DOI: 10.17516/1997-1397-2022-15-3-357-366
УДК 517.9

Idempotent Values of Commutators Involving Generalized Derivations

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Received 24.12.2021, received in revised form 28.01.2022, accepted 20.03.2022

Abstract. In the present article, we characterize generalized derivations and left multipliers of prime rings involving commutators with idempotent values. Precisely, we prove that if a prime ring of characteristic different from 2 admits a generalized derivation G with an associative nonzero derivation g of R such that $[G(u), u]^n = [G(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where L a noncentral Lie ideal of R and $n > 1$ is a fixed integer, then one of the following holds:

(i) R satisfies s_4 and there exists $\lambda \in C$, the extended centroid of R such that $G(x) = ax + xa + \lambda x$ for all $x \in R$, where $a \in U$, the Utumi quotient ring of R,

(ii) there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.

As an application, we describe the structure of left multipliers of prime rings satisfying the condition $([T^m(u), u])^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where $m, n > 1$ are fixed integers. In the end, we give an example showing that the hypothesis of our main theorem is not redundant.

Keywords: prime ring, Lie ideal, generalized derivation, GPI.

Citation: G.S. Sandhu, Shakir Ali, Idempotent Values of Commutators Involving Generalized Derivations, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 357–366. DOI: 10.17516/1997-1397-2022-15-3-357-366.

1. Introduction

A celebrated result of Wedderburn states that: *Every finite division ring is commutative and also any Boolean ring is a commutative ring.* In 1945, Jacobson [15] generalized this result by proving the following: *Any ring in which every element satisfies an equation of the form $x^{n(x)} = x$, is commutative, where $n(x) > 1$ is an integer related to x.* In this vein Herstein [11] proved the following theorem: *If R is a ring with center $Z(R)$, and if $x^n - x \in Z(R)$ for all $x \in R$, then R is commutative, where $n > 1$ is a fixed integer*, which is of course a generalization of the classical theorem due to Jacobson. Later, Herstein [12] established the commutativity of rings that satisfy the condition $[x, y]^n = [x, y]$, where $n(x, y) > 1$ is an integer. These results

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have inspired the development of several techniques to explore the conditions that force a ring to be commutative, for instance, generalizing Herstein's conditions, using certain polynomial constraints, using restrictions on automorphisms, introducing identities involving derivations and generalized derivations etc. For more details and references one can see a well organized survey paper by Pinter–Lucke [20].

An additive mapping $T : R \to R$ is said to be a left (resp. right) multiplier if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both a left as well as a right multiplier of R, then it is said to be a multiplier of R (cf.; [23] and [25] for details). An additive mapping $g : R \to R$ is called a derivation if $g(xy) = g(x)y + xg(y)$ holds for all $x, y \in R$. An additive mapping G is called a generalized derivation if there is a derivation g of R satisfying $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $F(x) = ax + xb$ for all $x \in R$, where a and b are fixed element of R. Moreover, the concept of generalized derivation includes both the concepts of derivation and left multiplier. Hence, the concept of generalized derivation is a natural generalization of the concept of derivation and left multiplier. Further, generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example Hvala [13] and Lee [18], where further references can be looked). In the present paper, we describe the structure of generalized derivations and left multipliers of prime rings under some specific situations.

The study of commutators involving derivations goes back to 1957, when Posner [21] proved that a prime ring R admits a nonzero derivation d satisfying $[d(x), x] = 0$ for all $x \in R$, is commutative. Since then, this result has been generalized in many directions. In 2000, Carini and Filippis [6] studied the nilpotent values of commutators involving derivations of prime rings. Precisely, they proved that: *Let R be a prime ring of characteristic different from 2, L a noncentral Lie ideal of R, d a nonzero derivation of R and $n \geqslant 1$ is a fixed integer. If $[d(x), x]^n = 0$ for all $x \in L$, then R is commutative.* In 2006, Filippis [9] extended this result to the class of generalized derivations as follows: *Let R be a prime ring of characteristic different from 2, L a noncentral Lie ideal of R and $n \geqslant 1$ is a fixed integer. If R admits a generalized derivation F associated with a derivation d such that $[F(x), x]^n = 0$ for all $x \in L$, then either R satisfies s_4, the standard identity in four noncommuting variables or there exists $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$.* Therefore, it is natural to look at the idempotent elements of the set $E = \{[\varphi(x), x] : x \in L\}$, where φ is a mapping and L is a subset of a prime ring R. Recently, Scudo and Ansari [22] considered this problem with generalized derivations of prime rings. In fact, they proved the following theorem:

