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Dihedral Group of Order 8 in an Autotopism Group of a Semifield Projective Plane of Odd Order

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Abstract. We investigate the well-known hypothesis of D. R. Hughes that the full collineation group of a non-Desarguesian semifield projective plane of a finite order is solvable (the question 11.76 in Kourovka notebook was written down by N. D. Podufalov). The spread set method allows us to prove that any non-Desarguesian semifield plane of order p^N, where $p \equiv 1 \pmod 4$ is prime, does not admit an autotopism subgroup isomorphic to the dihedral group of order 8. As a corollary, we obtain the extensive list of simple non-Abelian groups which cannot be the autotopism subgroups.

Keywords: semifield plane; spread set; Baer involution; homology; autotopism.

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Introduction

A projective plane is called a *semifield plane* if its points and lines are coordinatized by a *semifield*, that is a non-associative ring $Q = (Q, +, \cdot)$ with identity where the equations $ax = b$ and $ya = b$ are uniquely solved for any $a, b \in Q \setminus \{0\}$. The study of finite semifields and semifield planes started more than a century ago with the first examples constructed by L. E. Dickson [1].

By the mid-1950s, some classes of finite semifield planes had been found. All of them had the common property that the collineation group (automorphism group) is solvable. So D. R. Hughes conjectured in 1959 in his report that any finite projective plane coordinatized by a non-associative semifield has the solvable collineation group. This hypothesis is presented in the monography [2, Ch. VIII, Sec. 6]; it is proved also that the hypothesis is reduced to the solvability of an autotopism group as a group fixing a triangle. In 1990 the problem was written down by N. D. Podufalov in the Kourovka notebook ([3], the question 11.76).

We represent the approach to study Hughes' problem based on the classification of finite simple groups and the theorem of J. G. Thompson on minimal simple groups. The spread set method allows us to identify the conditions when the semifield plane with certain autotopism subgroup exists. This method can be used also to construct examples, including computer calculations. The elimination of some simple groups as autotopism subgroups follows to the progress in solving the problem.

It is shown by the author in [4, 5], that an autotopism of order two has the matrix representation convenient for calculations and reasoning. These marices are used further to represent the

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elementary abelian 2-subgroups and 2-elements in the autotopism group [6,7]. Also it was proved that any non-Desarguesian semifield plane of odd order cannot admit an autotopism subgroup isomorphic to the alternating group A_5 [8].

Here we use the spread set method to prove that any semifield plane of order p^N, p is prime and $p-1$ is divisible by 4, cannot admit an autotopism subgroup isomorphic to the dihedral group D_8 of order 8, see Theorem 2.1. The proof is based on a concretization of a geometric sense of autotopisms of order 2 and 4, it uses also the matrix representation of autotopisms of order 4. Obviously, the presence of this group in almost all simple non-Abelian groups allows us to exclude an extensive list from possible autotopism subgroups.

1. Definitions and preliminary results

We use main definitions, according [2,9], see also [6], for notifications.

Consider a linear space W, n-dimensional over the finite field $GF(p^s)$ (p be prime) and the subset of linear transformations $R \subset GL_n(p^s) \cup \{0\}$ such that:

1) R consists of p^{ns} square $(n \times n)$-matrices over $GF(p^s)$;
2) R contains the zero matrix 0 and the identity matrix E;
3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a non-singular matrix.

The set R is called a *spread set* [2]; it is an image of an injective mapping θ from W: $R = \{\theta(y) \mid y \in W\}$. Determine the multiplication on W by the rule $x * y = x \cdot \theta(y)$ $(x, y \in W)$. Then $\langle W, +, * \rangle$ is a right quasifield of order p^{ns} [9,10]. Moreover, if R is closed under addition then $\langle W, +, * \rangle$ is a semifield. This semifield coordinatizes the projective plane π of order p^{ns} such that:

1) the affine points are the elements (x, y) of the space $W \oplus W$;
2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in W\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in W\} \quad (m \in W);$$

3) the set of all cosets to the subgroup is the singular point;
4) the set of all singular points is the singular line;
5) the incidence is set-theoretical.

