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Determination of Non-stationary Potential Analytical with Respect to Spatial Variables

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Abstract. The inverse problem of determining coefficient before the lower term of the hyperbolic equation of the second order is considered. The coefficient depends on time and n spatial variables. It is supposed that this coefficient is continuous with respect to variables t, x and it is analytic in other spatial variables. The problem is reduced to the equivalent integro-differential equations with respect to unknown functions. To solve this equations the scale method of Banach spaces of analytic functions is applied. The local existence and global uniqueness results are proven. The stability estimate is also obtained.

Keywords: inverse problem, Cauchy problem, fundamental solution, local solvability, Banach space.

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1. Introduction and problem formulation

The inverse problem of determining coefficient $a(t, x, y)$, $t \in \mathbb{R}$, $(x, y) = (x, y_1, \ldots, y_m) \in \mathbb{R}^{1+m}$, before the lower term of the hyperbolic equation is studied in this paper. The problem is considered in the class of coefficients that are continuous with respect to variables t, x and it is analytic in variable y. It is known that such problems are referred to as multidimensional inverse problems. For multidimensional inverse problems there are only special cases for which solvability is established. One of such classes of functions in which local solvability takes place is the class of analytic functions. The technique used here is based on the scale method of Banach spaces of analytic functions developed by L. V. Ovsyannikov [1, 2] and L. Nirenberg [3]. This method was first applied to the problem of solvability of multidimensional inverse problems by V. G. Romanov [4–6].

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This method was used to study multidimensional inverse problems of determining the convolution kernel in parabolic and hyperbolic integro-differential equations of the second order; theorems of local unique solvability of inverse problems in the class of functions with finite smoothness with respect to time variable and analytic with respect to spatial [7–13]. variables. This paper generalizes the results given in [4] (Sec. 3) for the case of non-stationary potential.

Let us consider the problem of determining a pair of functions u and a that satisfy the following equations

$$u_{tt} - u_{xx} - \triangle u - a(t,x,y)u = g(y)\delta(x)\delta'(t-t_0),\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0, \tag{1}$$

$$u\Big|_{t<0} \equiv 0, \tag{2}$$

where $\triangle$ is the Laplace operator with respect to variables $(y_1,\dots,y_m) = y$, $\delta(\cdot)$ is the Dirac delta function, $\delta'(\cdot)$ is the derivative of the Dirac delta function, t_0 is a problem parameter. Therefore $u = u(t,x,y,t_0)$, and $g(y)$ is a given smooth function so that $g(y) \neq 0$ for $y \in \mathbb{R}^m$.

It is required to find potential $a(x,t,y)$ in (1) if the solution of problem (1)–(3) is known for $x = 0$, i.e., the condition

$$u(t,0,y,t_0) = f(t,y,t_0),\ t > 0,\ t_0 > 0 \tag{3}$$

is given.

Following monograph [4, sec. 3], we consider the Banach space $A_s(r)$ $s > 0$ of functions $\varphi(y)$, $y \in \mathbb{R}^m$ which are analytic in the neighbourhood of the origin and they satisfy the following relation

$$\|\varphi\|_s(r) := \sup_{|y|<r} \sum_{|\alpha|=0}^{\infty} \frac{s^{|\alpha|}}{\alpha!} |D^\alpha \varphi(y)| < \infty.$$

Here $r > 0,\ s > 0$ and

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \dots \partial y_m^{\alpha_m}},\ \alpha := (\alpha_1,\dots,\alpha_m),$$

$$|\alpha| := \alpha_1 + \dots + \alpha_m,\ \alpha! := (\alpha_1)! \dots (\alpha_m)!.$$

In what follows, parameter r is fixed while parameter s is variable. Then, it is formed a scale of Banach spaces $A_s(r),\ s > 0$ of analytic functions. The following property is obvious: if $\varphi(y) \in A_s(r)$ then $\varphi(y) \in A_{s'}(r)$ for all $s' \in (0,s)$. Consequently, $A_s(r) \subset A_{s'}(r)$ if $s' \in (0,s)$ and the following inequality is valid

$$\|D^\alpha \varphi\|_{s'} \leqslant c_\alpha \frac{\|\varphi\|_s(r)}{(s-s')^{|\alpha|}}$$

for any α with constant c_α which depends only on α.

Solution of problem (1), (2) is considered in the form

$$u(t,x,y,t_0) = \frac{1}{2} g(y)\delta(t-t_0-|x|) + v(t,x,y,t_0).$$

Substituting this expression into (1) and taking into account that $(1/2)g(y)\delta(t-t_0-|x|)$ satisfies (in a general meaning) equation $u_{tt} - u_{xx} = g(y)\delta(x)\delta'(t-t_0)$, we obtain the following problem for function v:

$$
\begin{aligned}
v_{tt} - v_{xx} = \triangle v + \frac{1}{2}\triangle g(y)\delta(t - t_0 - |x|)+ \\
+ a(t,x,y)\left[\frac{1}{2}g(y)\delta(t - t_0 - |x|) + v(t,x,y,t_0)\right],\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0,
\end{aligned} \tag{4}
$$

$$
v\mid_{t<0} \equiv 0. \tag{5}
$$

In the next section inverse problem (4), (5) and (2) is replaced with the equivalent integro-differential equations. In what follows, we assume that function a is even in x.

