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Centralizers in the Diagonal Direct Limits of the Symmetric Groups and of the Monomial Groups

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Abstract. Centralizers of finite subgroups in the direct limit of the finite and finitary symmetric groups via strictly diagonal embeddings is characterized in 2015 by Güven, Kegel and Kuzucuoğlu. In this paper this idea is extended to the case of diagonal embeddings. In addition, a new class of infinite limit monomial groups is constructed via diagonal embedding.

Keywords: Centralizer, simple locally finite, diagonal embedding, level-preserving.

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The class of groups which is constructed as the direct limit of symmetric groups is widely studied. The concept first introduced by Zaleskiy [7], after that Kegel–Wehrfritz [2] and Kroshko–Suschansky in [3] studied these class. The classification of limit symmetric groups of diagonal type is given in [5].

Since the centralizers played an essential role for the classification of finite simple groups, it is natural to ask: What is the structure of centralizers of elements and finite subgroups in the locally finite simple groups constructed as the direct limit of symmetric groups?
For the limit groups of strictly diagonal type the answer is given in [1]. In the first section of this paper, the structure of centralizers of elements and subgroups in the symmetric groups of diagonal type will be given.

1. Diagonal embeddings

Definition 1. *An embedding ϕ of the permutation group (G, X) into the permutation group (H, Y) is called* ***diagonal embedding*** *if $(\phi(G), \Delta)$ is isomorphic to (G, X) for any orbit Δ of G having more than 1 element. In addition, if all orbits of $\phi(G)$ have more than 1 element, then the embedding is called* ***strictly diagonal embedding.***

Consider the embedding of finite symmetric groups as follows:

$$d(r, k) : S_n \longrightarrow S_{nr+k}.$$

For any $\alpha \in S_n$, $d(r, k)(\alpha) \in S_{nr+k}$ is determined as follows:

$$((i-1)r + t)^{d(r,k)(\alpha)} = (i^\alpha - 1)r + t \text{ where } 1 \leqslant t \leqslant r, \quad 1 \leqslant i \leqslant n.$$

Hence, if $\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$, then

$$d(r,k)(\alpha) = \left(\begin{smallmatrix} 1 & 2 & \cdots & r \\ (i_1-1)r+1 & (i_1-1)r+2 & \cdots & i_1 r \end{smallmatrix} \middle| \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \middle| \begin{smallmatrix} (n-1)r+1 & (n-1)r+2 & \cdots & nr \\ (i_n-1)r+1 & (i_n-1)r+2 & \cdots & i_n r \end{smallmatrix} \middle| \begin{smallmatrix} nr+1 \\ nr+1 \end{smallmatrix} \middle| \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \middle| \begin{smallmatrix} nr+k \\ nr+k \end{smallmatrix} \right)$$

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Lemma 2. *$d(r,k)$ is a diagonal embedding.*

Proof. First let us determine the forms of arbitrary orbits of $d(r,k)(S_n)$ in the set $\{1,2,\ldots,nr+k\}$. Since the action is trivial on the points i where $nr+1 \leqslant i \leqslant nr+k$, the orbit $\Delta_i = \{i^{d(r,k)(\alpha)} \mid \alpha \in S_n\} = \{i\}$. The non-trivial orbits are of the form $\Delta_i = \{i, r+i, 2r+i, \cdots, (n-1)r+i\}$ for all $1 \leqslant i \leqslant r$. Note that the length of the orbits are n. Define a map, $\sigma : \{1,2,\ldots,n\} \to \Delta_i$ where $\sigma(j) = (j-1)r+i$.

Now, the group $(S_n, \{1,2,\ldots,n\})$ is permutationally isomorphic to $(d(r,k)(S_n), \Delta_i)$ as follows:

For any $j \in \{1,2,\ldots,n\}$ and $\alpha \in S_n$,

$$\sigma(j)^{d(r,k)(\alpha)} = ((j-1)r+i)^{d(r,k)(\alpha)} = (j^\alpha - 1)r + i = \sigma(j^\alpha).$$

Hence, the embedding $d(r,k)$ is a diagonal embedding. □

For an infinite sequence of integer tuples $\chi = \langle (1,k_0), (n_1,k_1), \ldots \rangle$, the sequences of diagonal maps,

$$S_{k_0} \xrightarrow{d(n_1,k_1)} S_{n_1k_0+k_1} \xrightarrow{d(n_2,k_2)} S_{(n_1k_0+k_1)n_2+k_2} \xrightarrow{d(n_3,k_3)} \ldots$$

will define a direct limit group S_χ.

