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Coupled Fixed Point Theorems Via Mixed Monotone Property in A_b-metric Spaces & Applications to Integral Equations

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Abstract. In this paper, we establish some results on the existence and uniqueness of coupled common fixed point theorems in partially ordered A_b-metric spaces. Examples have been provided to justify the relevance of the results obtained through the analysis of extant theorem. Further, we also find application to integral equations via fixed point theorems in A_b-metric spaces.

Keywords: Coupled fixed point, Mixed weakly monotone property, A_b-metric space, Integral equation.

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1. Introduction and preliminaries

The study of fixed point theory comes from wider area of non-linear function analysis. However, its study began almost a century ago in the field of algebraic topology. Fixed point theorems find applications in proving the existence and uniqueness of the solutions of certain differential and integral equations that arise in physical, engineering and other optimization problems. In the study of fixed point theory, some of the generalizations of metric space are 2-metric space, D-metric space, D^*-metric space, G-metric space, S-metric space, Rectangular metric or metric-like space, Partial metric space, Cone metric space. In 1989, I. A. Bakhtin [2] introduced the concept

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of b-metric space. Consequent upon the introduction of b-metric space, many generalizations of metric spaces came into existence. In 2015, M. Abbas et al. [1] introduced the concept of n-tuple metric space and studied its topological properties. M. Ughade et al. [15] introduced the notion of A_b-metric spaces as a generalized form of n-tuple metric space. Subsequently N. Mlaiki et al. [11] obtained unique coupled common fixed point theorems in partially ordered A_b-metric spaces.

In this paper, we use the notion of a mixed weakly monotone pair of maps to state a coupled common fixed point theorem on partially ordered A_b-metric spaces. We prove some unique coupled common fixed point theorems in partially ordered A_b-metric space and also provide example to support our results.

First we recall some notions, lemmas and examples which will be useful to prove our results.

Definition 1.1 (M. Abbas et al. [1]). *Let $\Im$ be a non empty set and $n(\geqslant 2)$ be a positive integer. A function $A : \Im^n \to [0,\infty)$ is called an A-metric on $\Im$, if for any ζ_i, $a \in \Im$. $i = 1, 2, \ldots, n$, the following conditions hold.*

(i) $A(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) \geqslant 0$,

(ii) $A(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) = 0$ *if and only if* $\zeta_1 = \zeta_2 = \cdots = \zeta_{n-1} = \zeta_n$,

(iii) $A(\zeta_1, \zeta_2, \ldots \zeta_{n-1}, \zeta_n) \leqslant [A(\zeta_1, \zeta_1, \ldots, \zeta_{1_{(n-1)}}, a) + A(\zeta_2, \zeta_2, \ldots, \zeta_{2_{(n-1)}}, a) + + \cdots + A(\zeta_{n-1}, \zeta_{n-1}, \ldots, \zeta_{n-1_{(n-1)}}, a) + A(\zeta_n, \zeta_n, \ldots, \zeta_{n_{(n-1)}}, a)]$.

The pair $(\Im, A)$ is called an A-metric space.

Definition 1.2 (T. G. Bhaskar et al. [6]). *Let X be a non empty set. A b-metric on X is a function $d : X^2 \to [0,\infty)$ such that the following conditions hold for all $x, y, z \in X$.*

(i) $d(x, y) = 0 \iff x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) there exists $s \geqslant 1$, *such that* $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Definition 1.3 (M. Ughade et al. [14]). *Let $\Im$ be a non empty set and $n \geqslant 2$. Suppose $b \geqslant 1$ is a real number. A function $A_b : \Im^n \to [0,\infty)$ is called an A_b -metric on $\Im$, if for any $\zeta_i, a \in \Im$, $i = 1, 2, \ldots, n$, the following conditions hold.*

(i) $A_b(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) \geqslant 0$,

(ii) $A_b(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) = 0$ *if and only if* $\zeta_1 = \zeta_2 = \cdots = \zeta_{n-1} = \zeta_n$,

(iii) $A_b(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) \leqslant b[A_b(\zeta_1, \zeta_1, \ldots, \zeta_{1_{(n-1)}}, a) + A_b(x_2, x_2, \ldots, x_{2_{(n-1)}}, a) + \ldots + A_b(\zeta_{n-1}, \zeta_{n-1}, \ldots, \zeta_{n-1_{(n-1)}}, a) + A_b(\zeta_n, \zeta_n, \ldots, \zeta_{n_{(n-1)}}, a)]$.