Theorem 1.1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, U the Utumi qotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If G is a generalized derivation of R with an associated derivation d of R such that $[G(u), u]^n = [G(u), u]$ for all $u \in L$, where $n > 1$ a fixed integer, then one of the following holds:*

(i) R satisfies the s_4 identity and there exists $a \in U$ and $\lambda \in C$ such that $G(x) = ax + xa + \lambda x$ for all $x \in R$.

(ii) there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.

In this line of investigation, Filippis et al. [10] obtained the following result on multilinear polynomials: *Let R be a prime ring with char$(R) \neq 2$, C the extended centroid of R, d a nonzero derivation of R, $f(x_1, \cdots, x_n)$ a multilinear polynomial over C, I a nonzero right ideal of R and*

$m > 1$ *a fixed integer such that*

$$[d(f(x_1, \cdots, x_n)), f(x_1, \cdots, x_n)]^m = [d(f(x_1, \cdots, x_n)), f(x_1, \cdots, x_n)]$$

for all $x_1, \cdots, x_n \in I$. *Then either* $[f(x_1, \cdots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I *or* $d(I)I = (0)$.

Very recently, Ashraf et al. [2] studied a related problem for automorphisms of prime rings. Specifically, they proved the following theorem: *Let* R *be a prime ring with char*$(R) \neq 2, 3$ *and* L *a noncentral Lie ideal of* R. *If* σ *is an automorphism of* R *such that* $[\sigma(x), x]^m = [\sigma(x), x]$ *for all* $x \in L$, *where* $m > 1$ *a fixed integer, then* R *is commutative.*

The main objective of this paper is to study the above mentioned problem for the set $[L, L] = \{[x, y \mid x, y \in L\}$, where L is a noncentral Lie-ideal of a prime ring R. In fact, we describe the structure of generalized derivations and left multipliers of prime rings with idempotent values on commutators. Precisely, we prove the following results:

Theorem 1.2. *Let* $n > 1$ *be a fixed integer. Next, let* R *be a prime ring with char*$(R) \neq 2$, U *the Utumi quotient ring,* C *is the extended centroid and* L *is a noncentral Lie ideal of* R. *If* R *admits a generalized derivation* G *associated with a derivation* g *such that* $[G(u), u]^n = [G(u), u]$ *for all* $u \in \{[x, y] : x, y \in L\}$, *then one of the following holds:*

(i) R *satisfies* s_4 *and there exists* $\lambda \in C$ *such that* $G(x) = ax + xa + \lambda x$ *for all* $x \in R$, *where* $a \in U$.

(ii) there exists $\gamma \in C$ *such that* $G(x) = \gamma x$ *for all* $x \in R$.

Further, as an application, we describe the structure of left multipliers of prime rings. In particular, we establish the following:

Theorem 1.3. *Let* $m, n > 1$ *be fixed integers. Next, let* R *be a prime ring with char*$(R) \neq 2$, U *the Utumi quotient ring,* C *is the extended centroid of* R *and* L *is a noncentral Lie ideal of* R. *If* T *is a left multiplier of* R *such that* $([T^m(u), u])^n = [T^m(u), u]$ *for all* $u \in \{[x, y] : x, y \in L\}$, *then there exists* $\gamma \in C$ *such that* $T(x) = \gamma x$ *for all* $x \in R$.