2. Reduction of the problem to integro-differential equations

According to the d'Alembert formula the solution of problem (4)–(5) satisfies the integral equation

$$
\begin{aligned}
v(t,x,y,t_0) = \frac{1}{2}\iint\limits_{\Delta(t,x)}\Big\{\triangle v(\tau,\xi,y,t_0) + \frac{1}{2}\triangle g(y)\delta(\tau - t_0 - |\xi|)+ \\
+ a(\tau,\xi,y)\left[\frac{1}{2}g(y)\delta(\tau - t_0 - |\xi|) + v(\tau,\xi,y,t_0)\right]\Big\}d\xi d\tau,\ (t,x,y) \in \mathbb{R}^{2+m},\ t_0 > 0,
\end{aligned} \tag{6}
$$

where

$$
\Delta(t,x) = \{(\tau,\xi)|\ 0 \leqslant \tau \leqslant t - |x - \xi|,\ x - t \leqslant \xi \leqslant x + t\}.
$$

Let

$$
Q_T := \{(t,t_0)|\ 0 \leqslant t_0 \leqslant t \leqslant T\},\ T > 0,
$$

$$
\Omega_T := \{(t,x)|\ 0 \leqslant |x| \leqslant t \leqslant T - |x|\},
$$

$$
\Upsilon_T := \Big\{(t,x,t_0)\Big|\ |x| + t_0 \leqslant t \leqslant T - |x|,\ 0 \leqslant t_0 \leqslant t \leqslant T\Big\}.
$$

Domain Υ_T in the space of variables x, t, t_0 is the pyramid with the base Ω_T and vertex $(0,T,T)$.

It follows from (6) that function $v(t,x,y,t_0)$ satisfies the integral equation

$$
\begin{aligned}
v(t,x,y,t_0) = \frac{\triangle g(y)}{4}(t - t_0) + \frac{g(y)}{4}\int\limits_{\frac{x-(t-t_0)}{2}}^{\frac{x+(t-t_0)}{2}} a(t_0 + |\xi|,\xi,y)d\xi+ \\
+\frac{1}{2}\iint\limits_{\square(t,x,t_0)}\left[\triangle v(\tau,\xi,y,t_0) + a(\tau,\xi,y)v(\tau,\xi,y,t_0)\right]d\tau d\xi,\quad (t,x,t_0) \in \Upsilon_T,\quad y \in \mathbb{R}^m,
\end{aligned} \tag{7}
$$

where $\theta(t) = 1,\ t \geqslant 0,\ \theta(t) = 0,\ t < 0$, and $\square(x,t,t_0)$ is domain in the form of a rectangle in the plane of variables (τ,ξ) for each fixed t_0 formed by characteristics passing through the points $(0,t_0)$ and (x,t) of the differential operator $\partial^2/\partial t^2 - \partial^2/\partial x^2$:

$$
\square(x,t,t_0) := \left\{(\xi,\tau)\Big|\ |\xi| + t_0 < \tau < t - |x - \xi|,\ \frac{x - (t - t_0)}{2} < \xi < \frac{x + t - t_0}{2},\ 0 < t_0 < t\right\}.
$$

Obviously, the equalities $f(t,y,t_0) = u(t,0,y,t_0) = v(t,0,y,t_0)$, $t > t_0$ are true. Besides, $f(t_0 + 0,y,t_0) = v(t,0,y,t_0)\big|_{t=t_0+0} = 0$.

First note that if $a(t,-x,y)=a(t,x,y)$ then $v(t,-x,y,t_0)=v(t,x,y,t_0)$. Taking the derivative with respect to t of the both sides of equation (7), we obtain

$$v_t(t,x,y,t_0)=$$
$$=\frac{\triangle g(y)}{2}+\frac{g(y)}{8}\left[a\left(\frac{x+t+t_0}{2},\frac{x+t-t_0}{2},y\right)+a\left(\frac{-x+t+t_0}{2},\frac{x-t+t_0}{2},y\right)\right]+$$
$$+\frac{1}{2}\int\limits_{\frac{x-(t-t_0)}{2}}^{\frac{x+(t-t_0)}{2}}\left[\triangle v\left(t-|x-\xi|,\xi,y,t_0\right)+a\left(t-|x-\xi|,\xi,y\right)v\left(t-|x-\xi|,\xi,y,t_0\right)\right]d\xi,$$
$$(t,x,t_0)\in\Upsilon_T,\ y\in\mathbb{R}^m.$$