The construction and the classification of the groups S_χ are done in [5]. Nowadays, these groups are called **limit symmetric groups of diagonal type**.

Assume the sequence χ is given. Then for $i > 0$ set $\lambda(0) := k_0$, $\lambda(1) = k_0n_1 + k_1$ and $\lambda(i) := \lambda(i-1)n_i + k_i$. Then the group S_χ is the direct limit of the finite symmetric groups $S_{\lambda(i)}$ and if the image of $S_{\lambda(i)}$ in the direct limit group is denoted by $S(\chi,i)$, then $S_\chi = \bigcup_{i=0}^{\infty} S(\chi,i)$.

Remark 3. *Note that if each $k_i = 0$ for all $i > 0$ the embeddings will be strictly diagonal. Hence the group S_χ will be isomorphic to limit symmetric group $S(\xi)$ of [1, 3] where $\xi = < k_0, n_1, n_2, \ldots >$.*

1.1. Centralizers of elements in S_χ

In this section, our aim is to obtain the structure of centralizers of arbitrary elements in the locally finite group S_χ. It turns out that the centralizer contains limit monomial groups.

Finite monomial groups are studied by Ore in [6]. He investigated some properties of monomial groups and determine all finite dimensional normal subgroups of the class. Starting with the finite monomial groups and using the strictly diagonal embeddings, one can find the limit monomial groups, which is constructed by Kuzucuoğlu, Oliynyk and Suschansky in [4]. They classified all the limit monomial groups by using the lattice of Steinitz numbers and find the structure of centralizer of elements in limit monomial groups.

The monomial group of degree n over a group H is denoted by $\Sigma_n(H)$. By [6], the monomial group is isomorphic to $S_n \ltimes \underbrace{(H \times \ldots \times H)}_{\text{n-times}}$ or in the wreath product notation, $\Sigma_n(H) \cong H \wr S_n$. For any sequence ξ consisting of primes, by taking strictly diagonal embeddings of finite monomial groups $\Sigma_n(H)$ we have the limit monomial groups which is denoted by $\Sigma_\xi(H)$. For the notations and definitions see [4]. If we take H to be the identity group, then $\Sigma_\xi(1)$ will be the limit symmetric group $S(\xi)$. The centralizers of elements in the limit monomial groups are studied in [4, Theorem 2.6].

Let $\chi = \langle (1,k_0), (n_1,k_1), (n_2,k_2), \ldots \rangle$ and $S_\chi = \bigcup_{i=0}^{\infty} S(\chi,i)$. For any element α in S_χ, we have a smallest integer n such that $\alpha \in S(\chi,n)$.

Definition 4. *For* $\alpha \in S_\chi = \bigcup_{i=0}^{\infty} S(\chi, i)$*, let* n *be the smallest integer such that* $\alpha \in S(\chi, n)$*. Then the* ***principal beginning*** α_0 *of* α *is the element in the finite symmetric group* $S_{\lambda(n)}$ *of which the image in the group* S_χ *is* α*.*

Note that the definition of principal beginning is similar to the case of limit symmetric groups, see [1, p.1922].

Definition 5. *The* ***short cycle type*** *of an element* $\alpha_0 \in S_n$ *is* $t(\alpha_0) = (r_1, \ldots, r_t)$ *where* r_i *is the number of* i*-cycles appearing in the cycle decomposition of* α_0 *for* $1 \leqslant i \leqslant t \leqslant n$ *and* t *is taken to be the biggest cycle length that appears in the decomposition.*

