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Some Solutions of the Euler System of an Inviscid Incompressible Fluid

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Abstract. We consider a system of two-dimensional Euler equations describing the motions of an inviscid incompressible fluid. It reduces to one non-linear equation with partial derivatives of the third order. A group of point transformations allowed by this equation is found. Some invariant solutions and solutions not related to invariance are constructed. The solutions found describe vortices, jet streams, and vortex-like formations.

Keywords: Euler equations, group of point transformations, invariant solutions, vortices, jets.

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Introduction

It is well known that the system of two-dimensional Euler equations

$$u_t + uu_x + vu_y + p_x = 0, \qquad v_t + uv_x + vv_y + p_y = 0, \qquad u_x + v_y = 0 \tag{1}$$

describes plane motions of an inviscid incompressible fluid [1]. Here u, v are the components of the velocity vector, p is the pressure. The symmetry group of the system (1) was found by A. Rodionov and V. Andreev. They found a new non-local operator and constructed some [2] invariant solutions. Moreover. they studied the invariant properties of the system in Lagrangian coordinates. Very non-trivial and interesting solutions in Lagrangian variables are constructed in the monograph [3]. At present, the question of the integrability of the system (1) by the method of the inverse scattering problem remains open.

It is very interesting to study axisymmetric flows with swirl [1]. The transformation group is admitted by these equations in Euler and Lagrangian coordinates is also presented in [2]. Few solutions are known for this model.

In this paper, the Euler system (1) is converted to one equation for the stream function. An infinite group of symmetries for this equation is found, and invariant solutions describing single vortices and kinks are constructed. Two kinks solutions and ones corresponding to the infinite group are given. A new solution of the stationary equation of axisymmetric flows with swirl, known in the physical literature as the Grad-Shafranov equation [4], is found.

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1. Symmetry groups and invariant solutions

It is well known [1] that the system of equations (1) can be reduced to one equation with partial derivatives of the third order

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = 0, \qquad (2)$$

where ψ is a stream function, Δ is the two-dimensional Laplacian operator and the lower indices denote differentiation by the corresponding variables. If the solution of the equation (2) is known, then the components of the velocity vector are reconstructed by the formulas $u = \psi_y,\ v = -\psi_x$.

Standard methods [5] can be used to find an Lie symmetry algebra of the equation (2). It is generated by the following operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\psi\frac{\partial}{\partial \psi},$$

$$X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_5 = 2ty\frac{\partial}{\partial x} - 2tx\frac{\partial}{\partial y} + (x^2+y^2)\frac{\partial}{\partial \psi},$$

$$X_6 = f(t)\frac{\partial}{\partial \psi}, \quad X_7 = g(t)\frac{\partial}{\partial x} + yg(t)'\frac{\partial}{\partial \psi}, \quad X_8 = h(t)\frac{\partial}{\partial y} - xh(t)'\frac{\partial}{\partial \psi},$$

where f, g and h are arbitrary functions of t. The first four operators generate well-known transformations: the translation in the t-direction, two scaling symmetries and rotation in $\mathbb{R}^2(x, y)$. The operator X_5 is responsible for the transition to a coordinate system rotating with a constant angular velocity. It generates the transformation

$$\tilde{t} = t, \quad \tilde{x} = x\cos at - y\sin at, \quad \tilde{y} = x\sin at + y\cos at, \quad \tilde{\psi} = \psi + a(x^2+y^2)/2, \ \forall a \in \mathbb{R}.$$

The operator X_6 defines the time shift of the function ψ: $\psi \longrightarrow \psi + f(t)$, and the operators X_7, X_8 give the generalized Galilean boosts

$$\tilde{t} = t, \ \tilde{y} = y, \ \tilde{x} = x + g, \ \tilde{\psi} = \psi + yg',$$

$$\tilde{t} = t, \ \tilde{x} = x, \ \ \tilde{y} = y + h, \ \tilde{\psi} = \psi - xh'.$$

The infinite subalgebra generated by the three operators X_6, X_7, X_8 induces action on the solutions of the equation (2)

$$\psi(t, x, y) \longrightarrow \psi(t, x-g, y-h) + yg' - xh' + m, \qquad (3)$$

where m is an arbitrary function of t.