Setting $x=0$ in this relation and using evenness of functions $a(t,x,y),\ v(t,x,y,t_0)$ with respect to x, we obtain the equality

$$f_t(t,y,t_0)=\frac{\triangle g(y)}{2}+\frac{g(y)}{4}a\left(\frac{t-t_0}{2},\frac{t+t_0}{2},y\right)+$$
$$+\int_0^{\frac{t-t_0}{2}}\left[\triangle v\left(t-\xi,\xi,y,t_0\right)+a\left(t-\xi,\xi,y\right)v\left(t-\xi,\xi,y,t_0\right)\right]d\xi,\ (t,t_0)\in Q_T,\ y\in\mathbb{R}^m.$$

Substituting $|x|$ for $(t-t_0)/2$ and t for $(t+t_0)/2$ and solving with respect to $a(t,x,y)$, we rewrite this equation in the form

$$a(t,x,y)=\frac{2\triangle g(y)}{g(y)}+\frac{4}{g(y)}f_t\left(t+|x|,y,t-|x|\right)-\frac{4}{g(y)}\int_0^{|x|}\Big[\triangle v\left(t+|x|-\xi,\xi,y,t-|x|\right)+$$
$$+a\left(t+|x|-\xi,\xi,y\right)v\left(t+|x|-\xi,\xi,y,t-|x|\right)\Big]d\xi,\ (t,x)\in\Omega_T,\ y\in\mathbb{R}^m. \qquad (8)$$

Thus, in order to find the value of function a at the point (t,x,y) it is necessary to integrate function $a(t,x,y)$ itself over the segment with boundaries $(t+|x|,0,y,0)$, $(t,|x|,y,\ 0)$ and function $v(t,x,y,t_0)$ over the segment with boundaries $(t+|x|,0,y,\ t-|x|)$, $(\ t,|x|,y,\ t-|x|)$ which belong to domain $\Upsilon_T\times\mathbb{R}^m$.

Note that function v, even with respect to $x=0$, satisfies the condition $\partial v/\partial x|_{x=0}$. Taking into account this fact and considering equations (4), (5), (3) for v in the domain $x>0$, we obtain

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-\triangle v-a(t,x,y)v=0,\ 0<x<t-t_0,\ y\in\mathbb{R}^m,$$
$$v\Big|_{x=0}=f(t,y,t_0),\ \frac{\partial v}{\partial x}\Big|_{x=0}=0,\ 0<t-t_0\leqslant T,\ y\in\mathbb{R}^m$$

. Then in accordance with the d'Alembert formula which gives the Cauchy problem solution with an initial data at $x=0$ we find

$$v(t,x,y,t_0)=v_0(t,x,y,t_0)+\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big[\triangle v\left(\tau,\xi,y,t_0\right)+a\left(\tau,\xi,y\right)v\left(\tau,\xi,y,t_0\right)\Big]d\xi d\tau, \qquad (9)$$

where

$$v_0(t,x,y,t_0)=\frac{1}{2}\left[f(t+x,y,t_0)+f(t-x,y,t_0)\right],$$

$$\triangle'(t,x) := \{(\tau,\xi) \mid 0<\xi<x, |\tau-t|<x-\xi\}, \quad 0<x<t-t_0<T-x, \quad y\in\mathbb{R}^m.$$

Considering (8) for $x \geqslant 0$, we have

$$a(t,x,y) = a_0(t,x,y) - \\ -\frac{4}{g(y)}\int_0^x \Big[\triangle v\,(t+x-\xi,\xi,y,t-x) + a(t+x-\xi,\xi,y)v(t+x-\xi,\xi,y,t-x)\Big]d\xi, \tag{10}$$

where

$$a_0(t,x,y) = \frac{4}{g(y)}f_t(t+x,y,t-x) + \frac{2\triangle g(y)}{g(y)}, \quad 0\leqslant x\leqslant t\leqslant T-x, \quad y\in\mathbb{R}^m.$$

The system of equations (9), (10) is a closed integro-differential equations for functions a, v. Note that operator $\triangle$ for function v appears in the system only under the integral sign.

Next we consider system (9), (10) in domain

$$D_T = \Upsilon'_T\times\mathbb{R}^m, \quad \Upsilon'_T = \{(t,x,t_0)|\ 0\leqslant x+t_0\leqslant t\leqslant T-x\}.$$

3. The main results and proofs

Let $C_{(t,x,t_0)}(\Upsilon'_T;\ A_{s_0})$ denote the class of functions with values in A_{s_0} ($s_0>0$) which are continuous with respect to variables (t,x,t_0) in domain Υ'_T. For fixed (t,x,t_0) the norm of function $v(t,x,y,t_0)$ in A_{s_0} is denoted by $\|v\|_{s_0}(t,x,t_0)$. The norm of function v in $C_{(t,x,t_0)}(\Upsilon'_T;\ A_{s_0})$ is defined by the equality

$$\|v\|_{C_{(t,x,t_0)}(\Upsilon'_T;\,A_{s_0})} = \sup_{(t,x,t_0)\in\Upsilon'_T}\|v\|_{s_0}(t,x,t_0).$$

Let $C_{(t,x)}(G_T;\ A_{s_0})$ be a class of functions with values in A_{s_0} which are continuous with respect to variables (t,x) in domain $G_T=\{(t,x)|\ 0\leqslant x\leqslant t\leqslant T-x\}$. For fixed (t,x) the norm of function $a(t,x,y)$ in A_{s_0} is denoted by $\|a\|_{s_0}(t,x)$. The norm of function a in $C_{(t,x)}(G_T;\ A_{s_0})$ is defined as

$$\|a\|_{C_{(t,x)}(G_T;\,A_{s_0})} = \sup_{(t,x)\in G_T}\|a\|_{s_0}(t,x).$$

Let us also denote the class of functions with values in A_{s_0} which are continuous with respect to t, t_0 in domain Q_T by $C(Q_T;A_{s_0})$.

