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Existence and Uniqueness of the Solution to a Class of Fractional Boundary Value Problems Using Topological Methods

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Abstract. This paper investigates the existence and uniqueness of solutions to boundary value problems involving the Caputo fractional derivative in Banach space by topological structures with some appropriate conditions. It is based on the application of topological methods and fixed point theorems. Moreover, some topological properties of the solutions set are considered. Finally, an example is provided to illustrate the main results.

Keywords: fractional derivatives and integrals; topological properties of mappings, fixed point theorems.

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Introduction

Fractional differential equations have been verified to be effective modeling of many phenomena in several fields of science, for more details see Kilbas et al. [9], Miller and Ross [10], Podlubny [11], Agnieszka and Delfim [3]. Topological methods are one of the important tools that require weakly compact conditions rather than strongly compact conditions. In fact, the use of topological methods closely to study the existence of solutions for fractional differential equations in the last decades, see [6, 7, 13–16]. The fractional differential equations in Banach space are finding increasing consideration by many researchers such as Agarwal et al. [1, 2], Balachandran and Park [4], Benchohra et al. [5] and Zhang [19]. In 2006, Zhang [20], considered the existence of positive solutions to nonlinear fractional boundary value problems by applying the properties of the Green function and fixed point theorem on cones. In 2009, Benchohra et al. [5], examined the existence and uniqueness of solutions to fractional boundary value problems with nonlocal conditions by fixed point theorem. In 2012, Wang et. al [17,18], obtained the necessary and sufficient conditions for the fractional boundary value problems via a coincidence degree for condensing maps in Banach spaces. In 2015, the result was extended to the case for

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solutions to a fractional order multi point boundary value problem by Khan and Shah [8], who intentioned sufficient conditions for the existence of outcomes for a boundary value problem. In 2017, Samina et al. [12], studied the existence to solutions for nonlinear fractional Hybrid differential equations through some results about the existence of solutions and the Kuratowski's measure of non-compactness.

Stimulated by some of the mentioned results, our aim of this paper is to generate some new results about the following boundary value problem (BVP) for fractional differential equations involving Caputo fractional derivative with topological methods and fixed point theorems in Banach space $\mathcal{X}$.

$$\begin{cases} {}^c\mathcal{D}^q x(t) = \xi(t, x(t)) & t \in \mathcal{J} = [0, \tau],\ 0 < q \leqslant 1, \\ \beta x(0) + \gamma x(\tau) = \mu, \end{cases} \tag{1}$$

where ${}^c\mathcal{D}^q$ is the Caputo fractional derivative, $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is a continuous function. $\mathcal{C}(\mathcal{J}, \mathcal{X})$ will be a Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$ with the norm $\|x\|_c := sup\{\|x(t)\| : x \in \mathcal{C}(\mathcal{J}, \mathcal{X})\}$ for $t \in \mathcal{J}$. β, γ, μ are real constants satisfy $\beta + \gamma \neq 0$.

1. Preliminaries

In this section, we recall some of the basic definitions, propositions and basic theorem that will be used in this paper.

Definition 1.1 ([10]). *The q^{th} fractional order integral of a continuous function ξ on the closed interval $[a, b]$, is defined as*

$$\mathcal{I}_a^q \xi(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \xi(s) ds, \tag{2}$$

where Γ is the gamma function.

Definition 1.2 ([10]). *The q^{th} Riemann–Liouville fractional-order derivative of a continuous function ξ on the closed interval $[a, b]$, is defined as*

$$(\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-q-1} \xi(s) ds, \tag{3}$$

where $n = [q] + 1$ and $[q]$ is the integer part of q.

Definition 1.3 ([10]). *For a given continuous function ξ on the closed interval $[a, b]$, the Caputo fractional order derivative of ξ, is defined as*

$$({}^c\mathcal{D}_{a+}^q \xi)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} \xi^{(n)}(s) ds, \tag{4}$$

where $n = [q] + 1$.

