r \in \{1, \ldots, q-1\}$ such that $pr = 1 \mod q$. Then r is unique, $\gcd(r, q) = 1$. Define $\sigma := e^{2\pi i/q}$ and $\omega := \omega^p$. Then $\omega^r = \sigma$.

Theorem 1. *If* $a \in \mathcal{A}_{p/q}$ *then*

$$a(x_1 + 1, x_2) = V_0^{-r} a(x_1, x_2) V_0^r \ \textit{ and }\ a(x_1, x_2 + 1) = U_0^r a(x_1, x_2) U_0^{-1}. \tag{4}$$

Conversely, if $a \in \mathcal{P}_q$ *satisfies (4), then* $a \in \mathcal{A}_{p/q}$.

Proof. (3) and (8) give $V^{-r} U V^r = \sigma U$ and $U^r V U^{-r} = \sigma V$. If $a = U^m V^n$, then

$$a(x_1 + 1, x_2) = \sigma^m a(x_1, x_2) = V_0^{-r} a(x_1, x_2) V_0^r;\ a(x_1, x_2 + 1) = \sigma^n a(x_1, x_2) = U_0^r a(x_1, x_2) U_0^{-r}.$$

The first assertion follows since $\text{span}\{U^m V^n : m, n \in \mathbb{Z}\}$ is dense in $\mathcal{A}_{p/q}$. Conversely, if $a \in \mathcal{P}_q$, then (1), Lemma 3, and Weierstrass' approximation theorem implies that there exist unique $c(m, n, j, k) \in \mathbb{C}$ with

$$a(x_1, x_2) \sim \sum_{(m,n) \in \mathbb{Z}^2} \sum_{j,k=0}^{q-1} c(m, n, j, k)\, e^{2\pi i (m x_1 + n x_2)/q}\, U_0^j V_0^k$$

where $\sim$ denotes Fourier series. Then (4) gives $c\,(m, n, j, k)\,\sigma^m = c\,(m, n, j, k)\,\sigma^j$ and $c(m, n, j, k)\sigma^n = c(m, n, j, k)\sigma^k$. Since $\sigma^q = 1,\ c(m, n, j, k) = 0$ unless $j = m \mod q$and $k = n$ mod q. Define $c(m, n) := c(m, n, m \mod q, n \mod q)$. Then $a \in A_{p/q}$ since

$$a \sim \sum_{(m,n) \in \mathbb{Z}^2} c(m, n) U^m V^n.$$

Representations $\rho_1, \rho_1 : \mathcal{A} \mapsto \mathcal{B}(H)$ of a C^*-algebra $\mathcal{A}$ are unitarily equivalent if there exists $U \in \mathcal{U}(H)$ such that $\rho_2(a) = U \rho_1(a) U^{-1},\ a \in \mathcal{A}$. □

Theorem 2. *If $\mathcal{A} \in \mathfrak{C}_{p/q}$ is generated by $\{U, V\}$ with $UV = \omega VU$ and $\rho : \mathcal{A} \to \mathcal{B}(H)$ is an irreducible representation then:*

1. *dim $H = q$ so $B(H) = \mathcal{M}_q$,*
2. *there exist $z_1, z_2 \in \mathbb{T}$ such that $\rho = \rho_{z_1,z_2}$ where $\rho_{z_1,z_2}(U^jV^k) := z_1^j z_2^k U_0^j V_0^k$.*
3. *$\rho_{z_1',z_2'}$ is unitarity equivalent to ρ_{z_1,z_2} iff $(z_1'/z_1)^q = (z_2'/z_2)^q = 1$.*