To construct and study finite semifields, we use a prime field $\mathbb{Z}_p$ as a basic field. In this case the mapping θ is presented using linear functions only; it greatly simplifies the reasoning and calculations (also computer).

The solvability of a collineation group $Aut\,\pi$ for a semifield plane is reduced [2] to the solvability of an autotopism group Λ (collineations fixing a triangle). Without loss of generality, we can assume that autotopisms are determined by linear transformations of the space $W \oplus W$:

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

here the matrices A and B satisfy the condition (for instance, see [11])

$$A^{-1}\theta(m)B \in R \quad \forall \theta(m) \in R. \tag{1}$$

The collineations fixing a closed configuration have special properties. It is well-known [2], that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central*, or *perspectivity*, if it fixes a line pointwise (the *axis*) and a point linewise (the *center*). If the center is incident to the axis then a collineation is called an *elation*, and a *homology* in another case. The order of any elation is a factor of the order $|\pi|$ of a projective plane, and the order of any homology is a factor of $|\pi| - 1$. All the perspectivities in an autotopism group are homologies in the case when a semifield plane is of odd order. They form the cyclic subgroups [12] which are normal in Λ, and contain three involution homologies:

$$h_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, \quad h_2 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad h_3 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}.$$

Obviously these homologies are all in the center of Λ.

A collineation of a finite projective plane π is called a *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|}$ (*Baer subplane*). We use the following results on the matrix representation of a Baer involution $\tau \in \Lambda$ and of a spread set obtained earlier in [5].

Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime). If its autotopism group Λ contains the Baer involution τ then $N = 2n$ is even and we can choose the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \tag{2}$$

where $L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$ and the Baer subplane π_τ fixed by τ is the set of points

$$\pi_\tau = \{(0, \dots, 0, x_1, \dots, x_n, 0, \dots, 0, y_1, \dots, y_n) \mid x_i, y_i \in \mathbb{Z}_p\}.$$

In this base the spread set $R \subset GL_{2n}(p) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}, \tag{3}$$

where $V \in Q$, $U \in K$; Q, K are the spread sets in $GL_n(p) \cup \{0\}$, m, f are additive injective functions from K and Q into $GL_n(p) \cup \{0\}$, $m(E) = E$. Note that throughout the article, the blocks-submatrices have the same dimension by default.

It is shown by author in [6,7], that the order of a semifield plane provides a natural restriction to the order of an elementary abelian 2-subgroup and to the order of 2-element in an autotopism group. We will use some results and so we state it here in the more convenient form.

Theorem 1.1. *Let π be a semifield plane of order p^N, p be prime, $p \equiv 1 \pmod 4$, $\tau \in \Lambda$ is a Baer involution.*

1. If α is an autotopism of order 4 and $\alpha^2 = \tau$ then the restriction of α onto the Baer subplane π_τ is a Baer involution.

2. If $\sigma \neq \tau$ is a Baer involution in $C_\Lambda(\tau)$ then the restriction of σ onto the Baer subplane π_τ is a homology if $\sigma = h_i\tau$ $(i = 1, 2, 3)$ or a Baer involution.

Theorem 1.2. *Let π be a semifield plane of order p^N, p be prime, $p \equiv 1 \pmod 4$, α is an autotopism of order 4, $\tau = \alpha^2$ is a Baer involution. Then N is divisible by 4, and the base of the linear space can be chosen such that τ is (2) and*

$$\alpha = \begin{pmatrix} iL & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & iL & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{4}$$

where $i \in \mathbb{Z}_p$, $i^2 = -1$. The spread set R of the plane π is formed by matrices

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} m_1(U_2) & m_2(V_2) & f_1(V_1) & f_2(U_1) \\ m_3(V_2) & m_4(U_2) & f_3(U_1) & f_4(V_1) \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}, \qquad (5)$$

where any block-submatrix is $(N/4 \times N/4)$-dimensional, $V_1 \in Q_1$, $U_1 \in K_1$, $V_2 \in Q_2$, $U_2 \in K_2$, the matrix sets Q_1, K_1, Q_2, K_2 are the spread sets of semifield planes of order $p^{N/4}$, all the functions are additive.