2. Preliminaries

A ring R is said to be a prime if for any $a, b \in R$; $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive mapping $g : R \to R$ is called a derivation if $g(xy) = g(x)y + xg(y)$ for all $x, y \in R$. For a fixed element $a \in R$, a mapping $x \mapsto [a, x]$ is a well-known example of a derivation, which is called the inner derivation induced by a. By generalized derivation, we mean an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xg(y)$, where g is a derivation of R associated with F. For any $x, y \in R$, the symbol $[x, y]$ denotes the Lie product (or commutator) $xy - yx$. An additive subgroup L of R is known as Lie ideal of R if $[x, r] \in L$ for all $x \in L$ and $r \in R$. The Utumi quotient ring of R is denoted by U and C the extended centroid of R. For more detail of these objects and generalized polynomial identities, we refer the reader to [3]. By s_4, we denote the standard identity in four noncommuting variables, which is defined as follows:

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)},$$

where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$. It is known that by the standard PI-theory, a prime ring R satisfying s_4 can be characterized in a number of ways, as follows: Let R be a prime ring with C its extended centroid. Then the following assertions are equivalent:

- $\dim_C(RC) \leqslant 4$.
- R satisfies s_4.
- R is commutative or R embeds into $M_2(F)$, for a field F.
- R is algebraic of bounded degree 2 over C.
- R satisfies $[[x^2, y], [x, y]]$ (see [5, Lemma 1]).

In order to prove this result, we need the following remarks:

Remark 1 ([18], Theorem 3)**.** *Every generalized derivation of R can be uniquely extended to a generalized derivation of U and assumes the form that $F(x) = ax + g(x)$ for some $a \in U$ and a derivation g of U.*

Remark 2 ([7], Theorem 2)**.** *If I is a two-sided ideal of R, then I, R and U satisfy the same generalized polynomial identities with coefficients in U.*

3. Main results

We begin our discussions with the following lemma.

Lemma 1. *Let $R = M_k(C)$ be the ring of $k \times k$ matrices over a field C with char$(R) \neq 2$ and $a, b \in R$. If $k = 2$ and $n > 1$ a fixed integer such that*

$$\big([a[u,v] + [u,v]b, [u,v]]\big)^n = [a[u,v] + [u,v]b, [u,v]]$$

for all $u, v \in [R, R]$, then $b - a$ is central.

Proof. By the given hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned}&\big([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]]\big)^n - \\ &\quad -[a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]].\end{aligned} \tag{1}$$

Let us assume that $b - a = \sum_{i,j=1}^{k} \alpha_{ij} e_{ij}$, where $\alpha_{ij} \in C$ and e_{ij} denotes the standard matrix unit with (i,j)-th place 1 and 0 elsewhere. For $i \neq j$, we choose $x_1 = e_{ii}$, $x_2 = e_{ij}$, $x_3 = e_{ij}$ and $x_4 = e_{ji}$. With this, we have $[x_1, x_2] = e_{ij}$, $[x_3, x_4] = e_{ii} - e_{jj}$ and hence $[[x_1, x_2], [x_3, x_4]] = -2e_{ij}$. In this view, it follows from (1) that

$$4^n\big([ae_{ij} + e_{ij}b, e_{ij}]\big)^n - 4[ae_{ij} + e_{ij}b, e_{ij}] = 0.$$

Performing the computations and using the fact that char$(R) \neq 2$ and $n > 1$, we obtain $e_{ij}(b-a)e_{ij} = 0$, where $i \neq j$. It implies that $\alpha_{ji} = 0$ for all $i \neq j$, hence $b - a$ is a diagonal matrix. For any C-automorphism ξ of R, $\xi(b-a)$ enjoys the same property as $b - a$ does; i.e.,

$$\begin{aligned}&\big([\xi(a)[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]\xi(b), [[x_1,x_2],[x_3,x_4]]]\big)^n - \\ &\quad -[\xi(a)[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]\xi(b), [[x_1,x_2],[x_3,x_4]]].\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in R$, implies that $\xi(b-a)$ is a diagonal matrix. In particular, let $\xi(x) = (1 + e_{ij})x(1 - e_{ij})$, where $i \neq j$, then we see that the (j,i)-th entry of $\xi(b-a)$ is zero, i.e.,

$$0 = (\xi(b-a))_{ij} = \alpha_{ij} - \alpha_{ii} + \alpha_{jj} - \alpha_{ji} = -\alpha_{ii} + \alpha_{jj}.$$