Let us consider the stationary equation (2)

$$\psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = 0. \qquad (4)$$

The left-hand side of this equation is the Jacobian determinant of ψ and $\Delta\psi$. Therefore, any solution to the equation

$$\Delta\psi = \omega(\psi)$$

satisfies (4). It can be shown that equation (4) admits a symmetry algebra generated by six operators

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \frac{\partial}{\partial y}, \qquad Y_3 = \frac{\partial}{\partial \psi}, \qquad Y_4 = \psi\frac{\partial}{\partial \psi},$$

$$Y_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Let us proceed to the construction of solutions related to the symmetries of the equations.
Vortices. A solution of the equation (4), which is invariant under the rotation transformation, has the form $\psi = F(x^2+y^2)$, where F is an arbitrary smooth function. Hence, according to (3), the function

$$\psi = F((x-g)^2 + (y-h)^2) + yg' - xh'$$

also the solution of this equation for any smooth functions $g(t), f(t)$.

Define the functions F, g, h as follows

$$F = \frac{1}{(x^2+y^2)^6 + 0.2}, \qquad g = \sin(t) + 0.5t, \qquad h = 3\cos(4t) + 0.1t.$$

Then the pattern of streamlines at time $t = 0$ looks like in Fig. 1, where we see a single vortex. At $t = 1.2$ the vortex disappears (Fig. 2), but at time $t = 2.4$ it appears again. We get a

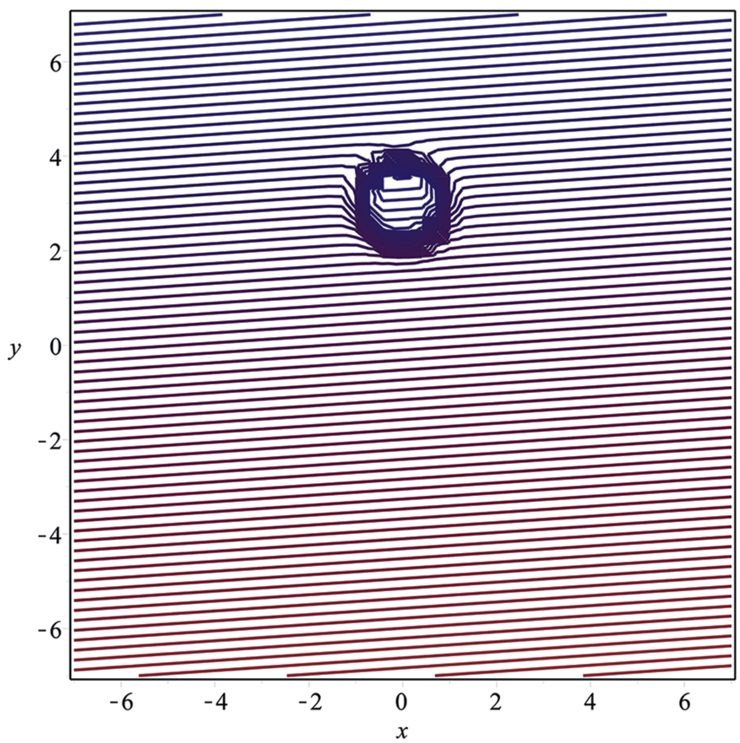

Fig. 1. Vortex at time t=0.

"flickering" vortex: it either appears or disappears. It is easy to choose the functions F, f, g so that the vortex will exist all the time. To do this, it is enough to leave the function F the same, and change the functions g and h a little: $g = \sin(t) + 1.5t,\ h = \cos(4t) + t$.

If one chooses the function

$$\psi = \frac{1}{x^2+y^2},$$

then we obtain a steady flow with a singularity at the origin of coordinates. This singular solution is analogous to the classical point vortex in a irrotational flow [1], but this flow is rotational.