The pair $(\Im, A_b)$ is called an A_b-metric space.

Note: In practice we write A for A_b when there is no confusion.

Example 1.4 (M. Ughade et al. [14]). *Let $\Im = [1,\infty)$ and $n \geqslant 2$. Define $A_b : \Im^n \to [1,\infty)$ by $A_b(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \zeta_n) = \sum_{i=1}^{n} \sum_{i<j} |\zeta_i - \zeta_j|^2$, for all $\zeta_i \in \Im$, $i = 1, 2, \ldots, n$. Then $(\Im, A_b)$ is an A_b-metric space with b=2.*

Lemma 1.5 (M. Ughade et al. [14]). *Let $(\Im, A)$ be A_b metric space, so that $A : \Im^n \to [0,\infty)$ for some $n \geqslant 2$. Then $A(\underbrace{\zeta, \zeta, \ldots, \zeta}_{(n-1)times}, y) \leqslant bA(\underbrace{y, y, \ldots, y}_{(n-1)times}, \zeta)$, for all $\zeta, y \in \Im$.*

Lemma 1.6 (M. Ughade et al. [14]). *Let $(\Im, A)$ be A_b metric space, so that $A : \Im^n \to [0,\infty)$ for some $n \geqslant 2$. Then* $A(\underbrace{\zeta,\zeta,\ldots,\zeta}_{(n-1)times},z) \leqslant (n-1)b\ A(\underbrace{\zeta,\zeta,\ldots,\zeta}_{(n-1)times},y) + b^2\ A(\underbrace{y,y,\ldots,y}_{(n-1)times},z)$, *for all* $\zeta, y, z \in \Im$.

Lemma 1.7 (M. Ughade et al. [14]). *Let (X,A) be A_b metric space. Then (X^2, D_A) is A_b-metric space on $X \times X$ with D_A defined by*
$D_A((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) = A(x_1,x_2,\ldots,x_n) + A(y_1,y_2,\ldots,y_n)$ *for all* $x_i, y_i \in X$, $i,j = 1,2,\ldots,n$.

Definition 1.8. *Let (X,A) be A_b-metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$, if* $A(\underbrace{x_n,x_n,\ldots,x_n}_{(n-1)times},x) \to 0$ *as $n \to \infty$. That is, to each $\varepsilon \geqslant 0$ there exist $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have* $A(\underbrace{x_n,x_n,\ldots,x_n}_{(n-1)times},x) \leqslant \varepsilon$ *and we write* $\lim\limits_{n\to\infty} x_n = x$.

Note: x is called the limit of the sequence $\{x_n\}$.

Lemma 1.9 (N. Mlaiki et al. [11]). *Let (X,A) be A_b-metric space. If the sequence $\{x_n\}$ in X converges to a point x, then the limit x is unique.*

Definition 1.10. *Let (X,A) be A_b-metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if* $A(\underbrace{x_n,x_n,\ldots,x_n}_{(n-1)times},x_m) \to 0$ *as $n,m \to \infty$. That is, to each $\varepsilon \geqslant 0$, there exists $N \in \mathbb{N}$ such that for all $n,m \geqslant N$, we have* $A(\underbrace{x_n,x_n,\ldots,x_n}_{(n-1)times},x_m) \leqslant \varepsilon$.

Lemma 1.11 (N. Mlaiki et al. [11]). *Every convergent sequence in a A_b-metric space is a Cauchy sequence.*

Definition 1.12. *A A_b-metric space (X,A) is said to be complete, if every Cauchy sequence in X is convergent.*

Definition 1.13 (M. E. Gordji et al. [7]). *Let $(X, \leqslant)$ be a partially ordered set and $f, g : X \times X \to X$ be mappings. We say that (f,g) has the mixed weakly monotone property on X, if for any $x, y \in X$,*
$x \leqslant f(x,y),\ y \geqslant f(y,x) \implies f(x,y) \leqslant g((f(x,y),f(y,x)),\ f(y,x) \geqslant g((f(y,x),f(x,y))$
and
$x \leqslant g(x,y),\ y \geqslant g(y,x) \implies g(x,y) \leqslant f((g(x,y),g(y,x)),\ g(y,x) \geqslant f(g(y,x),g(x,y))$.