Theorem 1.1 (Banach contraction mapping principle, [21]). *Let $\mathcal{X}$ be a complete metric space, and $\psi : \mathcal{X} \to \mathcal{X}$ is a contraction mapping with a contraction constant $\mathcal{K}$, then ψ has a unique fixed point.*

Theorem 1.2 (Schaefer's fixed point theorem, [21]). *Let $\mathcal{K}$ be a non-empty convex, closed and bounded subset of a Banach space $\mathcal{X}$. If $\psi : \mathcal{K} \to \mathcal{K}$ is a complete continuous operator such that $\psi(\mathcal{K}) \subset \mathcal{X}$, then ψ has at least one fixed point in $\mathcal{K}$.*

Definition 1.4 ([14, 21]). *Let $\Omega \subset \mathcal{X}$ and $\mathcal{F} : \Omega \to \mathcal{X}$ be a continuous bounded map. One can say that $\mathcal{F}$ is α-Lipschitz if there exists $k \geqslant 0$ such that*

$$\alpha(\mathcal{F}(B)) \leqslant k\alpha(B) \ \ (\forall)\ B \subset \Omega \ \textit{bounded.}$$

In case, $k < 1$, then we call $\mathcal{F}$ is a strict α-contraction. One can say that $\mathcal{F}$ is α-condensing if

$$\alpha(\mathcal{F}(B)) < \alpha(B) \ \ (\forall)\ B \subset \Omega \ \textit{bounded with}\ \alpha(B) > 0.$$

We recall that $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz if there exists $k > 0$ such that

$$\|\mathcal{F}_x - \mathcal{F}_y\| \leqslant k\|x - y\| \ \ (\forall)\ x, y \subset \Omega,$$

and if $k < 1$ then $\mathcal{F}$ is a strict contraction.

Proposition 1.1 ([14, 21]). *If $\mathcal{F}, \mathcal{G} : \Omega \to \mathcal{X}$ are α-Lipschitz maps with the constants k, $k^{'}$ respectively, then $\mathcal{F} + \mathcal{G} : \Omega \to \mathcal{X}$ is α-Lipschitz with constant $k + k^{'}$.*

Proposition 1.2 ([14, 21]). *If $\mathcal{F} : \Omega \to \mathcal{X}$ is compact , then $\mathcal{F}$ is α-Lipschitz with zero constant.*

Proposition 1.3 ([14, 21]). *) If $\mathcal{F} : \Omega \to \mathcal{X}$ is Lipschitz with a constant k, then $\mathcal{F}$ is α-Lipschitz with the same constant k.*

2. Main results

Definition 2.1. *If $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies the equation ${}^c\mathcal{D}^q x(t) = \xi(t, x(t))$ almost everywhere on $\mathcal{J}$, and the condition $\beta x(0) + \gamma x(\tau) = \mu$ then x is said to be a solution of the fractional BVP (1).*

In order to discuss the existence and uniqueness solutions to BVP(1), we require the following assumptions:
$[H1]$ $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is continuous.
$[H2]$ For each $t \in \mathcal{J}$ and all $x, y \in \mathcal{X}$, there exists constant $\delta > 0$ such that

$$\|\xi(t, x) - \xi(t, y)\| \leqslant \delta\|x - y\|$$

and

$$\frac{\delta\tau^q(1 + \frac{|\gamma|}{|\beta+\gamma|})}{\Gamma(q + 1)} < 1.$$

$[H3]$ For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_1, \delta_2 > 0$, $q_1 \in [0, 1)$ such that

$$\|\xi(t, x)\| \leqslant \delta_1\|x\|^{q_1} + \delta_2.$$

Lemma 2.1. *The fractional integral equation*

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}\xi(s, x(s))ds - \frac{1}{\beta + \gamma}[\frac{\gamma}{\Gamma(q)}\int_0^\tau (\tau - s)^{q-1}\xi(s, x(s))ds - \mu],\ t \in \mathcal{J} \quad (5)$$

has a solution $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ if and only if x is a solution of the fractional BVP (1).