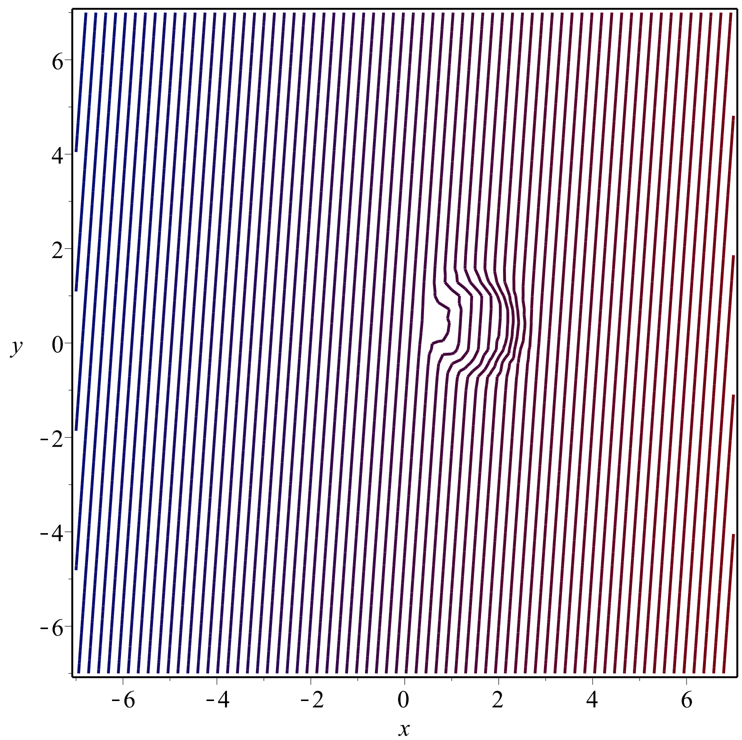

Fig. 2. The disappearance of the vortex at t=1.2

Using the action of the transformation group (3), it is not difficult to obtain a nonstationary vortex with a singular point.

Kink and soliton. The solution of the equation (4) invariant under the combination of translations is of the form

$$\psi = F(ax + by), \quad a, b \in \mathbb{R},$$

where F is an arbitrary smooth function. If we take the function F equal to $\arctan(ax + by)$, then its graph is a two-dimensional kink (step). Using the Galilean transformations, we obtain a nonstationary solution of the equation (2)

$$\psi = yg' - xh' + \arctan(\exp(x - g + y - h)),$$

where $g = 0.5t, h = 0.1t$. In this case, the graphs of the velocity components at different times are similar to a soliton and an antisoliton with variable amplitudes, respectively.

"Two soliton" solution. Using computer algebra systems, it is not difficult to find the following stationary solution

$$\psi = \arctan\left(\frac{f_1 + f_2}{1 + s_{12} f_1 f_2}\right) \tag{5}$$

of the equation (4). Here f_1, f_2 are functions equal to $\exp(k_i x + n_i y + m_i)$ $(i = 1, 2)$, and k_i, n_i, m_i, s_{12} are parameters that satisfy two relations

$$n_2^2 + k_2^2 = n_1^2 + k_1^2, \qquad s_{12} = \frac{(n_1 - n_2)^2}{(k_1 + k_2)^2}.$$

A typical graph of the function ψ given by the formula (5) looks like two stationary kinks for real parameter values (Fig.3). The streamline pattern represents the interaction of two jets. Using the

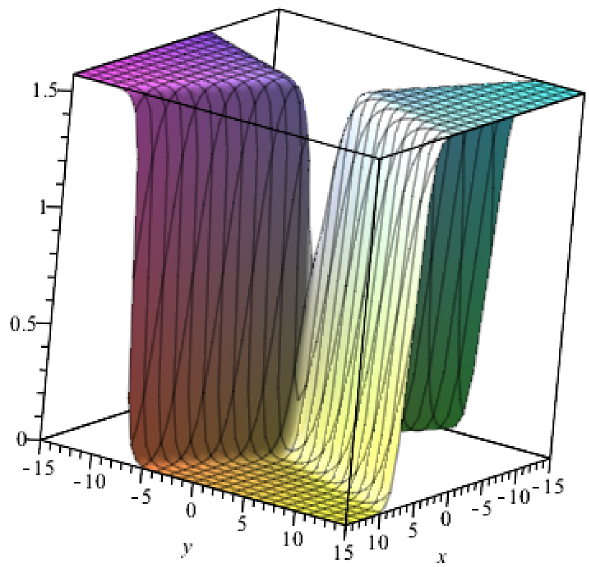

Fig. 3. Two stationary kinks

generalized Galilean subgroup, one can construct non-stationary solutions of the equation (2).

Another way to construct stationary solutions of the equation (2) follows from the next statement. Let ϕ be the solution Laplace equation $\Delta\phi = 0$, then the function

$$\psi = \log(s\Delta(\log\phi)), \quad \forall s \in \mathbb{R},$$

satisfies the equation (2). One can obtain nonstationary solutions using the symmetry group of the equation (2). This representation for the stream function is due to the fact that the Liouville equation

$$\Delta\psi = \exp(\psi),$$

admits an infinite group of transformations.