Proof. Boca gives a proof in ([1], p. 5, Lemma 1.8, p. 7, Theorem 1.9). We give a proof based on Schur's lemma. Let $\mathcal{C} \subset \mathcal{A}$ be the C^*-subalgebra generated by $\{U^q, V^q\}$. Since ρ is irreducible and $\rho(\mathcal{C})$ commmutes with $\rho(\mathcal{A})$, there exists a $*$-homomorphism $\gamma : \mathcal{C} \mapsto \mathbb{C}$ such that $\rho(c) = \gamma(c)I$, $c \in \mathcal{C}$. Choose $h \in H\backslash\{0\}$ and define $H_1 := \text{span}\,\{\rho(U^jV^k)h; 0 \leqslant j,k \leqslant q-1\}$. Since H_1 is closed, ρ-invariant, $H_1 \neq \{0\}$, and ρ is irreducible, $H = H_1$. Since $\dim H \leqslant q^2$, $\rho(V)$ has an eigenvector b with eigenvalue $\lambda \in \mathbb{T}$ and $||b|| = 1$. Define $z_2 := \lambda\omega$. Choose $z_1 \in \mathbb{T}$ so $z_1^q = \gamma(U^q)$ and define $b_j := z_1^j \rho(U^{-j})b$, $1 \leqslant j \leqslant q$. Then $\rho(V)b_j = z_2\omega^{j-1}b_j$, $j = 1, \ldots, q$, and $\rho(U)b_1 = z_1 b_q$, and $\rho(U)b_j = z_1 b_{j-1}$, $2 \leqslant j \leqslant q$. Therefore $\{b_1, \ldots, b_q\}$ is a basis for H, and (7) implies that $\rho(U) = z_1U_0$, and $\rho(V) = z_2V_0$ with respect to this basis. This proves assertions 1 and 2. Assertion 3 follows since the set of eigenvalues of $\rho(U)$ is $\{z_1\omega^j,\, 0 \leqslant j \leqslant q-1\}$, the set of eigenvalues of $\rho(V)$ is $\{z_2\omega^j,\, 0 \leqslant j \leqslant q-1\}$, and the set of eigenvalues determines unitary equivalence. $\square$

Theorem 3. *$\mathcal{A}_{p/q} \subset \mathcal{B}(H)$ is the universal C^*-algebra in $\mathfrak{C}_{p/q}$.*

Proof. Assume that $\mathcal{B} \in \mathfrak{C}_{p/q}$. Then there exists a Hilbert space H_1 and $U_1, V_1 \in \mathcal{B}(H_1)$ with $U_1V_1 = \omega V_1U_1$ and $\mathcal{B}$ is generated by $\{U_1, V_1\}$. It suffices to construct a continuous $*$-homomorphism $\varphi : \mathcal{A}_{p/q} \mapsto \mathcal{B}$ satisfying $\varphi(U) = U_1$ and $\varphi(V) = V_1$. Define dense $*$-subalgebras

$$\widetilde{\mathcal{A}_{p/q}} := \text{ span } \{U^jV^k : j,k \in \mathbb{Z}\} \subset \mathcal{A}_{p/q}, \quad \widetilde{\mathcal{B}} := \text{ span } \{U_1^jV_1^k : j,k \in \mathbb{Z}\} \subset \mathcal{B},$$

and a $*$-homomorphism $\widetilde{\varphi} : \widetilde{\mathcal{A}_{p/q}} \mapsto \widetilde{\mathcal{B}}$ by $\widetilde{\varphi}(U^jV^k) := U_1^jV_1^k$. To extend $\widetilde{\varphi}$ to $*$-homomorphism $\varphi : \mathcal{A}_{p/q} \mapsto \mathcal{B}$ it suffices to show that for every Laurent polynomial of two variables $p(u,v)$ the following inequality is satisfied $||p(U_1,V_1)|| \leqslant ||p(U,V)||$ since $p(U_1V_1) = \widetilde{\varphi}(p(U,V))$. Then ([13], Corollary I.9.11), which follows directly from the Gelfand-Naimark-Segal construction, implies that there exists an irreducible representation $\rho_1 : \mathcal{B} \mapsto \mathcal{M}_q$ and $v \in H_1$ with $||v|| = 1$ such that $||p(U_1,V_1)|| = ||\rho_1(p(U_1,V_1))v||$. Theorem 2 implies that $\rho_1(U_1) = z_1U_0$ and $\rho_1(V_1) = z_2V_0$ for some $z_1, z_2 \in \mathbb{T}$. Let $\rho : \mathcal{A}_{p/q} \mapsto \mathcal{M}_q$ be the irreducible representation defined by Theorem 2 so $\rho(U) = z_1U_0$ and $\rho(V) = z_2V_0$. Since $\rho_1 \circ \widetilde{\varphi} =$ the restriction of ρ to $\widetilde{\mathcal{A}_{p/q}}$, (2) and (3) imply that