Theorem 6. *Let* $\alpha \in S_\chi$, $\chi = \langle (1, k_0), (n_1, k_1), \ldots \rangle$ *and let* $\alpha_0 \in S_{\lambda(l-1)}$ *be the principal beginning of* α *and* $t(\alpha_0) = (r_1, r_2, \ldots, r_k)$ *be the short cycle type of* α_0*. Then the centralizer of* α *in* S_χ*;*

$$C_{S_\chi}(\alpha) \cong \underset{i=2}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i) \times S_{\chi'}$$

where $\xi_i = (r_i, n_l, n_{l+1}, \ldots)$*, for all* $i \geqslant 2$*,* $\chi' = \langle (1, r_1), (n_l, k_l), \ldots \rangle$ *and* C_i *is the cyclic group of order* i*.*

Proof. Let $\alpha_0 \in S_{\lambda(l-1)}$ be the principal beginning of α. Now we know the cycle type of α_0 and there are r_i many i cycles and r_1 many fixed points.

Note that, since $\alpha_0 = x_{1,0} x_{2,0} \ldots x_{k,0}$ where $x_{i,0}$ is the product of i-cycles in the cycle decomposition of α_0, and $\alpha = x_1 x_2 \ldots x_k$ where the principal beginning of x_i is $x_{i,0}$ for $1 \leqslant i \leqslant k$, by using the same method as in [1], we have

$$C_{S_\chi}(\alpha) = \underset{i=1}{\overset{k}{Dr}} C_{S_\chi}(x_i).$$

Therefore, it is enough to find the centralizer of an element with a fixed cycle type.

Observe that for any element x with principal beginning $x_0 \in S(\chi, l-1)$ which is a product of i-cycles $i \geqslant 2$, the embedding of x_0 into $S(\chi, l)$ is strictly diagonal. So by [1, Theorem 3] and [4, Corollary 2.7], we have $C_{S_\chi}(x_i) = \Sigma_{\xi_i}(C_i)$ where $\xi_i = (r_i, n_l, n_{l+1} \ldots)$ and $\Sigma_{\xi_i}(C_i)$ is the limit monomial group over the cyclic group C_i of order i.

For the centralizer of x_1 which is identity but is formed with the fixed points of α_0 in level l, we have r_1 many fixed points and any element in symmetric group, S_{r_1}, on r_1 points will commute with α_0. The embedding of S_{r_1} into $S(\chi, l)$ is diagonal and the image is isomorphic to a subgroup of the symmetric group, $S_{r_1 n_l + k_l}$. Continuing like that we will have the diagonal embeddings of finite symmetric groups which is isomorphic to $S_{\chi'}$ where $\chi' =< (1, r_1), (n_l, k_l), \ldots >$. Hence,

$$C_{S_\chi}(\alpha) \cong \underset{i=2}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i) \times S_{\chi'}$$

where $\xi_i = (r_i, n_l, n_{l+1}, \ldots)$, for all $i \geqslant 2$, $\chi' = \langle (1, r_1), (n_l, k_l), \ldots \rangle$. □

Corollary 7. *If* $k_0 = 1, k_i = 0$ *for all* $i > 0$*, then as* $S_\chi = S(\xi)$ *we get*

$$C_{S(\xi)}(\alpha) \cong \underset{i=1}{\overset{k}{Dr}} \Sigma_{\xi_i}(C_i)$$

where $\xi_i = (r_i, n_l, n_{l+1}, \ldots)$*, for all* $i \geqslant 1$*.*

For a finite group $F \leqslant S_\chi$, with a similar argument conducted in [1, Theorem 6], one may obtain the structure of the centralizer $C_{S_\chi}(F)$.

2. Construction of an infinite group via diagonal embeddings of monomial groups

In this section we will construct a new class of infinite groups by embedding monomial groups via diagonal embeddings of previous section. This new type is a generalization of limit monomial groups of strictly diagonal type, see [4].