Theorem 3.1. *Let $f(+t_0,y,t_0)=0$, $|g(y)|\geqslant g_0>0$, g_0 is a known number and*

$$\left\{\frac{1}{g(y)},\ \frac{\triangle g(y)}{g(y)}\right\}\in A_{s_0};\ \{f(t,y,t_0),\ f_t(t,y,t_0)\}\in C(Q_T;A_{s_0}),$$

in addition, the relations

$$\max\left\{2\left\|\frac{\triangle g(y)}{g(y)}\right\|_{s_0},\ \max_{(t,t_0)\in Q_T}\|f(t,y,t_0)\|_{s_0},\ \max_{(t,t_0)\in Q_T}\left\|\frac{4f_t(t,y,t_0)}{g(y)}\right\|_{s_0}\right\}\leqslant\frac{R}{2}$$

are valid for some fixed $s_0>0$, R. Then there is such a number $b\in(0,T/(2s_0))$, $b=b(s_0,R,T)$ that for each $s\in(0,s_0)$ in domain $D_T\cap\{(t,x,y,t_0):0\leqslant x+t_0\leqslant b(s_0-s)\}$ there exists the unique solution of equations (9), (10) *and $v(t,x,y,t_0)\in C_{(t,x,t_0)}(P_{sT};\ A_{s_0})$,*

$a(t,x,y) \in C_{(t,x)}\left(K_{sT};\ A_{s_0}\right)$, *where* $P_{sT} = \Upsilon'_T \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\}$, $K_{sT} = G_T \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\}$, *moreover*

$$\|v - v_0\|_s (t,x,t_0) \leqslant R,\ (t,x,t_0) \in P_{sT},$$

$$\|a - a_0\|_s (t,x) \leqslant \frac{R}{s_0 - s},\ (t,x) \in K_{sT}.$$

Proof. Under the conditions of Theorem 1 we have

$$v_0 \in C_{(t,x,t_0)}\left(\Upsilon'_T;\ A_{s_0}\right),\quad a_0 \in C_{(t,x)}\left(G_T;\ A_{s_0}\right),$$

$$\|v_0\|_s(t,x,t_0) \leqslant R,\ (t,x,t_0) \in \Upsilon'_T,\quad \|a_0\|_s(t,x) \leqslant R,\quad (t,x) \in G_T,\quad 0 < s < s_0.$$

Let b_n be the member of the monotone decreasing sequence that is defined by the equalities

$$b_{n+1} = \frac{b_n}{1 + 1/(n+1)^2},\ n = 0,\ 1,\ 2,\ \ldots.$$

Let

$$b = \lim_{n\to\infty} b_n = b_0 \prod_{n=0}^{\infty} \left(1 + 1/(n+1)^2\right)^{-1}.$$

The number $b_0 \in (0, T/(2s_0))$ is chosen in an appropriate way. For the system of equations (9), (10) the process of successive approximations is constructed according to the following scheme

$$v_{n+1}(t,x,y,t_0) = v_0(t,x,y,t_0)+$$
$$+\frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big[\triangle v_n(\tau,\xi,y,t_0) + a_n(\tau,\xi,y)\, v_n(\tau,\xi,y,t_0)\Big]\, d\tau d\xi,\ 0 \leqslant x \leqslant t - t_0 \leqslant T - x,$$

$$a_{n+1}(t,x,y) = a_0(t,x,y)-$$
$$-\frac{4}{g(y)}\int_0^x \Big[\triangle v_n(t+x-\xi,\xi,y,t-x) + a_n(t+x-\xi,\xi,y) v_n(t+x-\xi,\xi,y,t-x)\Big] d\xi,$$
$$0 \leqslant x \leqslant t \leqslant T - x.$$

Function $s'_n(x)$ is defined by the formula

$$s'_n(x) = \frac{s + \nu^n(x)}{2},\quad \nu^n(x) = s_0 - \frac{x}{b_n}. \tag{11}$$