$$||p(U_1,V_1)|| = ||\rho_1(p(U_1,V_1))v|| \leqslant ||\rho(p(U,V))|| \leqslant ||p(U,V)||$$

which concludes the proof. $\square$

3. Bundle topology and isomorphism classes

Define $\mathbb{E}_1$ to be the Cartesian product $[0,1]^2 \times \mathcal{M}_q$ with the identification

$$(1, x_2, M) = (0, x_2, V_0^{-r}MV_0^r), \quad x_2 \in [0,1],\ M \in \mathcal{M}_q$$

and

$$(x_1, 1, M) = (x_1, 0, U_0^rMU_0^{-r}), \quad x_1 \in [0,1],\ M \in \mathcal{M}_q$$

and define the algebra bundle $\pi_1 : \mathbb{E}_1 \mapsto \mathbb{T}^2$ by

$$\pi_1(x_1, x_2, M) = (e^{2\pi i x_1}, e^{2\pi i x_2}), \quad (x_1, x_2, M) \in \mathbb{E}_1.$$

A map $s : \mathbb{T}^2 \mapsto \mathbb{E}_1$ is called a section if it is continuous and $\pi_1 \circ s = I$ where I denotes the identity map on $\mathbb{T}^2$. Since for every $p \in \mathbb{T}^2$, the fiber $\pi_1^{-1}(p) = \mathcal{M}_q$, the set of sections under pointwise operations is a C^*-algebra. The theorems above show that this algebra is isomorphic to $\mathcal{A}_{p/q}$. Furthermore, since points in $\mathbb{T}^2$ correspond to unitary equivalence classes of irreducible representations, isomorphism of algebras induces homeomorphisms of $\mathbb{T}^2$. In order to compute isomorphism classes of universal rational rotation C^*-algebras it is convenient to use a slightly different bundle representation of $\mathcal{A}_{p/q}$. Define $W \in \mathcal{P}_q$

$$W(x_1, x_2) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i x_1/q} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{2\pi i (q-1) x_1/q} \end{bmatrix}$$

and $\mathcal{A}'_{p/q} := W \mathcal{A}_{p/q} W^{-1}$, which is $*$-isomorphic to $\mathcal{A}_{p/q}$. Then $\mathcal{A}'_{p/q}$ is represented as the algebra of sections of the algebra bundle $\pi_2 : \mathbb{E}_2 \mapsto \mathbb{T}^2$ where $\mathbb{E}_2$ is the Cartesian product $\mathbb{T} \times [0,1] \times \mathcal{M}_q$ with the identification

$$(z_1, 1, M) = (z_1, 0, G^r M G^{-r}), \quad z_1 \in \mathbb{T}, \ M \in \mathcal{M}_q$$

and

$$G(z_1) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & z_1 \end{bmatrix} U_0.$$

G^r is the clutching function of the bundle. Let $G_{in} : \mathbb{T} \mapsto Aut_q^*$ be the map defined by conjugation by G. Using the arguments for vector bundles in [18], it can be shown that the isomorphism classe of $\mathcal{A}_{p/q}$ is determined the homotopy class of $G_{in}^r : \mathbb{T} \mapsto Aut_q^*$. Since $\pi_1(G_{in}) = -1$, $\pi_1(G_{in}^r) = -r$ which gives:

Theorem 4. *$\mathcal{A}_{p/q}$ is isomorphic to $\mathcal{A}_{p'/q'}$ iff $q' = q$ and either $p' = p$ or $p' = q - p$.*

4. Requisite results

4.1. Hilbert spaces and adjoints

H is a Hilbert space with inner product $< \cdot, \cdot >: H \times H \mapsto \mathbb{C}$, norm $||v|| := \sqrt{< v, v >}$, and metric $d : H \times H \to [0, \infty)$ defined by $d(v, w) := ||v - w||$. $\mathcal{B}(H)$ is the Banach algebra of bounded operators on H (continuous linear maps from H to H) with operator norm

$$||a|| := \sup\{ \, ||av|| \, : \, v \in H, \, ||v|| = 1 \, \}.$$

The dual space H^* is the set of continuous linear functions $L : H \mapsto \mathbb{C}$. For $w \in H$ define $L_w \in H^*$ by $L_w v := < v, w >, \quad v \in H$.