Let H be an arbitrary subgroup. Consider the general linear group $GL(n, Z[H])$ over the integral group ring $Z[H]$. A monomial substitution corresponds to a matrix such that it has only one non-zero component from the group H in each row and column. In other words, if $\{x_1, \dots x_n\}$ are basis elements, then a monomial substitution over H is represented as

$$\rho = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_n x_{i_n} \end{pmatrix}.$$

If $\eta = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ k_1 x_{j_1} & k_2 x_{j_2} & \cdots & k_n x_{j_n} \end{pmatrix}$ is another substitution, then with the multiplication

$$\eta\rho = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 k_{i_1} x_{j_{i_1}} & h_2 k_{i_2} x_{j_{i_2}} & \cdots & h_n k_{i_n} x_{j_{i_n}} \end{pmatrix}$$

and the inverse

$$\rho^{-1} = \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\ h_1^{-1} x_1 & h_2^{-1} x_2 & \cdots & h_n^{-1} x_n \end{pmatrix}$$

monomial substitutions forms a subgroup which is isomorphic to the wreath product $H \wr S_n$ where S_n is the symmetric group on n letters. This subgroup $H \wr S_n$ is called **complete monomial group**, see [4]. Since $H \wr S_n \cong \underbrace{H \times H \ldots \times H}_{\text{n-times}} \rtimes S_n$, every monomial substitution can be written uniquely as a product

$$\rho = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_n x_{i_n} \end{pmatrix} = [h_1, h_2, \dots h_n] \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix}$$

where $[h_1, h_2, \dots h_n]$ is called multiplication and

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{i_1} & x_{i_2} & \cdots & x_{i_n} \end{pmatrix}$$

is called permutation.

Let $H \wr S_n$ be the complete monomial group. Consider the embedding

$$d(r,k) : \Sigma_n(H) = H \wr S_n \longrightarrow \Sigma_{nr+k}(H) = H \wr S_{nr+k}$$

which sends any element of the form $[h_1, h_2, \dots h_n]\pi$ in $H \wr S_n$ to the element in $H \wr S_{nr+k}$,

$$[\underbrace{h_1, \dots, h_1}_{\text{r-times}}, \dots, \underbrace{h_n, \dots, h_n}_{\text{r-times}}, \underbrace{1, \dots, 1}_{\text{k-times}}] d(r,k)(\pi)$$

where $d(r,k)(\pi)$ is the diagonal embedding used in Section 1 for constructing S_χ.

We will show the map is also a homomorphism from $\Sigma_n(H)$ to $\Sigma_{nr+k}(H)$. Let $u = [h_1, \dots h_n]\pi$ and $v = [k_1, \dots k_n]\sigma$ be two elements of $\Sigma_n(H)$. The image equals to

$$d(r,k)(uv) = d(r,k)([h_1 k_{1^\pi}, \dots, h_n k_{n^\pi}]\pi\sigma) =$$
$$= [\underbrace{h_1 k_{1^\pi}, \dots, h_1 k_{1^\pi}}_{\text{r-times}}, \dots, \underbrace{h_n k_{n^\pi}, \dots, h_n k_{n^\pi}}_{\text{r-times}}, \underbrace{1, \dots, 1}_{\text{k-times}}] d(r,k)(\pi\sigma). \qquad (*)$$

On the other hand,

$$d(r,k)(u)d(r,k)(v) =$$
$$= [\underbrace{h_1,\ldots,h_1}_{\text{r-times}},\ldots,\underbrace{h_n,\ldots,h_n}_{\text{r-times}},\underbrace{1,\ldots,1}_{\text{k-times}}]d(r,k)(\pi)[\underbrace{k_1,\ldots,k_1}_{\text{r-times}},\ldots,\underbrace{k_n,\ldots,k_n}_{\text{r-times}},\underbrace{1,\ldots,1}_{\text{k-times}}]d(r,k)(\sigma) =$$
$$= [\underbrace{h_1 k_{1^{d(r,k)(\pi)}},\ldots,h_1 k_{1^{d(r,k)(\pi)}}}_{\text{r-times}},\ldots,\underbrace{h_n k_{n^{d(r,k)(\pi)}},\ldots,h_n k_{n^{d(r,k)(\pi)}}}_{\text{r-times}},\underbrace{1,\ldots,1}_{\text{k-times}}]d(r,k)(\pi\sigma). \qquad (**)$$

To show $d(r,k)$ is a homomorphism on the wreath products, it is enough to show the equality of components of multiplication parts. Note that when we write $k_{i^{d(r,k)(\pi)}}$ we mean the action of $d(r,k)(\pi)$ on the indicies of the multiplication i.e.

$$[\underbrace{k_1,\ldots,k_1}_{\text{r-times}},\ldots,\underbrace{k_n,\ldots,k_n}_{\text{r-times}},\underbrace{1,\ldots,1}_{\text{k-times}}] = [k'_1,k'_2,k'_3,\ldots k'_{nr},1,\ldots,1]$$

where for all $1 \leqslant i \leqslant n$ when t runs through the set $\{1,\ldots r\}$, the equality $k'_{(i-1)r+t} = k_i$ holds.