Let us introduce the following notations: $p_n = v_{n+1} - v_n$, $q_n = a_{n+1} - a_n$, $n = 0,\ 1,\ 2, \ldots.$ Then $p_n,\ q_n$ satisfy the relations

$$p_0(t,x,y,t_0) = \frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big[\triangle v_0(\tau,\xi,y,t_0) + a_0(\tau,\xi,y)\, v_0(\tau,\xi,y,t_0)\Big] d\tau d\xi,\quad (t,x,y,t_0) \in D_T,$$

$$q_0(t,x,y) = -\frac{4}{g(y)}\int_0^x \Big\{\triangle v_0(t+x-\xi,\xi,y,t-x)+$$
$$+ a_0(t+x-\xi,\xi,y) v_0(t+x-\xi,\xi,y,t-x)\Big\} d\xi,\quad (t,x,y) \in G_T \times \mathbb{R}^m;$$

$$p_{n+1}(t,x,y,t_0) = \frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big\{\triangle p_n(\tau,\xi,y,t_0)+$$
$$+\, q_n\,(\tau,\xi,y)\, v_{n+1}(\tau,\xi,y,t_0) + a_n\,(\tau,\xi,y)\, p_n\,(\tau,\xi,y,t_0)\Big\}\, d\tau d\xi, \quad (t,x,y,t_0)\in D_T,$$

$$q_{n+1}(t,x,y) = -\frac{4}{g(y)}\int_0^x \Big\{\triangle p_n(t+x-\xi,\xi,y,t-x)+$$
$$+\, q_n(t+x-\xi,\xi,y)v_{n+1}(t+x-\xi,\xi,y,t-x) + a_n(t+x-\xi,\xi,y)p_n(t+x-\xi,\xi,y,t-x)\Big\}d\xi,$$
$$(t,x,y)\in G_T\times\mathbb{R}^m.$$

Let us show that $b_0 \in \left(0, \dfrac{T}{2s_0}\right)$ can be chosen so that the following inequalities be valid for all $n = 0,\, 1,\, 2, \cdots$:

$$\lambda_n = \max\Bigg\{\sup_{(t,x,s)\in\hat{F}_n}\left[\|p_n\|_s(t,x,t_0)\frac{\nu^n(x)-s}{x}\right], \sup_{(t,x,s)\in F_n}\left[\|q_n\|_s(t,x)\frac{(\nu^n(x)-s)^2}{x}\right]\Bigg\} < \infty, \quad (12)$$

$$\|\tilde{v}_{n+1}-v_0\|_s(t,x,t_0)\leqslant R, \quad \|a_{n+1}-a_0\|_s(t,x)\leqslant \frac{R}{s_0-s}, \quad (13)$$

where

$$\hat{F}_n = \{(t,x,t_0,s)|\ \ (t,x,t_0)\in\Upsilon'_T, \ \ 0\leqslant x+t_0 < b_n(s_0-s), \ \ 0<s<s_0\},$$
$$F_n = \{(t,x,s)|\ (t,x)\in G_T,\ 0\leqslant x < b_n(s_0-s),\ 0<s<s_0\}.$$

Indeed, using the relations for p_n, q_n, one can find

$$\|p_0\|_s(t,x,t_0)\leqslant \frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big[\|\triangle v_0\|_s\,(\tau,\xi,t_0)+\|a_0\|_s\,(\tau,\xi)\,\|v_0\|_s\,(\tau,\xi,t_0)\Big]d\tau d\xi\leqslant$$
$$\leqslant \frac{1}{2}\iint\limits_{\triangle'(t,x)}\left[\frac{Rc_0}{(s'_0(\xi)-s)^2}+R^2\right]d\tau d\xi.$$

Here c_0 is a positive constant such that

$$\|\triangle v_0\|_s\leqslant c_0\frac{\|v_0\|_{s'_n}}{(s'_n-s)^2}, s'_n>s>0,\ n=0,1,2,\ldots$$

. It is easy to check that $c_0 = 4m$.

Taking function $s'_n(\xi)$ from (11) for $n=0$, we have

$$\|p_0\|_s(t,x,t_0)\leqslant \frac{1}{2}\int_0^x (x-\xi)\left[\frac{4Rc_0}{(\nu^0(\xi)-s)^2}+R^2\right]d\xi\leqslant$$
$$\leqslant \frac{1}{2}R\left[4c_0+s_0^2R\right]\int_0^x\frac{(x-\xi)d\xi}{(\nu^0(\xi)-s)^2}\leqslant$$
$$\leqslant \frac{1}{2}b_0R\left[4c_0+s_0^2R\right]\frac{x}{\nu^0(x)-s},\ (t,x,s)\in\hat{F}_0.$$

In a similar way we obtain

$$\|q_0\|_s(t,x) \leqslant 4g_0 \int_0^x \left[\frac{4Rc_0}{(\nu^0(\xi)-s)^2} + R^2\right] d\xi \leqslant$$
$$\leqslant 4g_0 R\left(4c_0 + s_0^2 R\right)\frac{x}{(\nu^0(x)-s)^2},\ (t,x,s)\in F_0.$$

These estimates imply that inequality (12) is valid for $n=0$. Moreover,we find

$$\|\tilde v_1 - \tilde v_0\|_s(t,x,t_0) = \|p_0\|_s(t,x,t_0) \leqslant \frac{\lambda_0 x}{\nu^0(x)-s} \leqslant \frac{\lambda_0 b_1}{1-b_1/b_0} = b_0\lambda_0,\ (t,x,t_0,s)\in \hat F_1,$$

$$\|a_1 - a_0\|_s(t,x) = \|a_0\|_s(t,x) \leqslant \frac{\lambda_0 x}{(\nu^0(x)-s)^2} \leqslant \frac{4b_0\lambda_0}{s_0 - s},\ (t,x,s)\in F_1.$$

Choosing b_0 so that $4b_0\lambda_0 \leqslant R$, one can conclude that inequalities (13) are satisfied for $n=0$.