Lemma 2. *If $L \in H^*$ then there exists a unique $w \in H$ such that $L = L_w$.*

Proof. Rudin gives a direct proof ([28], Theorem 4.12). If $\mathfrak{B}$ is an orthonormal basis for H and $w := \sum_{b \in \mathfrak{B}} \overline{Lb}\, b$, then since for every $v \in H$, $v = \sum_{b \in \mathfrak{B}} < v, b > b$, it follows that

$$Lv = \sum_{b \in \mathfrak{B}} < v, b > \, Lb = \left\langle v, \sum_{b \in \mathfrak{B}} \overline{Lb}\; b \right\rangle = < v, w >= L_w\, v.$$

Lemma 2 ensures the existence of adjoints. For $a \in \mathcal{B}(H)$ define its adjoint $a^* \in \mathcal{B}(H)$ by $L_{a^*w} := L_w \circ a, \;\; w \in H$ where $\circ$ denotes composition of functions. Therefore

$$< a\; v, w >=< v, a^* w >, \;\; v, w \in H.$$

Clearly $a^{**} = a$, $(ab)^* = b^* a^*$, and the Cauchy–Schwarz inequality gives

$$||a^*|| = \sup\{\, | < a^* v, w > | : v, w \in H, \, ||v|| = ||w|| = 1\} =$$

$$= \sup\{\, | < v, aw > | : v, w \in H, \, ||v|| = ||w|| = 1\} = ||a||$$

and

$$\begin{aligned} ||a^* a|| &= \sup\{\, | < a^* av, w > | : v, w \in H, \, ||v|| = ||w|| = 1\} = \\ &= \sup\{\, | < av, aw > | : v, w \in H, \, ||v|| = ||w|| = 1\} = ||a||^2. \end{aligned} \tag{5}$$

(5) is called the C^*-identity. It makes $\mathcal{B}(H)$ equipped with the adjoint a C^*-algebra. The identity operator $I \in \mathcal{B}(H)$ is defined by $Iv := v$ for all $v \in H$.

$$\mathcal{U}(H) := \{U \in \mathcal{B}(H) : UU^* = U^* U = I\},$$

the set of unitary operators, is a group under multiplication. A subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is a C^*-algebra if it is closed in the metric space topology on $\mathcal{B}(H)$ and $a^* \in \mathcal{A}$ whenever $a \in \mathcal{A}$. The intersection of any nonempty collection of C^*-subalegras of $\mathbb{B}(H)$ is a C^*-algebra. If $S \subset \mathcal{B}(H)$ the intersection of all C^*-subalgebras of $\mathcal{B}(H)$ that contain S is the C^*-algebra generated by S. □

4.2. Matrix algebras

For $m, n \in \mathbb{N}$, $\mathbb{C}^{m \times n}$ denotes the set of m by n matrices with complex entries and $\mathbb{C}^n := \mathbb{C}^{n \times 1}$. The adjoint of $a \in \mathbb{C}^{m \times n}$ is the matrix $a^* \in \mathbb{C}^{n \times m}$ defined by $a^*_{j,k} := \overline{a_{k,j}}$. $\mathbb{C}^n$ is a Hilbert space with scalar product $< v, w > := \; w^* v, \;\; v, w \in \mathbb{C}^n$. Clearly

$$\mathcal{B}(\mathbb{C}^n) = \mathcal{M}_n$$

where for $a \in \mathcal{M}_n$ the adjoint of a as an operator corresponds to the adjoint of a as a matrix. I_n denotes the n by n identity matrix whose diagonal entries equal 1 and other entries equal 0. The operator norm of $a \in \mathcal{M}_n$ is $||a|| = \sqrt{\text{spectral radius } a^* a}$ where the spectral radius is the largest moduli of the eigenvalues of a matrix. Thus $\mathcal{M}_n$ is a C^*-algebra. It is also a Hilbert space a Hilbert space of dimension n^2 with inner product

$$< a, b > := \text{Trace } b^* a \tag{6}$$

and orthonormal basis $e_{j,k}$:= matrix with 1 in row j and column j with all other enties = 0. Fix $p, q \in \mathbb{N}$ with $p \leqslant q-1$ and $\gcd(p,q) = 1$. Define $U_0, V_0 \in \mathcal{M}_q$ by

$$U_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad V_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega^{q-1} \end{bmatrix}, \tag{7}$$