It implies $\alpha_{ii} = \alpha_{jj}$ with $i \neq j$. It forces that $b - a$ central element in R. $\square$

Lemma 2. *Let $R = M_k(C)$ be the ring of all $k \times k$ matrices over the field C with char$(R) \neq 2$ and $q \in R$. If $k \geqslant 2$, and $n > 1$ a fixed integer such that*

$$\left([q[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]\right)^n = [q[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]$$

for all $x_1, x_2, x_3, x_4 \in R$, then $q \in Z(R)$.

Proof. Let $q \in R$, i.e., $q = \sum_{r,s=1}^{k} q_{rs}e_{rs}$, where $q_{rs} \in C$ and e_{rs} denotes the usual matrix unit with (r,s)-th entry 1 and 0 elsewhere. For $i \neq j$, we choose $x_1 = e_{ii}$, $x_2 = e_{ij}$, $x_3 = e_{ij}$ and $x_4 = e_{ji}$. With this, we have $[[x_1,x_2],[x_3,x_4]] = -2e_{ij}$. In this view, our situation yields

$$4^n([qe_{ij}, e_{ij}])^n - 4[qe_{ij}, e_{ij}] = 0.$$

Since $n > 1$ and char$(R) \neq 2$, we get $e_{ij}qe_{ij} = 0$, where $i \neq j$. It implies that $q_{ji} = 0$ for all $i \neq j$, hence q is a diagonal matrix. With the same reasoning of Lemma 1, we find that $q \in Z(R)$. □

Proposition 1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, U the Utumi quotient ring and C the extended centroid of R. If for some $a, b \in U$ and a fixed integer $n > 1$,*

$$[a[u,v] + [u,v]b, [u,v]]^n = [a[u,v] + [u,v]b, [u,v]]$$

for all $u, v \in [R,R]$, then either R satisfies s_4 and $b - a \in C$ or $a, b \in C$.

Proof. By our assumption, R satisfies the generalized polynomial identity

$$\begin{aligned}&[a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]]^n = \\ &= [a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]].\end{aligned} \tag{2}$$

Let us assume that

$$\begin{aligned}\Omega(x_1,x_2,x_3,x_4) = {}&[a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]]^n - \\ &-[a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]].\end{aligned}$$

Since R and U satisfy the same generalized polynomial identities (see Remark 2), we have $\Omega(x_1,x_2,x_3,x_4) = 0$ for all $x_1,x_2,x_3,x_4 \in U$. In case C is infinite, then $\Omega(x_1,x_2,x_3,x_4) = 0$ for all $x_1,x_2,x_3,x_4 \in U \otimes_C \overline{C}$, where $\overline{C}$ denotes the algebraic closure of C. Since U and $U \otimes_C \overline{C}$ are centrally closed (see [8, Theorem 2.5, Theorem 3.5]), we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite, respectively. Therefore, we may assume that R is centrally closed over C, which is either finite or algebraically closed. If both $a, b \in C$, then we have nothing to prove. Therefore we assume that at least one of a and b is not in C. Then by Remark 2, $\Omega(x_1,x_2,x_3,x_4)$ is a nontrivial generalized polynomial identity for R. Now, with the aid of Martindale's theorem [19], R is a primitive ring having nonzero socle $\mathcal{H}$ with C as the associated division ring. In this sequel, a result due to Jacobson [14, p. 75] yields that R is isomorphic to a dense ring of linear transformations of some vector space V over C. For some positive integer k, let $\dim_C(V) = k < \infty$, then by density of R on V, $R \cong M_k(C)$. In view of our assumption $\dim_C(V) \neq 1$. Moreover, in case $\dim_C(V) = 2$, then R satisfies s_4 and $b - a \in C$ by Lemma 1.