For $1 \leqslant i \leqslant n$ the $((i-1)r+t)^{th}$ component of the multiplication part of v is the same as k_i where t runs through $\{1,\ldots r\}$. Since $((i-1)r+t)^{d(r,k)(\pi)} = (i^\pi - 1)r + t$, we get

$$k'_{((i-1)r+t)^{d(r,k)(\pi)}} = k'_{(i^\pi-1)r+t} = k_{i^\pi}.$$

Hence $(\star)$ and $(\star\star)$ equals.

Lemma 8. $d(r_1,k_1)d(r_2,k_2) = d(r_1r_2, k_1r_2 + k_2)$.

Proof. Together with [5, Lemma 2.5], an elementary computation will show the equality. □

Let $\chi =< (1,k_0),(n_1,k_1),\ldots >$ be an infinite sequence of positive numbers. Set $\lambda(0) = k_0$ and for $i \geqslant 1$ set $\lambda(i) = k_0n_1n_2\ldots n_i + k_1n_2\ldots n_i + \ldots + k_{i-1}n_i + k_i$. Starting with the complete monomial group $H \wr S_{k_0}$ consider the direct limit of the groups as follows,

$$\Sigma_{\lambda(0)}(H) = H \wr S_{\lambda(0)} \overset{d(n_1,k_1)}{\longrightarrow} \Sigma_{\lambda(1)}(H) = H \wr S_{\lambda(1)} \overset{d(n_2,k_2)}{\longrightarrow} \Sigma_{\lambda(2)}(H) = H \wr S_{\lambda(2)} \ldots$$

From the direct limit, we obtain the group $\Sigma_\chi(H) = \bigcup_{i=0}^{\infty} \Sigma_{\lambda(i)}(H)$ which is called **limit monomial group of diagonal type**. The group $\Sigma_\chi(H)$ is a subgroup of the infinite monomial group $H \wr S_\chi$. Since we have $\Sigma_\chi(H) = \bigcup_{i=0}^{\infty} \Sigma_{\lambda(i)}(H)$, for any element one can define **principal beginning** as in the same way it is defined in Definition 4.

Lemma 9. *If all $k_i = 0$ for all $i > 0$ the group $\Sigma_\chi(H)$ will be isomorphic to the limit monomial group of strictly diagonal type $\Sigma_\xi(H)$ where $\xi = (k_0, n_1, n_2, \ldots)$, see [4].*

Since all elements of the monomial substitutions can be written uniquely as the product of a multiplication and a permutation when we take the image of an element $[h_1,\ldots,h_n]\pi \in H \wr S_n$ inside the group $H \wr S_{nr+k}$ we see the embedding sends permutation parts to permutation parts and multiplication parts to multiplication parts via the diagonal embedding. Hence from the diagonal embeddings of symmetric groups we get the infinite direct limit group S_χ and from the embeddings of multiplication parts we obtain a subgroup B_χ of the infinite Cartesian product of H.

Then the group can be written as $\Sigma_\chi(H) = B_\chi \rtimes S_\chi$ where B_χ consists of periodic components of elements from the group H followed by 1's.

2.1. Centralizers of elements

In this section the centralizer of elements in the groups $\Sigma_\chi(H)$ is investigated.

Before the result, for a finite monomial group $\Sigma_n(H)$ one may see the following definitions.

A monomial substitution is called a **cycle** if the permutation part is a cycle. Obviously, every monomial substitution can be written as a product of disjoint cycles.

Let $u = [h_1, h_2, \dots h_m](i_1, i_2, i_3 \dots i_m)$. If we take m-th power of u, we get a multiplication $u^m = [h_1 h_2 \dots h_m, h_2 h_3 \dots h_m h_1, \dots, h_m h_1 \dots h_{m-1}]$.