Lemma 3. *$\{(1/\sqrt{q})U_0^jV_0^k : 0 \leqslant j,k \leqslant q-1\}$ is an orthonormal basis for $\mathcal{M}_q$ with the scalar product defined by (6). Furthermore,*

$$U_0V_o = \omega V_0U_0. \tag{8}$$

Proof. (8) is obvious. The first assertion follows since

$$< U_0^jV_0^k, U_0^mV_0^n > \text{ Trace } V_0^{-n}U_0^{-m}U_0^jV_0^k = \text{Trace } U_0^{j-m}V_0^{k-n} = \begin{cases} q \text{ if } j = m \text{ and } k = n, \\ 0 \text{ otherwise.} \end{cases}$$

Define the groups of unitary matrices $\mathcal{U}_n := \mathcal{U}(\mathbb{C}^n)$ and special unitary matrices $\mathcal{S}_n := \{a \in \mathcal{U}_n : \det a = 1\}$. Clearly U_0 and V_0 are unitary. Since $\det U_0 = \det V_0 = (-1)^{q-1}$, they are special unitary iff q is odd. A map $\psi : \mathcal{M}_n \mapsto \mathcal{M}_n$ is a homomorphism if it is linear and satisfies $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathcal{M}_n$ and an automorphism if is also a bijective. An automorphism ψ is a $*$-automorpism if $\psi(b^*) = \psi(b)^*$ for all $b \in \mathcal{M}_n$. Aut_n, Aut_n^* denote the group of all automorpisms, $*$-automorphisms of $\mathcal{M}_n$. $\psi \in Aut_n$ is called inner if there exists an invertible $a \in \mathcal{M}_n$ such that $\psi(b) = aba^{-1}$ for every $b \in \mathcal{M}_n$.

Theorem 5 (Skolem–Noether)**.** *Every $\psi \in Aut_n$ is inner.*

Proof. The algebra $\mathcal{M}_n$ is simple, meaning it has no two-sided ideals othe that itself ([29], 11.41), so the result follows from the classic Skolem–Noether theorem. An elementary constructive proof is given in [32]. □

Theorem 6. *If $\psi \in Aut_n^*$ then there exists $a \in \mathcal{U}_n$ such that $\psi(b) = aba^*$ for every $b \in \mathcal{M}_n$.*

Proof. Every $\psi \in Aut_n^*$ induces an irreducible representation $\psi : \mathcal{M}_n \to \mathcal{B}(\mathbb{C}^n)$ so Theorem 2 implies that there exists a basis $\{b_1, \ldots, b_n\}$ with respect to which $\psi(U_0)$ has the matrix representation z_1U_0 and $\psi(V_0)$ has the representation z_2V_0. Since $U_0^n = V_0^n = I$, $z_1^n = z_2^n = 1$ so without loss of generality this basis can be chosen to make $z_1 = z_2 = 1$ and then $\psi(a) = aba^{-1}-1$ where $ae_j = b_j$ and $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. This theorem can also be derived as a corollary of of Theorem 5. Clearly $\psi(I_n) = I_n$. Theorem 5 implies that there exists an invertible $a \in \mathcal{M}_n$ such that $\psi(b) = aba^{-1}$ for all $b \in \mathcal{M}_n$. Since ψ is a $*$-homomorphism $ab^*a^{-1} = (aba^{-1})^* = (a^{-1})^*b^*a^*$ hence $a^*ab^* = b^*a^*a$ for every $b \in \mathcal{M}_n$ which implies that $a^*a = c\,I_n$ for some $c > 0$. Replacing a by $a/\sqrt{c}$ gives the conclusion.

Corollary 1. *Let $\mathbb{T}_n \subset \mathbb{T}$ be the subgroup of n-th roots of unity. $\mathbb{T}_n\, I_n \subset \mathcal{S}_n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Aut_n^* is isomorphic to the quotient group $\mathcal{S}_n/\mathbb{T}_n\, I_n$. The fundamental group $\pi_1(Aut_n^*)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Proof. Assertion one is obvious. Define $\zeta : \mathcal{U}_n \mapsto Aut_n^*$ by $\zeta(a)(b) := aba^*$. ζ is a $*$-homomorphism, kernel $\zeta = \mathbb{T}\, I_n$, and Corollary 1 implies that ζ is onto. The first homomorphism theorem of group theory ([33], 7.2) implies that Aut_n^* is isomoprhic to $\mathcal{U}_n/\mathbb{T}\, I_n$. Since