We now assume that $\dim_C(V) \geqslant 3$. For any $v \in V$, we first show that the vectors v and bv are linearly C-dependent. In this view, we suppose that for some $0 \neq v$, the set $\{v, bv\}$ is linearly C-independent and show that a contradiction follows. Since $\dim_C(V) \geqslant 3$, there exists some $w \in V$ such that the set $\{v, bv, w\}$ is linearly C-independent. By the density of R, there exist $x_1, x_2, x_3, x_4 \in R$ such that

$$x_1v = 0; \quad x_2v = -w; \quad x_3v = 0; \quad x_4v = w;$$

$$x_1bv = v; \quad x_2bv = 0; \quad x_3bv = 0; \quad x_4bv = w;$$
$$x_1w = v; \quad x_2w = bv; \quad x_3w = v; \quad x_4w = 0.$$

With all this, our hypothesis implies that

$$\begin{aligned} 0 &= (([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]])^n - \\ &\quad -([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]]))v = \\ &= (2^n-2)v, \end{aligned}$$

Since $\operatorname{char}(R) \neq 2$, it leads a contradiction. Thus for any $v \in V$, the vectors v and bv are linearly C-dependent. Therefore, there exists some $\tau_v \in C$ such that $bv = \tau_v v$ for all $v \in V$. By a standard argument, one can easily check that τ_v is not depending on the choice of v, i.e., $bv = \tau v$ for all $v \in V$. In this view, we have

$$\begin{aligned} [b,u]v &= (bu)v - u(bv) \\ &= \tau uv - u\tau v \\ &= 0 \end{aligned}$$

for all $v \in V$. This argument shows that for each $u \in V$, $[b,u]$ acts faithfully as a linear transformation on the vector space V, and hence $[b,u] = 0$, i.e., $b \in Z(R)$. Now Eq. (2) implies that

$$[(a+b)[[x_1,x_2],[x_3,x_4]], [[x_1,x_2],[x_3,x_4]]]^n - [(a+b)[[x_1,x_2],[x_3,x_4]], [[x_1,x_2],[x_3,x_4]]] = 0.$$

for all $x_1, x_2, x_3, x_4 \in R$. In this case, we get $a+b \in C$ by Lemma 2. Hence, $a \in C$.

In case $\dim_C(V) = \infty$, by Wong [24, Lemma 2], R satisfies the generalized polynomial identity

$$([a[u,v] + [u,v]b, [u,v]])^n - [a[u,v] + [u,v]b, [u,v]] = 0.$$

In this case the conclusion follows from [22, Proposition]. It completes the proof. □

3.1. Proof of Theorem 1.2

It is well known that every generalized derivation G takes the form $G(x) = ax + g(x)$ for all $x \in R$, where $a \in U$ (see Remark 1). By [4, Lemma 1], there exists a nonzero ideal I of R such that $[I,I] \subseteq L$. Therefore our hypothesis gives

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]), [[x_1,x_2],[x_3,x_4]]]\big)^n = \\ &\quad = [a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]), [[x_1,x_2],[x_3,x_4]]] \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in I$. In light of Remark 2, we find that R satisfies the GPI

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]), [[x_1,x_2],[x_3,x_4]]]\big)^n - \\ &\quad -[a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]), [[x_1,x_2],[x_3,x_4]]]. \end{aligned} \tag{3}$$

If g is the U-inner derivation, i.e., for some $c \in U$, $g(x) = [c,x]$ for all $x \in R$. In this view, we have $G(x) = (a+c)x - xc$ for all $x \in R$. By Proposition 1, we are done.