Proof. First, suppose that $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies BVP(1), then we have to show that x is also a solution of FIE(5). We have,

$$x(t) - x(0) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}\xi(s, x(s))ds. \quad (6)$$

Then,

$$x(\tau) - x(0) = \frac{1}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds.$$

By the boundary conditions $\beta x(0) + \gamma x(\tau) = \mu$, we get

$$x(0) = \frac{\mu}{(\beta+\gamma)} - \frac{\gamma}{(\beta+\gamma)\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds, \quad \beta + \gamma \neq 0. \tag{7}$$

Replacing in equation(6), we get

$$x(t) = \frac{\mu}{(\beta+\gamma)} - \frac{\gamma}{(\beta+\gamma)\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \xi(s, x(s)) ds.$$

Conversely, suppose $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ satisfies FIE(5). If $t = 0$ then $\beta x(0) + \gamma x(\tau) = \mu$. For $t < \tau \in \mathcal{J}$ using the facts that the Caputo fractional derivative ${}^c\mathcal{D}_t^q$ is the left inverse of the fractional integral $\mathcal{I}_t^q$ and the Caputo derivative of the constant is zero, we can get ${}^c\mathcal{D}_t^q x(t) = \xi(t, x(t))$ which completes the proof. □

Lemma 2.2. *The operator $\mathcal{F} : \mathcal{C}(\mathcal{J}, \mathcal{X}) \to \mathcal{C}(\mathcal{J}, \mathcal{X})$ defined by;*

$$(\mathcal{F}x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \xi(s, x(s)) ds - \frac{1}{\beta+\gamma} \left[\frac{\gamma}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \xi(s, x(s)) ds - \mu \right]$$

is continuous and compact.

Proof. In order to prove the continuity and compactness of $\mathcal{F}$. Consider a bounded set $\mathcal{D}_r := \{\|x\| \leqslant r : x \in \mathcal{C}(\mathcal{J}, \mathcal{X})\}$. Let $\{x_n\}$ be a sequence of a bounded set $\mathcal{D}_r \subseteq \mathcal{C}(\mathcal{J}, \mathcal{X})$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. we have to show that $\|\mathcal{F}x_n - \mathcal{F}x\| \to 0$ as $n \to \infty$. It is obvious that $\|\xi(s, x_n(s)) - \xi(s, x(s))\| \to 0$ as $n \to \infty$ due to the continuity of ξ. Using [H3], we get for each $t \in \mathcal{J}$,

$$\|\xi(s, x_n(s)) - \xi(s, x(s))\| \leqslant \|\xi(s, x_n(s))\| + \|\xi(s, x(s))\| \leqslant 2(\delta_1 |r|^{q_1} + \delta_2).$$

As the function $s \to 2(\delta_1 |r|^{q_1} + \delta_2)$ is integrable for $s \in [0, t]$, $t \in \mathcal{J}$, by means of the Lebesgue Dominated Convergence theorem

$$\int_0^t (t - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds \to 0 \text{ as } n \to \infty.$$

Then, for all $t \in \mathcal{J}$

$$\|(\mathcal{F}x_n)(t) - (\mathcal{F}x)(t)\| \leqslant \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds +$$
$$+ \frac{|\gamma|}{|\beta+\gamma|\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds \to 0 \text{ as } n \to \infty.$$

Which implies that $\mathcal{F}$ is continuous.

Let $\{x_n\}$ be a sequence on a bounded set $\mathcal{M} \subset \mathcal{D}_r$, for every $x_n \in \mathcal{M}$.

$$\|\mathcal{F}x_n\| \leqslant \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|\xi(s, x_n(s))\| ds +$$
$$+ \frac{|\gamma|}{|\beta+\gamma|\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} \|\xi(s, x_n(s))\| ds + \frac{|\mu|}{|\beta+\gamma|}, \quad t < \tau \in \mathcal{J}.$$