$\mathcal{S}_n = (\mathbb{T}I_n)((\mathbb{T}_n I_n)$ and $\mathbb{T}_n\, I_n = \mathcal{S}_n \cap (\mathbb{T}I_n)$, the second isomorphism theorem of group theory ([33], 7.3) implies that Aut_n^* is isomporphic to $\mathcal{S}_n/\mathbb{T}_n I_n$. $\mathcal{S}_n$ is simply connected ([17], Proposition 13.11) hence since $\mathbb{T}_n I_n$ is discrete $\mathcal{S}_n$ is the univeral cover of $\mathcal{S}_n \cap (\mathbb{T}I_n)$ hence the discussion in ([18], 1.3) implies the last assertion.

4.3. Spectral Decomposition Theorem for Unitary Operators

$E \in \mathcal{B}(H)$ is called a projection if $E^* = E$ and $E^2 = E$. Then $P : H \mapsto PH$ is orthogonal projection. A collection of projections $\{E_\varphi : \varphi \in [0, 2\pi]$ is called a spectral family if $E_{\varphi_1} E_{\varphi_2} = E_{\varphi_2} E_{\varphi_1} = E_{\varphi_1}$ whenver $\varphi_1 \leqslant E_{\varphi 2}$.

Let $A, P, N \in \mathcal{B}(H)$. A is self–adjoint if $A^* = A$. $P \in \mathcal{B}(H)$ is positive if $< Pv, v > \geqslant 0$ for all $v \in H$. $N \in \mathcal{B}(H)$ is called normal if $AA^* = A^*A$. Clearly self-adjoint and unitary operators (or transformations) are normal. Furthermore eigenvalues of self–adjoint operators are real and eigenvalues of unitary operators have modulus 1. If dim $H < \infty$, then H admits an orthonormal basis of eigenvectors ([29], Theorem 9.33). Therefore every unitary matrix in $\mathcal{M}_n$ can be diagonalized and its diagonal entries have modulus 1. The following result, copied verbatim from the classic textbook by F. Riesz and B. Sz.-Nagy ([27], p. 281), extends this diagonalization to unitary operators on arbitrary Hilbert spaces.

Theorem 7. *Every unitary transformation U has a spectral decomposition*

$$U = \int_{-0}^{2\pi} e^{i\varphi} dE_\varphi,$$

where $\{E_\varphi\}$ is a spectral family over the segmen $0 \leqslant \varphi \leqslant 2\pi$. We can require that E_φ be continuous at the point $\varphi = 0$, that is, $E_0 = 0$; $\{E_\varphi\}$ will then be determined uniquely by U. Moreover, E_φ is the limit of a sequence of polynomials in U and U^{-1}.

Proof. The authors of [27] reference 1929 papers by von Neumann [25] and Wintner [37], 1935 papers by Friedricks and Wecken, and a 1932 book by Stone. They observe that the theorem can be deduced from the one on symmetric transformation ([27], p. 280) (since $U = A + iB$ where $A := (U + U^*)/2$ and $B := -i(U - U^*)/2$ are symmetric) or from the theorem on trigonometric moments ([27], Section 53), but they give a direct three page proof. We sketch their proof. For every trigomometric polynomial $p(e^{i\varphi}) = \sum_{-n}^{n} c_k e^{ik\varphi}$ we associate the transformation $p(U) := \sum_{-n}^{n} c_k U^k$. This gives a $*$-homomorphism of the algebra of trigonometric polynomials (where $*$ means complex conjugation) into the subalgebra of $\mathcal{B}(H)$ generated by U and $U^* = U^{-1}$. Clearly if $p(e^{i\varphi})$ is real–valued then $p(U)$ is self-adjoint. If $p(e^{i\varphi}) \geqslant 0$ the Riesz–Fejer factorization Lemma ([27], Section 53) implies that there exists a trigonometric polynomial $q(e^{i\varphi})$ with $p(e^{i\varphi}) = q(e^{i\varphi})\overline{q(e^{i\varphi})}$ hence $p(U) = q(U)q(U)^*$. Therefore $< p(U)v, v > = < q(U)v, q(U) > \geqslant 0, \quad v \in H$, hence $p(U)$ is a positive operator. For $0 \leqslant \psi \leqslant 2\pi$ let e_ψ be the characteristic function of $(0, \psi]$ extended to a 2π periodic function on $\mathbb{R}$. Let p_n be a monotonically sequence of positive trigonometric functions with $\lim_{n\to\infty} p_n(U)v = E_\psi v, \quad v \in H$ ($p_n(U)$ converges to $E_\psi \in \mathbb{B}(H)$ in the strong operator topology). E_ψ is a projection since $E_\psi^* = E_\psi$ and $E_\psi^2 = E_\psi$, so , and the set $\{E_\varphi : \varphi \in [0, 2\pi]$ is a spectral family. Since the functions e_ψ are upper semi–continuous $\lim_{\chi\to\psi,\chi>\psi} E_\chi = E_\psi$.