Definition 1.14. *Let X be a non-empty set and $f, g : X \times X \to X$ be maps on $X \times X$.*
(i) A point $(x,y) \in X \times X$ is called a coupled fixed pint of f, if $x = f(x,y)$ and $y = f(y,x)$
(ii) A point $(x,y) \in X \times X$ is said to be a common coupled fixed pint of f and g, if $x = f(x,y) = g(x,y)$ and $y = f(y,x) = g(y,x)$.

Note: (x,y) is said to be a Coupled coincidence point of f and g, if $f(x,y) = g(x,y)$ and $f(y,x) = g(y,x)$.

We observe that a common coupled fixed pint of f and g is necessarily a Coupled coincidence point of f and g.

2. Main results

Now we prove our first main result.

Theorem 2.1. *Let $(X, \leqslant, A)$ be a partially ordered, complete A_b-metric space and let $f, g : X \times X \to X$ be the mappings such that*

(i) the pair (f, g) has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant f(x_0, y_0)$, $f(y_0, x_0) \leqslant y_0$ or $x_0 \leqslant g(x_0, y_0)$, $g(y_0, x_0) \leqslant y_0$,

(ii) there is an α such that $\alpha b^2((n-1)b+1) < 1$ and

$$A(f(x,y), f(x,y), \dots, f(x,y), g(u,v)) + A(f(y,x), f(y,x), \dots, f(y,x), g(v,u)) \leqslant \alpha M,$$

where

$$\begin{aligned} M = \max\Bigg\{ & \Bigg[\big(1 + D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))\big) \times \\ & \times \frac{(D((u,v),(u,v),\dots,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),\dots,(x,y),(u,v)))}\Bigg], D\big((x,y),(x,y),\dots,(x,y),(u,v)\big), \\ & \big(D((x,y),(x,y),\dots(x,y),(f(x,y),f(y,x))) + D((u,v),(u,v),\dots(u,v),(g(u,v),g(v,u)))\big), \\ & \big(D((u,v),(u,v),\dots(u,v),(f(x,y),f(y,x))) + D((x,y),(x,y),\dots(x,y),(g(u,v),g(v,u)))\big)\Bigg\} \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,

(iii) if f or g is continuous.

Then f and g have a coupled common fixed point in X.

Proof. Let (x_0, y_0) be a given point in $X \times X$, satisfying (i). Write $x_1 = f(x_0, y_0)$, $y_1 = f(y_0, x_0)$, $x_2 = g(x_1, y_1)$, $y_2 = g(y_1, x_1)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ inductively

$$\begin{aligned} x_{2n+1} = f(x_{2n}, y_{2n}),\ y_{2n+1} = f(y_{2n}, x_{2n}) \\ x_{2n+2} = g(x_{2n+1}, y_{2n+1}),\ y_{2n+2} = g(y_{2n+1}, x_{2n+1}) \\ \text{for all } n \in \mathbb{N} \end{aligned} \tag{2.2}$$

Since $x_0 \leqslant f(x_0, y_0)$ and $y_0 \geqslant f(y_0, x_0)$ and since (f, g) has mixed weakly monotone property, we have

$x_1 = f(x_0, y_0) \leqslant g(f(x_0, y_0), f(y_0, x_0)) = g(x_1, y_1) = x_2 \implies x_1 \leqslant x_2$

and $x_2 = g(x_1, y_1) \leqslant f(g(x_1, y_1), g(y_1, x_1)) = f(x_2, y_2) = x_3 \implies x_2 \leqslant x_3$

also $y_1 = f(y_0, x_0) \geqslant g(f(y_0, x_0), f(x_0, y_0)) = g(y_1, x_1) = y_2 \implies y_1 \geqslant y_2$

and $y_2 = f(y_1, x_1) \geqslant f(g(y_1, x_1), g(x_1, y_1)) = f(y_2, x_2) = y_3 \implies y_2 \geqslant y_3$.