Consequently, by assumption [H3] we can deduce that

$$\|\mathcal{F}x_n\| \leqslant \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds+$$

$$+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds+\frac{|\mu|}{|\beta+\gamma|}\leqslant$$

$$\leqslant \frac{[\delta_1 r^{q_1}+\delta_2]}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}ds+\frac{|\gamma|}{|\beta+\gamma|}\int_0^\tau (\tau-s)^{q-1}ds\right]+\frac{|\mu|}{|\beta+\gamma|}, \quad t<\tau\in\mathcal{J}.$$

Thus,

$$\|\mathcal{F}x_n\| \leqslant \left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\tau^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}+\frac{|\mu|}{|\beta+\gamma|} := \mathcal{K}. \tag{8}$$

Therefore $(\mathcal{F}x_n)$ is uniformly bounded on $\mathcal{M}$. Hence, $\mathcal{F}(\mathcal{M})$ is bounded in $\mathcal{D}_r \subseteq \mathcal{C}(\mathcal{J},\mathcal{X})$. Now, we need to prove that $(\mathcal{F}x_n)$ is equicontinuous. For $t_1, t_2 \in \mathcal{J}, \epsilon > 0$ and $t_1 \leqslant t_2$, let $\rho = \rho(\epsilon) > 0$ such that $\|t_2 - t_1\| < \rho$. Consider

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\| \leqslant \left\|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}\xi(s,x_n(s))ds-\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}\xi(s,x_n(s))ds\right\| \leqslant$$

$$\leqslant \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|\xi(s,x_n(s))\|ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|\xi(s,x_n(s))\|ds.$$

Consequently, by assumption [H3] we get

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\| \leqslant$$

$$\leqslant \frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right](\delta_1\|x_n\|^{q_1}+\delta_2)ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}(\delta_1\|x_n\|^{q_1}+\delta_2)ds \leqslant$$

$$\leqslant \frac{(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q)}\left[\frac{t_1^q}{q}+\frac{2(t_2-t_1)^q}{q}-\frac{t_2^q}{q}\right] \leqslant \frac{2(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}(t_2-t_1)^q < \frac{2\rho^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)} \equiv \epsilon.$$

Therefore, $(\mathcal{F}x_n)$ is equicontinuous. Since $\mathcal{F}$ is uniformly bounded and equicontinuous on $\mathcal{C}(\mathcal{J},\mathcal{X})$, then applying the Arzela Ascoli theorem, we get that $\mathcal{F}(\mathcal{M})$ is a relatively compact subset of $\mathcal{C}(\mathcal{J},\mathcal{X})$. Hence, $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. □

Remark 2.1. *As we proved $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is compact in Lemma (2.2). Consequently, by Proposition (1.2) $\mathcal{F}$ is $\alpha-$Lipschitz with zero constant.*

Theorem 2.1. *Assume that $[H1]-[H3]$ hold then the fractional BVP (1) has at least one solution.*

Proof. It is clear that, the fixed points of the operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ are solutions of BVP (1). Since the operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is continuous and completely continuous then we will prove that $\mathcal{S}(\mathcal{F})=\{x\in\mathcal{C}(\mathcal{J},\mathcal{X}) : x=k\mathcal{F}x,\ \textit{for some } k\in(0,1)\}$ is bounded. For $x\in\mathcal{S}(\mathcal{F})$ and $k\in(0,1)$, we have

$$\|x\| = k\|\mathcal{F}x\| \leqslant \left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\tau^q(\delta_1 r^{q_1}+\delta_2)}{\Gamma(q+1)}+\frac{|\mu|}{|\beta+\gamma|}.$$

The above inequality with $q_1 < 1$ and equation (8), show that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. Thus, by Schaefer's fixed point theorem, we can conclude that $\mathcal{F}$ has at least one fixed point and the set of fixed points of $\mathcal{F}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. □

Remark 2.2. *If $[H1]-[H3]$ hold and $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X})\to\mathcal{C}(\mathcal{J},\mathcal{X})$ is a linear operator then the set of solutions of the fractional BVP(1) is convex.*