Given $\epsilon > 0$ choose $0 < \psi_0 < \psi_1 < \cdots < \psi_n = 2\pi$ with max $(\psi_{k+1} - \psi_k) \leqslant \epsilon$ and choose $\varphi_k \in [\psi_{k-1}, \psi_k]$, $k = 1, \ldots, n$. Then for $\varphi \in (\psi_{k-1}, \psi_k]$

$$\left| e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k} \left[e_{\psi_k} - e_{\psi_{k-1}} \right] \right| = |e^{i\varphi} - e^{i\varphi_k})| \leqslant |\varphi - \varphi_k| \leqslant \epsilon$$

with a similar inequality for $\varphi = 0$. Since this inequality holds for all $\varphi \in [0, 2\pi]$

$$\left\| U - \sum_{k=1}^{n} e^{i\varphi_k} \left(E_{\psi_k} - E_{\psi_{k-1}} \right) \right\| \leqslant \epsilon$$

A subspace $H_1 \subset H$ is called proper if $H_1 \neq \{0\}$ and $H_1 \neq H$. The following is an immediate consequence of Theorem 7

Corollary 2. *If $U \in \mathcal{U}(H)$ then either $U = \gamma I$ for some $\gamma \in \mathbb{T}$ or there exists a projection operator $E : H \mapsto H$ satisfying*

1. *E is the limit in the strong operator topology on $\mathcal{B}(H)$ of polynomials $p(U, U^{-1})$.*
2. *EH is a proper closed U-invariant subspace of H.*

4.4. Irreducible representations and Schur's lemma

A representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is a *-homomorphism $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$. A subspace $H_1 \subset H$ is called ρ-invariant if $\rho(a)H_1 \subset H_1$ for every $a \in \mathcal{A}$. ρ is irreducible iff it H has no closed proper ρ-invariant subspaces. The following result extends Shur's lemma for finite dimensional representations ([29], 11.33) for unitary operators.

Theorem 8 (Schur's Lemma)**.** *If $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$ is an irreducible representation and $U \in \mathcal{U}(H)$ commutes with $\rho(a)$ for every $a \in \mathcal{A}$, then there exists $\gamma \in \mathbb{T}$ with $U = \gamma I$.*

Proof. If the conclusion does not hold then Corollary 2 implies that there exists a projection E satisfying conditions (1) and (2). Condition (1) implies that $U\rho(a) = \rho(a)U$ for every $a \in \mathcal{A}$. The $EH \subset H$ is closed and ρ-invariant since for every $a \in \mathcal{A}$, $\rho(a)EH = E\rho(a)H$. Condition (2) asserts that EH is a proper subspace thus contradicting the hypothesis that ρ is irreducible, and concluding the proof.

Corollary 3. *If $\mathcal{A}$ is an commutative C^*-algebra generated by a set of unitary operators and $\rho : \mathcal{A} \mapsto \mathcal{B}(H)$ is irreducible then dim $H = 1$ and there exists a $*$-homomorphism $\Gamma : \mathcal{A} \to \mathcal{C}$ with $\rho(a) = \Gamma(a)I$ for every $a \in \mathcal{A}$.*

Proof. Follows from from Theorem 8 since if $u \in \mathcal{A}$ is unitary the $\rho(u) \in \mathcal{U}(H)$ and $\rho(u)$ commutes with $\rho(a)$ for every $a \in \mathcal{A}$.