2. Additional solutions

We proceed to the construction of other non-stationary solutions of the equation (2). We look for the function ψ in the form

$$\psi = F(kx + ny + m(t)) + r_0(t)x + r_1(t)y + \sum_{i=2}^{N} r_i(t)(kx + ny)^i, \tag{6}$$

where F is an arbitrary function, k, n are arbitrary constants, and m, r_j $(j = 0, \ldots, N)$ are unknown functions on t. Substituting the representation (6) into equation (2), we have a system of ordinary differential equations on functions $m(t), r_j(t)$. Solving this system, we obtain recurrence formulas for the functions m, r_i

$$m = \int S dt, \qquad r_N = C, \qquad r_{i-1} = i\int r_i S dt, \quad i = 2, \ldots, N,$$

where C is an arbitrary constant, $S = -kr_0 + nr_1$, and r_0, r_1 are arbitrary smooth functions on t.

One can look for a solution to the equation (2) in the form

$$\psi = F(k(t)x + n(t)y + m(t)) + \sum_{i+j>0} r_{ij}(t)x^i y^j.$$

Substituting the latter expression into equation (2) results in a system of nonlinear ordinary differential equations on functions $k(t), n(t), m(t), r_{ij}(t)$. Finding its solutions remains an open problem.

Let us now consider the stationary equation for the stream function, which describes an axisymmetric swirling flow [1],

$$\psi_{xx} + \psi_{rr} - \psi_r/r = r^2 G + H, \tag{7}$$

where G, H are arbitrary functions of ψ. In plasma physics, this equation is called the Grad-Shafranov equation [4]. Some of its solutions are presented in the monograph [2], where there is also a group classification of this equation. Shan'ko [6] found some functionally-invadant solutions of the equation (7).

We will look for a solution to the equation (7) in the form

$$\psi = S(x^2 + ar^2), \qquad a \in \mathbb{R},$$

where S is the function to be found. Substituting this representation into the equation (7), we obtain the relation

$$2S' + 4x^2 S'' + r^2(4a^2 S'' - G) - H = 0.$$

We introduce a new variable $q = x^2 + ar^2$ and rewrite the last relation as

$$2S' + 4qS'' - H + r^2(4a(a-1)S'' - G) = 0.$$

Two equations follow from this

$$2S' + 4qS'' = H(S), \qquad 4a(a-1)S'' = G(S). \tag{8}$$

If the function S is given, then from the last two equations (8) one can find the functions G and H. Suppose, for example, $S = 1/q$. Then from the first equation of the system (8) we have $H(q^{-1}) = 6q^{-2}$. So the function $H(\psi)$ is $6\psi^2$. Similarly, from the second equation of the system (8) we find the function $G(\psi) = 8a(a-1)\psi^3$. Therefore, the equation (7), with the found functions G, H, has a solution

$$\psi = \frac{1}{x^2 + ar^2}.$$

The components of the velocity vector, according to [1], are

$$u = \frac{-2a}{(x^2 + ar^2)^2}, \qquad v = \frac{2x}{r(x^2 + ar^2)^2}, \qquad w = \frac{\sqrt{A - 4\psi^3}}{r}, \qquad A \in \mathbb{R}.$$

This solution has a singularity at $r = 0$ and tends to zero at infinity.

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References

[1] G.Batchelor, An introduction to fluid dynamics, Cambridge University Press, 1970.

[2] V.Andreev, O.Kaptsov, V.Pukhnachov, A.Rodionov, Applications of group-theoretical methods in hydrodynamics, Kluwer Academic Publishers, 1998.

[3] A.A.Abrashkin, E.I.Yakubovich, Vortex Dynamics in Lagrangian Description, Moscow, FIZMATLIT, 2006 (in Russian).

[4] L.Woods, Theory of Tokamak Transport. New Aspects for Nuclear Fusion Reactor Design, Wiley-VCH, Weinheim, 2006

[5] L.V.Ovsyannikov, Group Analysis of Differential Equations, Academic Press, NY, 1982.

[6] Yu.V.Shan'ko, Exact solutions of axially symmetric Euler equations, *AppL Maths Mech*, **60**(1996), no. 3, 433–437

Некоторые решения системы Эйлера невязкой несжимаемой жидкости

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Аннотация. В работе изучается система двумерных уравнений Эйлера, описывающая движения невязкой несжимаемой жидкости. Она сводится к одному нелинейному уравнению с частными производными третьего порядка. Найдена группа точечных преобразований, допускаемых этим уравнением. Построены некоторые инвариантные решения и решения не связанные с инвариантностью. Найденные решения описывают вихри, струйные течения и вихреподобные образования.

Ключевые слова: уравнения Эйлера, группы преобразований, инвариантные решения, вихри, струи.