Theorem 2.2. *Assume that $[H1]-[H3]$ hold then the fractional BVP(1) has a unique solution $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$.*

Proof. According to theorem (2.1), the fractional BVP (1) has a solution $x\in\mathcal{C}(\mathcal{J},\mathcal{X})$. It is sufficient to show that $\mathcal{F}$ is a contraction mapping on $\mathcal{C}(\mathcal{J},\mathcal{X})$ by [H2] as follows, for $x,y\in\mathcal{C}(\mathcal{J},\mathcal{X})$, we get

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\|\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds+$$

$$+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau(\tau-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds\leqslant$$

$$\leqslant\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\delta\|x-y\|ds+\frac{|\gamma|}{|\beta+\gamma|\Gamma(q)}\int_0^\tau(\tau-s)^{q-1}\delta\|x-y\|ds,\leqslant$$

$$\leqslant\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\delta\tau^q}{\Gamma(q+1)}\|x-y\|,\quad\beta+\gamma\neq0.$$

Thus, $\mathcal{F}$ is the contraction mapping on $\mathcal{C}(\mathcal{J},\mathcal{X})$ with a contraction constant $\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)\frac{\delta\tau^q}{\Gamma(q+1)}$. By applying Banach's contraction mapping principle we can conclude that the operator $\mathcal{F}$ has a unique fixed point on $\mathcal{C}(\mathcal{J},\mathcal{X})$. Hence, BVP(1) has a unique solution in $\mathcal{C}(\mathcal{J},\mathcal{X})$. □

Example 2.1. *Consider the following fractional BVP*

$$\begin{cases} {}^c\mathcal{D}^{\frac{2}{3}}x(t)=\dfrac{|x(t)|}{(9+e^t)(1+|x(t)|)}, & t\in[0,1],\\ x(0)+x(1)=0. \end{cases}\tag{9}$$

Set $q=\dfrac{2}{3}$, for $(t,x)\in[0,1]\times[0,+\infty)$, we can define $\xi(t,x)=\dfrac{x}{(9+e^t)(1+x)}$. Also, for $t\in[0,1]$ we have $x(t)=\dfrac{1}{9+e^t}$. For $x,y\in[0,+\infty)$, then

$$\Big|\xi(t,x)-\xi(t,y)\Big|=\left|\frac{x}{(9+e^t)(1+x)}-\frac{y}{(9+e^t)(1+y)}\right|\leqslant$$

$$\leqslant\frac{1}{10}\left|\frac{x}{(1+x)}-\frac{y}{(1+y)}\right|\leqslant\frac{1}{10}\left|\frac{x-y}{(1+x)(1+y)}\right|\leqslant$$

$$\leqslant\frac{1}{10}|x-y|\Rightarrow\delta=\frac{1}{10},\quad t\in[0,1].$$

If $q=\dfrac{2}{3}$, $\Gamma(q+1)=\Gamma\left(\frac{5}{3}\right)=0.89$, we have

$$\frac{\delta\tau^q\left(1+\frac{|\gamma|}{|\beta+\gamma|}\right)}{\Gamma(q+1)}=\frac{0.15}{0.89}<1.$$

Hence, we see that all assumptions in theorem (2.1) are satisfied which means our results can be used to solve BVP (9).

Conclusion

We have confirmed some sufficient conditions for the existence and uniqueness of solutions for BVP(1) based on the fixed Point theorems as well as the topological technique of approximate solutions. In addition, we studied some topological properties of the solutions set. Finally, an example is provided to verify our results.

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Существование и единственность решений краевых задач с дробной производной с помощью топологических структур

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Аннотация. В статье исследуется существование и единственность решений краевых задач с дробной производной Капуто в банаховом пространстве с помощью топологических структур с некоторыми соответствующими условиями. Он основан на применении топологических методов и теорем о неподвижной точке. Кроме того, рассматриваются некоторые топологические свойства множества решений. Приведен пример, иллюстрирующий основные результаты.

Ключевые слова: дробные производные и интегралы; топологические свойства отображений, теоремы о неподвижной точке