By way of induction, one can show that inequalities (12) and (13) are also valid for other values of n if b_0 is chosen suitably. Let us assume that inequalities (12) and (13) hold for $n = 0, 1, 2, \ldots, i$. Then $(t,x,t_0,s)\in \hat F_{i+1}$ and we have

$$\|p_{i+1}\|_s(t,x,t_0) \leqslant \frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big\{\|\triangle p_i\|_s(\tau,\xi,t_0) +$$
$$+\|q_i\|_s(\tau,\xi)\,\|v_{i+1}\|_s(\tau,\xi,t_0) + \|a_i\|_s(\tau,\xi)\,\|p_i\|_s(\tau,\xi,t_0)\Big\}\, d\tau d\xi \leqslant$$
$$\leqslant \frac{1}{2}\iint\limits_{\Delta(t,x)}\left[\frac{c_0\lambda_i\xi}{(s_i'(\xi)-s)^2(\nu^i(\xi)-s)} + \frac{2R\lambda_i\xi}{(\nu^i(\xi)-s)^2} + \frac{\lambda_i\xi}{(\nu^i(\xi)-s)}\frac{R(1+s_0)}{(s_0-s)}\right] d\tau d\xi \leqslant$$
$$\leqslant \frac{\lambda_i}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right)\int_0^x \frac{(x-\xi)\xi d\xi}{(\nu^{i+1}(\xi)-s)^3} \leqslant$$
$$\leqslant \frac{\lambda_i}{2} b_0^2\left(4c_0 + 3Rs_0 + Rs_0^2\right)\frac{x}{\nu^{i+1}(x)-s}.$$

Here function s_i' is defined by equality (11) with $n=i$ and the inequalities

$$\|v_i\|_s(t,x,t_0) \leqslant 2R,\quad \|a_i\|_s(t,x) \leqslant R\frac{1+s_0}{s_0-s}$$

are used. The latter is valid by the induction hypothesis together with the obvious inequalities $b_i \leqslant b_0$ and $\nu^{i+1}(x) \leqslant \nu^i(x)$. Similar arguments for q_{i+1} lead to inequalities

$$\|q_{i+1}\|_s(t,x) \leqslant 4g_0\int_0^x \left\{\frac{c_0\lambda_i\xi}{(s_i'(\xi)-s)^2(\nu^i(\xi)-s)} + \frac{2R\lambda_i\xi}{\nu^i(\xi)-s} + \frac{\lambda_i R(1+s_0)\xi}{(\nu^i(\xi)-s)^2}\right\} d\xi \leqslant$$

$$\leqslant 4\lambda_i g_0\left[4c_0 + 3Rs_0^2 + Rs_0\right]\int_0^x \frac{\xi d\xi}{(\nu^{i+1}(\xi)-s)^3} \leqslant$$

$$\leqslant 4\lambda_i g_0 b_0\left[4c_0 + 3Rs_0^2 + Rs_0\right]\frac{x}{(\nu^{i+1}(x)-s)^2},\ (t,x,s)\in F_{i+1}.$$

The obtained estimates yield

$$\lambda_{i+1} \leqslant \lambda_i\rho,\quad \lambda_{i+1} < \infty,$$

$$\rho = b_0 \max\left[\frac{1}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right); 4g_0\left(4c_0 + 3Rs_0^2 + Rs_0\right)\right].$$

Moreover, we have

$$\|\tilde{v}_{i+2} - v_0\|_s(t,x,t_0) \leqslant \sum_{n=0}^{i+1} \|p_n\|_s(t,x,t_0) \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n x}{\nu^n(x) - s} \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n b_{i+2}}{1 - b_{i+2}/b_n} \leqslant$$
$$\leqslant \sum_{n=0}^{i+1} \lambda_n b_n (n+1)^2 \leqslant \lambda_0 b_0 \sum_{n=0}^{i+1} \rho^n (n+1)^2,\ (t,x,t_0,s) \in \hat{F}_{i+2},$$

$$\|a_{i+2} - a_0\|_s(t,x) \leqslant \sum_{n=0}^{i+1} \|q_n\|_s(t,x) \leqslant \sum_{n=0}^{i+1} \frac{\lambda_n x}{(\nu^n(x) - s)^2} \leqslant \frac{1}{s_0 - s} \sum_{n=0}^{i+1} \frac{\lambda_n b_{i+2}}{(1 - b_{i+2}/b_n)^2} \leqslant$$
$$\leqslant \frac{\lambda_0 b_0}{s_0 - s} \sum_{n=0}^{i+1} \rho^n (n+1)^4,\ (t,x,s) \in F_{i+2}.$$