The elements $\Delta_i = h_i h_{i+1} \dots h_m h_1 \dots h_{i-1} \in H$ for all $1 \leqslant i \leqslant m$ are called **determinants** of u.

One may observe that $\Delta_i^{h_i} = \Delta_{i+1}$ for all $1 \leqslant i \leqslant m$ and upto conjugacy every cycle have a unique determinant class in H.

Let $u = [h_1, h_2, \dots h_m](i_1, i_2, i_3 \dots i_m)$. With a suitable element inside the group $\Sigma_n(H)$, by taking conjugate, one can find an element of the form $[1, 1, \dots, 1, a](i_1, i_2, \dots, i_m)$ where a is an element of H and a is a determinant class. This conjugate is called a **normal form** of u.

Since centralizers of conjugate elements are conjugate, it is enough to work on centralizers of normalized form of elements.

Recall that for an element $\pi \in S_n$, the type of π is the number sequence $(r_1, r_2, \dots, r_t)$ where each r_i is the number of i-cycles in the cycle decomposition of π. Similar to this, one can define the type of a monomial substitution as the number sequence

$$(a_{11}r_{11}, a_{12}r_{12}, \dots, a_{1k_1}r_{1k_1}, \dots a_{t1}r_{t1}, a_{t2}r_{t2}, \dots, a_{tk_t}r_{tk_t}).$$

This sequence means that for a fixed i, the monomial substitution has r_{ij} many i-cycles of determinant class a_{ij}.

For a cycle of normal form $\rho = [1, 1, \dots, 1, a] \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ x_2 & x_3 & \cdots & x_1 \end{pmatrix}$ if we get conjugate of ρ with an arbitrary element

$u = [b_1, b_2, \dots, b_n] \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{j_1} & x_{j_2} & \cdots & x_{j_n} \end{pmatrix}$ we have

$u\rho u^{-1} = [b_1^{-1}b_2, b_2^{-1}b_3, \dots, b_m^{-1}b_1 a] \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_m} \\ x_{j_2} & x_{j_3} & \cdots & x_{j_1} \end{pmatrix}$.

Hence one may observe that conjugacy preserves the type of an element.

Lemma 10. *Let* $\rho = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ x_2 & x_3 & \cdots & ax_1 \end{pmatrix}$ *be a cycle in* $\Sigma_n(H)$ *with determinant class* $a \in H$. *Then the image of* ρ *under* $d(r,k)$ *in* $\Sigma_{nr+k}(H)$ *consists of cycles with the same determinant class.*

Proof. The result follows when we write the image

$$d(r,k)(\rho) == \begin{pmatrix} x_1 & x_{r+1} & \cdots & x_{(m-1)r+1} \\ x_{r+1} & x_{r+2} & \cdots & ax_1 \end{pmatrix} \dots \begin{pmatrix} x_r & x_{2r} & \cdots & x_{mr} \\ x_{2r} & x_{3r} & \cdots & ax_r \end{pmatrix}$$

note that the fixed symbols are not written to avoid the confusion. Hence embedding does not alter the determinant class, it only increases the cycle numbers. □

Centralizers in monomial groups are given by Ore.

Let ρ be a monomial substitution in $\Sigma_n(H)$ with determinant class a. Then by [6, p.20], the centralizer of ρ is isomorphic to the cyclic extension of $C_H(a)$ by $< \rho >$.

$C_{\Sigma_n(H)}(\rho) \cong C_H(a) < \rho >= C_a$.

Theorem 11. *Let* ρ *be a normal form of an element in* $\Sigma_\chi(H)$. *Assume the principal beginning* ρ_0 *is in* $\Sigma_{\lambda(t)}(H)$. *Write* ρ *as the product* $\rho = \rho_1\rho_2 \dots \rho_l$ *where for each* i, *the substitutions*

ρ_i consist of cycles with the same length and same determinant class a_i. Assume also that ρ_i consists of r_i many cycles of the same type. Without loss of generality assume ρ_l is the element consisting of one cycles with determinant class 1. Then the centralizer is

$$C_{\Sigma_\chi(H)}(\rho) \cong \Sigma_{\xi_1}(C_{a_1}) \times \Sigma_{\xi_2}(C_{a_2}) \ldots \times \Sigma_{\xi_{l-1}}(C_{a_{l-1}}) \times S_{\chi_l}$$

where $\xi_i = (r_i, n_{t+1}, n_{t+2}, \ldots)$ and $\chi_l =< (1, r_l), (n_{t+1}, k_{t+1}), (n_{t+2}, k_{t+2}), \ldots >$.