By induction,

$$\begin{aligned} i.e,\ x_0 \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_n \leqslant x_{n+1} \leqslant \dots \\ y_0 \geqslant y_1 \geqslant y_2 \geqslant \dots \geqslant y_n \geqslant y_{n+1} \geqslant \dots \\ \text{for all } n \in \mathbb{N} \end{aligned} \tag{2.3}$$

Now we show that these sequences are Cauchy.
Define $D_n : X \times X \to X$ by

$$\begin{aligned} D_n &= D((x_n, y_n), (x_n, y_n), \dots, (x_n, y_n), (x_{n+1}, y_{n+1})) \\ &= A(x_n, x_n, \dots, x_n, x_{n+1}) + A(y_n, y_n, \dots, y_n, y_{n+1}) \text{ for all } x_i, y_i \in X, i, j = 1, 2, \dots, n. \end{aligned}$$

Now

$$D_{2n+1} = A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2}) =$$
$$= A(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), \ldots, f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1}))+$$
$$+ A(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}), \ldots, f(y_{2n}, x_{2n}), g(y_{2n+1}, x_{2n+1})) \leqslant$$

$$\leqslant \alpha \max\Bigg\{ \Bigg[\big(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))\big)\times$$
$$\times \frac{(D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \ldots, (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))}{(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})))}\Bigg],$$
$$D\big((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\big),$$
$$\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))+$$
$$+ (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \ldots, (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))\big),$$
$$\big((D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \ldots, (x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))))+$$
$$+ (D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))\big)\Bigg\} \leqslant$$
$$\leqslant \alpha \max\{D\big((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \ldots, (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})\big),$$
$$D\big((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\big),$$
$$\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))+$$
$$+ (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \ldots, (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))\big),$$
$$\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \ldots, (x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2}))\big)\}.$$

By using Lemma 1.6, we have

$$D_{2n+1} \leqslant \alpha\big\{(n-1)b[A(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \ldots, y_{2n}, y_{2n+1})] + \\ + b^2[A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2})]\big\}. \tag{2.4}$$

Similarly, we get

$$A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2}) + A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) \leqslant \\ \leqslant \alpha\{(n-1)b[A(y_{2n}, y_{2n}, \ldots, y_{2n}, y_{2n+1}) + A(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1})] + \\ + b^2[A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2}) + A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2})]\}. \tag{2.5}$$

From (2.4) and (2.5) we have,

$$2D_{2n+1} = 2[A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2})] \leqslant \\ \leqslant 2\alpha\{(n-1)b[A(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \ldots, y_{2n}, y_{2n+1})] + \\ + b^2[A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2})]\}.$$

Therefore

$$D_{2n+1} \leqslant \alpha\{(n-1)b[A(x_{2n}, x_{2n}, \ldots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \ldots, y_{2n}, y_{2n+1})]+ \\ + b^2[A(x_{2n+1}, x_{2n+1}, \ldots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \ldots, y_{2n+1}, y_{2n+2})]\}. \tag{2.6}$$

It gives that

$$D_{2n+1} \leqslant \frac{\alpha(n-1)b}{1-\alpha b^2} D_{2n}. \tag{2.7}$$

Put $\beta = \dfrac{\alpha(n-1)b}{1-\alpha b^2}$, then $0 \leqslant \beta < 1$.
From (2.7),

$$D_{2n+1} \leqslant \beta D_{2n}.$$

Similarly we can show that

$$D_{2n+2} \leqslant \beta D_{2n+1} \text{ for } \ n = 0, 1, 2, \dots .$$

Hence

$$D_{n+1} \leqslant \beta D_n.$$

Therefore

$$D_{n+1} \leqslant \beta^{n+1} D_0. \tag{2.8}$$

Define

$$\begin{aligned} D_{n,m} &= D(\underbrace{(x_n, y_n), (x_n, y_n), \dots, (x_n, y_n)}_{(n-1)-times}, (x_m, y_m)) = \\ &= A(\underbrace{x_n, x_n, \dots, x_n}_{(n-1)-times}, x_m) + A(\underbrace{y_n, y_n, \dots, y_n}_{(n-1)-times}, y_m). \end{aligned}$$