Now we choose $b_0 \in \left(0, \frac{T}{2s_0}\right)$ so as to obtain

$$\rho < 1,\ \lambda_0 b_0 \sum_{n=0}^{\infty} \rho^n (n+1)^4 \leqslant R.$$

Then

$$\|v_{i+2} - v_0\|_s(t,x,t_0) \leqslant R, \quad (t,x,t_0,s) \in \hat{F}_{i+2},$$
$$\|a_{i+2} - a_0\|_s(t,x) \leqslant \frac{R}{s_0 - s},\ (t,x,s) \in F_{i+2}.$$

Since the choice of b_0 is independent of the number of approximations, all successive approximations v_n, a_n belong to

$$C_{(t,x,t_0)}\left(\hat{F}; A_s\right),\ \hat{F} = \bigcap_{n=0}^{\infty} \hat{F}_n$$

and

$$C_{(t,x)}\left(F; A_s\right),\ F = \bigcap_{n=0}^{\infty} F_n,$$

respectively. Moreover,

$$\|v_n - v_0\|_s(t,x,t_0) \leqslant R,\ (t,x,t_0,s) \in \hat{F},$$
$$\|a_n - a_0\|_s(t,x) \leqslant \frac{R}{s_0 - s},\ (t,x,s) \in F.$$

For $s \in (0, s_0)$ the series

$$\sum_{n=0}^{\infty}(v_n - v_{n-1}), \quad \sum_{n=0}^{\infty}(a_n - a_{n-1})$$

converge uniformly in the norm of the spaces

$$C_{(t,x,t_0)}\left(P_{sT}; A_s\right),\ P_{sT} = \Upsilon_T' \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\},$$

$$C_{(t,x)}\left(K_{sT}; A_s\right),\ K_{sT} = G_T \cap \{(t,x,t_0) : 0 \leqslant x + t_0 < b(s_0 - s)\}.$$

Therefore $v_n \to v$, $a_n \to a$ and the limit functions v, a are elements of $C_{(t,x,t_0)}(P_{sT}; A_s)$, $C_{(t,x)}(K_{sT}; A_s)$ respectively and they satisfy equations (9), (10).

Now we prove that this solution is unique. Let us assume that $(v,\, a)$ and $(\widehat{v},\, \widehat{a})$ are any two solutions that satisfy the inequalities

$$\|v - v_0\|_s(t,x,t_0) \leqslant R,\ (t,x,t_0,s) \in \hat{F},$$

$$\|a - a_0\|_s(t,x) \leqslant \frac{R}{s_0 - s},\ (t,x,s) \in F.$$

Let us denote $\tilde{p} = v - \widehat{v}$, $\tilde{q} = a - \widehat{a}$,

$$\lambda := \max\left\{ \sup_{(t,x,t_0,s)\in\hat{F}} \left[\|\tilde{p}\|_s(t,x,t_0)\frac{\nu(x)-s}{x}\right],\quad \sup_{(t,x,s)\in F}\left[\|\tilde{q}\|_s(t,x)\frac{(\nu(x)-s)^2}{x}\right]\right\} < \infty$$

, where $\nu(x) = s_0 - x/b$, $b = b_0 \prod\limits_{n=0}^{\infty}\left(1 + 1/(n+1)^2\right)^{-1}$. Then the following relations can be obtained for functions $\tilde{p}$, $\tilde{q}$

$$\tilde{p}(t,x,y,t_0) = \frac{1}{2}\iint\limits_{\triangle'(t,x)} \Big\{\triangle\tilde{p}\,(\tau,\xi,y,t_0) + \tilde{q}\,(\tau,\xi,y)\,\widehat{v}(\tau,\xi,y,t_0) + a\,(\tau,\xi,y)\,\tilde{p}\,(\tau,\xi,y,t_0)\Big\}\,d\tau d\xi,$$

$$(t,x,y,t_0) \in D_T,$$

$$\tilde{q}(t,x,y) = -\frac{4}{g(y)}\int_0^x \Big\{\triangle\tilde{p}\,(t+x-\xi,\xi,y,t-x) +$$

$$+\tilde{q}(t+x-\xi,\xi,y)\widehat{v}(t+x-\xi,\xi,y,t-x) + a(t+x-\xi,\xi,y)\tilde{p}(t+x-\xi,\xi,y,t-x)\Big\}d\xi,$$

$$G_T \times \mathbb{R}^m.$$

Let us show that $b_0 \in \left(0, \dfrac{T}{2s_0}\right)$ can be chosen so that the following inequalities are valid for all $n = 0,\, 1,\, 2, \ldots$. Applying the estimates given above to these equations, we find the inequality

$$\lambda \leqslant \lambda\rho',$$

$$\rho' := b\max\left[\frac{1}{2}\left(4c_0 + 3Rs_0 + Rs_0^2\right);\, \frac{4}{\|g\|_s}\left(4c_0 + 3Rs_0^2 + Rs_0\right)\right] < \rho < 1.$$