Note, that α is determined up to multiplying to involution homologies h_i from the center of Λ (see the proof in [7]). If we consider certain subgroup of Λ then we can ignore these homologies.

The second statement of the theorem 1.2 is missed in [7] because obviously but here we must reconstruct it due to the importance for the main result.

Indeed, we consider the condition (1) for the autotopism α and the matrix $\theta(V, U)$ (3):

$$\begin{pmatrix} -iL & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix} \begin{pmatrix} iL & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} Lm(U)L & -iLf(V)L \\ iLVL & LUL \end{pmatrix}.$$

Then we conclude that

$$LVL \in Q, \quad LUL \in K, \quad m(LUL) = Lm(U)L, \quad f(LVL) = -Lf(V)L, \qquad \forall V \in Q,\ \forall U \in K.$$

So the semifield planes of order $p^{N/2}$ with the spreads Q and K admit the Baer involution (2) and the matrices $V \in Q$, $U \in K$ are of the same form as (3):

$$V = \begin{pmatrix} \nu(U_1) & \psi(V_1) \\ V_1 & U_1 \end{pmatrix}, \qquad U = \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix}.$$

If we suppose that

$$m(U) = m(V_2, U_2) = \begin{pmatrix} m_1(V_2, U_2) & m_2(V_2, U_2) \\ m_3(V_2, U_2) & m_4(V_2, U_2) \end{pmatrix},$$

then from $m(-V_2, U_2) = Lm(V_2, U_2)L$ we obtain that the functions m_1, m_4 depend on the block U_2 and other functions on V_2. For the function $f(V)$ we use the condition $f(-V_1, U_1) = -Lf(V_1, U_1)L$ and complete the proof.

2. Main result

Theorem 2.1. *Any non-Desarguesian semifield plane π of order p^N, where $p > 2$ is prime and $p \equiv 1 \pmod 4$, does not admit an autotopism subgroup isomorphic to the dihedral group of order 8 without homologies.*

Proof. Let $H \simeq D_8$ be a subgroup of Λ, $H = \langle\alpha\rangle \rtimes \langle\sigma\rangle$, $|\alpha| = 4$, $|\sigma| = 2$, $\sigma\alpha\sigma = \alpha^{-1}$. The autotopism $\alpha^2 = \tau$ is a Baer involution, so we can choose the base of $2N$-dimensional linear space such that τ is the matrix (2), α is the matrix (4) and the spread set consists of matrices (5).

Further, σ is a Baer involution commuting with τ, and then we have

$$\sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix}, \quad A_1^2 = A_2^2 = B_1^2 = B_2^2 = E.$$

According the Theorem 1.1, the restriction of σ onto the Baer subplane π_τ is a Baer involution, so $A_2 \neq \pm E,\ B_2 \neq \pm E$. From the condition $\sigma\alpha\sigma = \alpha^{-1}$, we have

$$A_1 L A_1 = B_1 L B_1 = -L, \qquad A_2 L A_2 = B_2 L B_2 = L,$$

$$A_1 = \begin{pmatrix} 0 & A_{11} \\ A_{12} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & B_{11} \\ B_{12} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} B_{21} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

The restrictions of α and σ onto the Baer subplane π_τ are commuting Baer involutions and, once more from the Theorem 1.1 and [6], we can choose the base of π_τ such that $A_{21} = A_{22} = B_{21} = B_{22} = L$ and

$$\sigma = \begin{pmatrix} 0 & S & 0 & 0 & 0 & 0 & 0 & 0 \\ S^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & S & 0 & 0 \\ 0 & 0 & 0 & 0 & S^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \end{pmatrix}.$$

Here, for compactness, $S = A_{11}$, and $A_1^2 = E$ follows $A_{12} = S^{-1}$. The equality $B_1 = A_1$ we obtain from the condition (1) for σ and $\theta(V,U) = E \in R$:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1 B_1 & 0 \\ 0 & E \end{pmatrix} \in R \Rightarrow A_1 B_1 = E.$$