Now we have to show that $D_{n,m}$ is a Cauchy sequence.
By Lemma 1.6, for all $n, m \in \mathbb{N}$, $n \leqslant m$, we have

$$\begin{aligned} D_{n+1,m+1} = {} & A(x_{n+1}, x_{n+1}, \dots, x_{n+1}, x_{m+1}) + A(y_{n+1}, y_{n+1}, \dots, y_{n+1}, y_{m+1}) \leqslant \\ & \leqslant b(n-1)[A(x_{n+1}, x_{n+1}, \dots, x_{n+1}, x_{n+2}) + A(y_{n+1}, y_{n+1}, \dots, y_{n+1}, y_{n+2})] + \\ & + b^2[A(x_{n+2}, x_{n+2}, \dots, x_{n+2}, x_{m+1}) + A(y_{n+2}, y_{n+2}, \dots, y_{n+2}, y_{m+1})] = \\ & = b(n-1)D_{n+1} + b^2 b(n-1)[A(x_{n+2}, x_{n+2}, \dots, x_{n+2}, x_{n+3}) + \\ & + A(y_{n+2}, y_{n+2}, \dots, y_{n+2}, y_{n+3})] + \\ & + b^2 b^2[A(x_{n+3}, x_{n+3}, \dots, x_{n+3}, x_{m+1}) + A(y_{n+3}, y_{n+3}, \dots, y_{n+3}, y_{m+1})] = \\ & = b(n-1)D_{n+1} + b^3(n-1)D_{n+2} + b^5(n-1)D_{n+3} \cdots + \\ & + b^{2(m-n)-3}(n-1)[A(x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) + A(y_{m-1}, y_{m-1}, \dots, y_{m-1}, y_m)] + \\ & + b^{2(m-n)-1}(n-1)[A(x_m, x_m, \dots, x_m, x_{m+1}) + A(y_m, y_m, \dots, y_m, y_{m+1})]. \end{aligned}$$

From (2.8), we have that

$$\begin{aligned} D_{n+1,m+1} & \leqslant b(n-1)[\beta^{n+1} + b^2\beta^{n+2} + b^4\beta^{n+3} + \dots + b^{2(m-n)-2}\beta^m]D_0 \leqslant \\ & \leqslant b(n-1)\beta^{n+1}[1 + b^2\beta + (b^2\beta)^2 + \dots + (b^2\beta)^{(m-n-1)}]D_0 = \\ & = b(n-1)\beta^{n+1}[1 + \gamma + \gamma^2 + \dots + \gamma^{(m-n-1)}]D_0 \leqslant \\ & \leqslant b(n-1)\beta^{n+1}\left(\frac{1}{1-\gamma}\right)D_0 \\ & \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus

$$\lim_{n,m\to\infty} A(x_n, x_n, \dots, x_n, x_m) = \lim_{n,m\to\infty} A(y_n, y_n, \dots, y_n, y_m) = 0.$$

Therefore $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in X.
By the completeness of X, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Therefore $D_{n,m}$ is a Cauchy sequence.
Now we show that (x, y) is a coupled fixed point of f and g.
Without loss of generality, we may suppose that f is continuous, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, y_{2n}) = f\left(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}\right) = f(x, y)$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f\left(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}\right) = f(y, x).$$

Thus (x, y) is a coupled fixed point of f.
From (2.1), taking $x = u$ *and* $y = v$, we have,