5. Remarks

We relate concepts introduced to explain rational rotation algebras to general C^*-algebra theory, especially two breakthrough results obtained by teams of computer scientists.

Remark 1. Dauns [14] initiated a program to represent C^*-algebras by continuous sections over bundles over their primitive ideal spaces (kernels of irreducible representations equipped with the hull-kernel topology). The primitive ideal space of rational rotation algebras is homemorpic to the torus $\mathbb{T}^2$.

Remark 2. Bratteli, Elliot, Evans, and Kishimoto [5] represent fixed point C^*-subalgebras of $\mathcal{A}_{p/q}$ by algebras of sections of M_q-algebra bundles over the sphere S^2, which is the space of orbits of $\mathbb{T}^2$ under the map $g \mapsto g^{-1}$.

Remark 3. Elliot and Evans [15] derived the structure of irrational rotation algebras. They proved that if $p/q < \theta < p'/q'$, then $\mathcal{A}_\theta$ can be approximated by a C^*-subalgebra isomorphic to $C(\mathbb{T}) \otimes M_q \oplus C(\mathbb{T}) \otimes M_{q'}$. This approximation, combined with the continued fraction expansion of θ, represents $\mathcal{A}_\theta$ as an inductive limit of these subalgebras.

Remark 4. Williams [36] gives an extensive explanation of crossed product C^*-algebras, which include rotation algebras.

Remark 5. Kadison and Singer [21] formulated a problem about extending pure states. Such an extension is used in the Gelfand–Naimark–Segal construction which we used to prove Theorem 3. This problem was shown to be equivalent to numerous problems in functional analysis and signal processing [6], dynamical systems [23, 24], and other fields [3]. Weaver [35] gave a discrepancy–theoretic formulation that was proved in a seminal paper by three computer scientists: Marcus, Spielman, and Shrivastava [19].

Remark 6. Courtney [8, 9] proved that the class of residually finite dimensional C^*-algebras, those whose structure can be recovered from their finite dimensional representations, coincides with the class of algebras containing a dense set of elements that attain their norm under a finite dimensional representation, this set is the full algebra iff every irreducible representation is finite dimensional (as for rational rotation algebras), and related these concepts to Conne's embedding conjecture [7]. Her publications [10–12] cite many references that discuss equivalent formulations of this conjecture.

Remark 7. In January 2020 five computer scientists: Ji, Natarajan, Vidick, Wright and Yuen submitted a proof that the Conne's embedding conjecture is false. As of November 2021 their paper is still under peer review. However, the editors of the ACM decided, based on the enormous interest that their paper attracted, to publish it [20].

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Учебник по рациональному вращению C^*-Алгебры

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Аннотация. Алгебра вращений $\mathcal{A}_\theta$ — это универсальная C^*-алгебра, порожденная унитарными операторами U, V, удовлетворяющими коммутационному соотношению $UV = \omega VU$, где $\omega = e^{2\pi i\theta}$. Они рациональны, если $\theta = p/q$ с $1 \leqslant p \leqslant q-1$, в противном случае иррациональны. Операторы в этих алгебрах связаны с квантовым эффектом Холла [2, 26, 30], квантовыми системами [22, 34] и эффектным решением проблемы Тена Мартини [1]. Брабантер [4] и Инь [38] классифицировали C^*-алгебры рационального вращения с точностью до $*$-изоморфизма. Стейси [31] построила свои группы автоморфизмов. Они использовали известные специалистам методы: коциклы, скрещенные произведения, классы Диксмье-Дуади, эргодические действия, К-теорию и эквивалентность Мориты. Эта пояснительная статья определяет $\mathcal{A}_{p/q}$ как C^*-алгебру, порожденную двумя операторами в гильбертовом пространстве, и использует линейную алгебру, ряды Фурье и конструкцию Гельфанда–Наймарка–Сигала [16] для доказательства его универсальности. Затем он представляет его как алгебру сечений расслоения матричной алгебры над тором для вычисления его класса изоморфизма. Раздел примечаний связывает эти концепции с общей теорией операторной алгебры. Мы пишем для математиков, не являющихся экспертами в C^*-алгебре.

Ключевые слова: топология расслоения, конструкция Гельфанда–Наймарка–Сигала, неприводимое представление, спектральное разложение.