Consequently $\lambda = 0$. Therefore $v = \widehat{v}$, $a = \widehat{a}$. Theorem 1 is proved. □

Let us consider the set Γ of functions $f(t,y,t_0)$ representing the elements of $C(Q_T;\, A_{s_0})$, $s_0 > 0$ for which conditions of Theorem 1 are valid with R, T, s_0. Then we have the stability theorem

Theorem 3.2. *Let f, $\bar{f} \in \Gamma$. For the corresponding solutions (v,a) and $(\bar{v},\bar{a})$ of (9), (10), we have*

$$\|v - \bar{v}\|_s \leqslant cM,\ (t,x,t_0) \in P_{sT},\quad \|a - \bar{a}\|_s \leqslant \frac{cM}{s_0 - s},\ (t,x) \in K_{sT},\ 0 < s < s_0, \tag{14}$$

where

$$M = \max\left[\max\|f - \bar{f}\|_{s_0}(t,t_0),\ \max\|f_t - \bar{f}_t\|_{s_0}(t,t_0)\right],\ (t,t_0) \in Q_T,$$

and constant c depends on R, T, s_0.

Proof. Taking into account (9)–(10), we obtain the following equalities for the differences $v-\bar{v}=\tilde{v},\ a-\bar{a}=\tilde{a}$ and $f-\bar{f}=\tilde{f}$

$$\tilde{v}(t,x,y,t_0)=\tilde{v}_0(t,x,y,t_0)+\frac{1}{2}\iint\limits_{\triangle'(t,x)}\Big\{\triangle\tilde{v}\,(\tau,\xi,y,t_0)+ \\ +\tilde{a}\,(\tau,\xi,y)\,v(\tau,\xi,y,t_0)+\bar{a}\,(\tau,\xi,y)\,\tilde{v}\,(\tau,\xi,y,t_0)\Big\}\,d\tau d\xi,\ (t,x,y,t_0)\in D_T, \tag{15}$$

$$\tilde{a}(t,x,y)=\tilde{a}_0(t,x,y)-\frac{4}{g(y)}\int_0^x\Big\{\triangle\tilde{v}\,(t+x-\xi,\xi,y,t-x)+ \\ +\tilde{a}(t+x-\xi,\xi,y)v(t+x-\xi,\xi,y,t-x)+\bar{a}(t+x-\xi,\xi,y)\tilde{v}(t+x-\xi,\xi,y,t-x)\Big\}d\xi, \\ (t,x,y)\in G_T\times\mathbb{R}^m, \tag{16}$$

where

$$\tilde{v}_0(t,x,y,t_0)=\frac{1}{2}[\tilde{f}(t+x,y,t_0)+\tilde{f}(t-x,y,t_0)],\quad \tilde{a}_0(t,x,y)=\frac{4}{g(y)}\tilde{f}_t(t+x,y,t-x).$$

It is obvious that

$$\|\tilde{v}_0\|_{s_0}(t,x,t_0)\leqslant M,\ (t,x,t_0)\in P_{sT}, \\ \|\tilde{a}_0\|_{s_0}(t,x)\leqslant\frac{4}{\|g(y)\|_{s_0}}M,\ (t,x)\in K_{sT}. \tag{17}$$

We have from Theorem 1 that

$$\|v\|_s\leqslant 2R,\ \|a\|_s\leqslant\frac{R(1+s_0)}{s_0-s}.$$

Applying the method of successive approximations used for the proof of Theorem 1 to the system of equations (15)–(16) (it is linear with respect to $\tilde{v}$ and $\tilde{a}$), we find that the following inequalities are valid for solution of (15)–(16)

$$\|\tilde{v}-\tilde{v}_0\|_s(t,x,t_0)\leqslant c_1M,\ (t,x,t_0)\in P_{sT}, \\ \|\tilde{a}-\tilde{a}_0\|_s(t,x)\leqslant\frac{c_1M}{s_0-s},\ (t,x)\in K_{sT},\ 0<s<s_0,$$

where c_1 depends on R, T, s_0. Hence, taking into account (17), we find that inequalities (14) are true. Theorem 2 is proved. □

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Определение нестационарного потенциала, аналитического по пространственным переменным

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Аннотация. Изучена обратная задача определения коэффициента зависимости временных и n пространственных переменных для младшего члена гиперболического уравнения второго порядка. Предполагается, что этот коэффициент непрерывен по отношению к переменным t, x и аналитичен по другим пространственным переменным. Задача сводится к эквивалентной системе нелинейных интегро-дифференциальных уравнений относительно неизвестных функций. Для решения этих уравнений применяется метод шкал банаховых пространств аналитических функций. Доказаны теоремы локальной разрешимости и единственности в глобальном смысле. Получена оценка устойчивости обратной задачи.

Ключевые слова: обратная задача, фундаментальное решение, задача Коши, локальная разрешимость, устойчивость.