We now assume that g is not U-inner, in this case we call g an outer derivation. On expending (3), we get

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + [[g(x_1),x_2],[x_3,x_4]] + [[x_1,g(x_2)],[x_3,x_4]] + [[x_1,x_2],[g(x_3),x_4]] + \\ &+[[x_1,x_2],[x_3,g(x_4)]], [[x_1,x_2],[x_3,x_4]]]\big)^n - [a[[x_1,x_2],[x_3,x_4]] + [[g(x_1),x_2],[x_3,x_4]] + \\ &+[[x_1,g(x_2)],[x_3,x_4]] + [[x_1,x_2],[g(x_3),x_4]] + [[x_1,x_2],[x_3,g(x_4)]], [[x_1,x_2],[x_3,x_4]]] = 0. \end{aligned}$$

With aid of a result due to Kharchenko [16, Theorem 2], R and hence Usatisfies the GPI

$$\begin{gathered}\big([a[[x_1,x_2],[x_3,x_4]]+[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+\\+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n-[a[[x_1,x_2],[x_3,x_4]]+[[A,x_2],[x_3,x_4]]+\\+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]=0.\end{gathered}$$

In particular, we find

$$\big([a[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]\big)^n-[a[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]=0,$$

for all $x_1,x_2,x_3,x_4\in R$. In view of Lemma 2, it implies that $a\in C$. Thus the above relation reduces to

$$\begin{gathered}\big([[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n-\\-[[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]=0.\end{gathered}$$

Take $A=B=M=0$. It implies that

$$\big([[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n=[[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]].\tag{4}$$

Since it is a polynomial identity for R, in view of a result due to Lanski [17, Lemma 1], it follows that for a suitable field F, we have $R\cong M_k(F)$, moreover R and $M_k(F)$ satisfy same generalized polynomial identity. Since R is noncommutative, $k\geqslant 2$. Choose $x_1=e_{ij}$, $x_2=e_{ji}$, $x_3=e_{jj}$, $x_4=e_{ji}$, $N=-2e_{ij}$. With this, we have $[x_1,x_2]=e_{ii}-e_{jj}$, $[x_3,\ x_4]=e_{ji}$ and $[x_3,N]=2e_{ij}$. In this view, from (4), we have

$$(-1)^n8^n(e_{ii}-e_{jj})^n=8(e_{ii}-e_{jj}).\tag{5}$$

If $n=2$, we find $8^2(e_{ii}-e_{jj})^2=8(e_{ii}-e_{jj})$. It implies that $7e_{ii}=-9e_{jj}$ with $i\neq j$, a contradiction. Now we suppose that $n>2$. Right multiply the (5) by e_{ij}, we get

$$(-1)^n8^ne_{ij}=8e_{ij},\text{ i.e., }(-1)^n8^{n-1}e_{ij}=e_{ij}$$

with $i\neq j$, a contradiction. It completes the proof.

Following are immediate consequences of Theorem 1.2.

Corollary 1 ([22], Main Theorem)**.** *Let R be a noncommutative prime ring with char$(R)\neq 2$, U the Utumi qotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If G is a generalized derivation of R with an associated derivation d of R such that $[G(u),u]^n=$ $=[G(u),u]$ for all $u\in L$, where $n>1$ a fixed integer, then one of the following holds:*

(i) R satisfies the s_4 identity and there exists $a\in U$ and $\lambda\in C$ such that $G(x)=ax+xa+\lambda x$ for all $x\in R$.

(ii) there exists $\gamma\in C$ such that $G(x)=\gamma x$ for all $x\in R$.