Now we simplify the matrix σ changing the base by the block-diagonal transition matrix

$$T = \mathrm{diag}\,(E, S, E, E, E, S, E, E).$$

This modification preserves the matrices τ and α, but allows us to write σ in the more convenient form:

$$\sigma = \begin{pmatrix} 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \end{pmatrix}.$$

Consider the condition (1) for the spread set (5) and the Baer involution σ. For $V_2 = U_2 = 0$ we have:

$$\begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix} \begin{pmatrix} 0 & 0 & f_1(V_1) & f_2(U_1) \\ 0 & 0 & f_3(U_1) & f_4(V_1) \\ \nu(U_1) & \psi(V_1) & 0 & 0 \\ V_1 & U_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & f_3(U_1)L & f_4(V_1)L \\ 0 & 0 & f_1(V_1)L & f_2(U_1)L \\ L\psi(V_1) & L\nu(U_1) & 0 & 0 \\ LU_1 & LV_1 & 0 & 0 \end{pmatrix} \in R.$$

So, the matrices LU_1 and LV_1 belong to the spread sets Q_1 and K_1 for all $V_1 \in Q_1$, $U_1 \in K_1$. For instance, we have $L \in K_1$ if $V_1 = E$. The spread set K_1 of a semifield plane is closed under addition, so the non-zero degenerate matrix $L + E$ belongs to K_1, that is impossible. This contradiction proves the theorem.

Note that the absence of homologies in H is the natural condition for us because we investigate the existence problem for simple non-Abelian subgroups in the autotopism group Λ (for instance, minimal simple non-Abelian groups from the Thompson's list). Indeed, the homologies generate the normal subgroup of Λ; moreover, the involution homlogies are in the center of Λ.

Let G be a subgroup of Λ and S be the Sylow 2-subgroup of G. If two involutions in S does not commute then they generate the dihedral subgroup in S. Further, using the results of D. Goldschmidth [13] on strongly closed subgroups (see also D. Gorenstein [14, th. 4.128]), we conclude that D_8 is contained almost in all finite simple non-Abelian groups and list the exceptions.

Theorem 2.2. *Let π be a non-Desarguesian semifield plane of order p^N, where $p > 2$ is prime and $p \equiv 1 \pmod 4$. Then its autotopism group Λ does not contain a simple non-Abelian subgroup, except probably the following: $PSL(2, 2^n)$, $n \geqslant 2$, $PSU(3, 2^n)$, $n \geqslant 2$, $Sz(2^n)$, n is odd, $n > 1$, $PSL(2, q)$, $q \equiv \pm 3 \pmod 8$, J_1 or ${}^2G_2(3^n)$, n is odd, $n > 1$.*

Referring to the Thompson's list, we clarify also that the autotopism group Λ under the order condition above does not contain $PSL(2, 3^n)$, $n > 2$ is prime, $PSL(2, n)$, $n \equiv \pm 1 \pmod 8$ is prime, and $PSL(3, 3)$.

Conclusion

In order to study Hughes' problem on the solvability of the full collineation group of a finite non-Desarguesian semifield plane, the author considers it possible to use the obtained results to further investigations. The method applied will probably be useful to study other small autotopism subgroups under the conditions on the plane order.

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Группа диэдра порядка 8 в группе автотопизмов полуполевой проективной плоскости нечетного порядка

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Аннотация. Изучается известная гипотеза Д. Хьюза о разрешимости полной группы автоморфизмов конечной недезарговой полуполевой проективной плоскости (также вопрос 11.76 Н. Д. Подуфалова в Коуровской тетради). Метод регулярного множества позволяет доказать, что недезаргова полуполевая плоскость порядка p^N, где p — простое, $p-1$ делится на 4, не допускает подгрупп автотопизмов, изоморфных диэдральной группе порядка 8. В качестве следствия выделяется обширный список простых неабелевых групп, не являющихся подгруппами автотопизмов.

Ключевые слова: полуполевая плоскость, регулярное множество, бэровская инволюция, гомология, автотопизм.