Proof. Since conjugation does not change the type, the centralizer can be written as the direct product of centralizers of the cycles of different cycle type and length. Hence it is enough to find the centralizer for an arbitrary ρ_i.

Since the principal beginning ρ_0 is in $\Sigma_{\lambda(t)}$ we may assume $C_{\Sigma_\xi}(\rho_i) = \bigcup_{j=1}^{\infty} C_{\Sigma_{\lambda(j+t)}}(\rho_i)$.

For ρ_l the element is nothing but the identity element formed with the fixed points. Therefore, $C_{\Sigma_{\lambda(t)}}(\rho_l) = S_{r_l}$. With the embedding $d(n_{t+1}, k_{t+1})$ the image will be again identity with $r_l n_{t+1} + k_{t+1}$ many fixed points and permutation formed with those $r_l n_{t+1} + k_{t+1}$ points will centralize the image. Hence $C_{\Sigma_{\lambda(t+1)}}(\rho_l) = S_{r_l n_{t+1}+k_{t+1}}$. Continuing the embedding we will have S_{χ_l} as the centrlizer of ρ_l where $\chi_l =< (1, r_l), (n_{t+1}, k_{t+1}), (n_{t+2}, k_{t+2}), \ldots >$.

As for the other parts, by [6, p.20], since ρ_i has r_i many cycles of the same type we get

$$C_{\Sigma_{\lambda(t)}}(\rho_i) \cong C_H(a_i) < \rho_i > \wr S_{r_i} = C_{a_i} \wr S_{r_i} = \Sigma_{r_i}(C_{a_i}).$$

By Lemma 10, the embedding $d(n_{t+1}, k_{t+1})$ only increases the cycle number and does not affect the type. Hence with the embedding $d(n_{t+1}, k_{t+1})$, we have

$$C_{\Sigma_{\lambda(t+1)}}(\rho_i) \cong C_{a_i} \wr S_{r_i n_{t+1}} = \Sigma_{r_i n_{t+1}}(C_{a_i}).$$

Continuing the embeddings, monomial groups will be embedded into the monomial groups so we will have

$$C_{\Sigma_\chi}(\rho_i) = \Sigma_{\xi_i}(C_{a_i})$$

where $\xi_i = (r_i, n_{t+1}, n_{t+2}, \ldots)$.

Therefore,

$$C_{\Sigma_\chi(H)}(\rho) \cong \Sigma_{\xi_1}(C_{a_1}) \times \Sigma_{\xi_2}(C_{a_2}) \ldots \times \Sigma_{\xi_{l-1}}(C_{a_{l-1}}) \times S_{\chi_l}$$

where $\xi_i = (r_i, n_{t+1}, n_{t+2}, \ldots)$ and $\chi_l =< (1, r_l), (n_{t+1}, k_{t+1}), (n_{t+2}, k_{t+2}), \ldots >$. □

Corollary 12. *If all $k_i = 0$ except for k_0, then the group $\Sigma_\chi(H)$ will be isomorphic to the limit monomial group of [4]. Hence, Theorem 11 also gives the centralizer of an element in limit monomial groups.*

Corollary 13. *If the group H is the identity group, then $\Sigma_\chi(H)$ will become S_χ. Hence Theorem 11 gives also the results obtained by Theorem 6.*

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Централизаторы в диагональных прямых пределах симметрических групп и мономиальных групп

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Аннотация. Централизаторы конечных подгрупп в прямом пределе конечных и финитарных симметрических групп через строго диагональные вложения охарактеризованы в 2015 г. Гювеном, Кегелем и Кузукуоглу. В статье эта идея распространяется на случай диагональных вложений. Кроме того, посредством диагонального вложения строится новый класс бесконечных предельных мономиальных групп.

Ключевые слова: централизатор, простое локально конечное диагональное вложение, сохраняющее уровень.