Corollary 2. *Let $n>1$ are fixed integers. Next, let R be a prime ring with char$(R)\neq 2$, U the Utumi quotient ring, C the extended centroid of R and L a noncentral Lie ideal of R. If T is a left multiplier of R such that $([T(u),u])^n=[T(u),u]$ for all $u\in\{[x,y]:x,y\in L\}$, then there exists $\lambda\in C$ such that $T(x)=\lambda x$ for all $x\in R$.*

Proof. It is well know that every left multiplier is generalized derivation with $g = 0$. Hence, G takes the form $G(x) = ax$ for all $x \in R$ and some $a \in U$. The given hypothesis gives that R satisfies

$$[a[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]]^n = [a[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]].$$

Set $\Omega(x_1, x_2, x_3, x_4) = [[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]$, a multilinear polynomial in the variables x_1, x_2, x_3, x_4. Thus, R satisfies

$$[a\Omega(x_1, x_2, x_3, x_4), \Omega(x_1, x_2, x_3, x_4)]^n = [a\Omega(x_1, x_2, x_3, x_4), \Omega(x_1, x_2, x_3, x_4)].$$

In light of [1, Lemma 3.10], it follows that $a \in C$, which completes the proof. $\square$

3.2. Proof of Theorem 1.3

Let $m,\ n \geqslant 1$ be fixed integers and $T : R \to R$ be left multiplier such that $[T^m(u), u]^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where L is noncentral Lie ideal of R. Then, by using induction on m, it is straightforward to check that T is a left multiplier of a ring R if and only of T^m is a left multiplier of R. Hence, direct application of Corollary 3.5 yields the required result. This completes the proof of Theorem 1.3.

We conclude this article with the following example which demonstrates that Theorem 1.2 does not holds for arbitrary rings.

Example 1. *Let $\mathbb{H}$ denotes the ring of quaternions and*

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d \in \mathbb{H} \right\},$$

and $L = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, d \in \mathbb{H} \right\}$ *be the noncentral Lie ideal of R. It can be seen that R is not a prime ring. Let us define a mappings $g,\ G : R \to R$ such that* $G\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$ *and* $g\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$. *It is easy to check that G is a nonzero generalized derivation with an associative derivation g of R and satisfying the identity $[G(u), u]^n = [G(u), u]$ for all $u \in [L, L]$. Since $\mathbb{H}$ is a noncommutative rings, it is not difficult to accomplish that R does not satisfy the identity $[[x^2, y], [x, y]]$, which is equivalent to s_4 (see [5, Lemma 1]), consequently R does not satisfy s_4. Therefore, neither R satisfies s_4 nor G takes the form $G(x) = \lambda x$ for all $x \in R$ and some $\lambda \in C$. Hence, the assumption of primeness in Theorem 1.2 can not be omitted.*

The research of the second author is supported by SERB-DST Project India under Grant no. MTR/2019/000603.

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Идемпотентные значения коммутаторов с обобщенными дифференцированиями

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Аннотация. В настоящей статье мы характеризуем обобщенные дифференцирования и левые мультипликаторы первичных колец, включающие коммутаторы с идемпотентными значениями. А именно, мы доказываем, что если первичное кольцо характеристики, отличной от 2, допускает обобщенное дифференцирование G с ассоциативным ненулевым дифференцированием g кольца R такое, что $[G(u), u]^n = [G(u), u]$ для всех $u \in \{[x, y] : x, y \in L\}$, где L — нецентральный идеал Ли R, а $n > 1$ — фиксированное целое число, то выполняется одно из следующих утверждений:

(i) R удовлетворяет s_4 и существует λ $\in$C, расширенный центр тяжести R, такой, что $G(x) = ax + xa + \lambda x$ для всех $x \in R$, где $a \in U$, фактор-кольцо Утуми кольца R,

(ii) существует $\lambda \in C$, такое, что $G(x) = \gamma x$ для всех $x \in R$.

В качестве приложения опишем строение левых мультипликаторов первичных колец, удовлетворяющих условию $([T^m(u), u])^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, где $m, n > 1$ — фиксированные целые числа. В заключение приведем пример, показывающий, что условие нашей основной теоремы не является избыточным.

Ключевые слова: первичное кольцо, идеал Ли, обобщенный вывод, GPI.