$$\begin{aligned}
& A(x, x, , \dots, x, g(x, y)) + A(y, y, \dots, y, g(y, x)) = \\
= \; & A(f(x, y), f(x, y), \dots, f(x, y), g(x, y)) + A(f(y, x), f(y, x), \dots, f(y, x), g(y, x)) \leqslant \\
\leqslant \; & \alpha \max\left\{\left[(1 + D((x, y), (x, y) \dots (x, y), (x, y))) \frac{(D((x, y), (x, y) \dots (x, y), (g(x, y), g(y, x))))}{(1 + D((x, y), (x, y) \dots (x, y), (x, y)))}\right],\right. \\
& D\big((x, y), (x, y), \dots, (x, y), (x, y)\big), \big(D((x, y), (x, y), \dots, (x, y), (x, y)) + \\
& + D((x, y), (x, y), \dots, (x, y), (g(x, y), g(y, x)))\big), \big(D((x, y), (x, y), \dots, (x, y), (x, y)) + \\
& \left. + D((x, y), (x, y), \dots, (x, y), (g(x, y), g(y, x)))\big)\right\} \leqslant \\
\leqslant \; & \alpha b \; ((g(x, y), g(y, x)), (g(x, y), g(y, x)), \dots, (g(x, y), g(y, x)), (x, y)).
\end{aligned}$$

Since $\alpha b < 1$, we have $(g(x, y), g(y, x)) = (x, y)$.
Therefore $g(x, y) = x$ and $g(y, x) = y$.
Therefore (x, y) is a coupled fixed point of g.
Thus (x, y) is a coupled common fixed point of f and g. □

Theorem 2.2. *Let $(X, \leqslant, A)$ be a partially ordered, complete A_b-metric space and $f, g : X \times X \to X$ be the mappings such that*

(i) the pair (f, g) has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant f(x_0, y_0), f(y_0, x_0) \leqslant y_0$ or $x_0 \leqslant g(x_0, y_0), g(y_0, x_0) \leqslant y_0$,
(ii) there is an α such that $\alpha b^2((n-1)b + 1) < 1$ and

$$A(f(x, y), f(x, y), \dots, f(x, y), g(u, v)) + A(f(y, x), f(y, x), \dots, f(y, x), g(v, u)) \leqslant \alpha M$$

where

$$\begin{aligned}
M = \; & \max\left\{\left[(1 + D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x))))\right.\right. \\
& \left.\frac{(D((u, v), (u, v), \dots, (u, v), (g(u, v), g(v, u))))}{(1 + D((x, y), (x, y), \dots, (x, y), (u, v)))}\right], \\
& D\big((x, y), (x, y), \dots, (x, y), (u, v)\big), \big(D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x))) + \\
& + D((u, v), (u, v), \dots, (u, v), (g(u, v), g(v, u)))\big), \big(D((u, v), (u, v), \dots, (u, v), (f(x, y), f(y, x))) + \\
& \left. + D((x, y), (x, y), \dots, (x, y), (g(u, v), g(v, u)))\big)\right\}
\end{aligned}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,
(iii) X has the following properties

(a) if $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \leqslant x$ for all $n \in \mathbb{N}$,
(b) if $\{y_n\}$ is a decreasing sequence with $y_k \to y$, then $y \leqslant y_n$ for all $n \in \mathbb{N}$.
Then f and g have coupled common fixed points in X.

Proof. Suppose X satisfies (a) and (b), by (2.3) we get $x_n \leqslant x$ and $y_n \geqslant y$ for all $n \in \mathbb{N}$. Applying Lemmas 1.5 and 1.6, we have

$$\begin{aligned} &D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))) \leqslant \\ &\leqslant\ b(n-1)D((x,y),(x,y),\dots,(x,y),(x_{2n+2},y_{2n+2})+ \\ &\quad +\ b^2D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),\dots,(x_{2n+2},y_{2n+2}),(f(x,y),f(y,x)))= \\ &=\ b(n-1)D((x,y),(x,y),\dots,(x,y),(x_{2n+2},y_{2n+2}))+ \\ &\quad +\ b^2D((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),\dots \\ &\quad \dots,(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x))). \end{aligned} \tag{2.9}$$

By (2.1), we get

$$\begin{aligned} &A\big((g(x_{2n+1},y_{2n+1})),(g(x_{2n+1},y_{2n+1})),\dots,(g(x_{2n+1},y_{2n+1})),(f(x,y))\big)+ \\ +\ &A\big((g(y_{2n+1},x_{2n+1})),(g(y_{2n+1},x_{2n+1})),\dots,(g(y_{2n+1},x_{2n+1})),(f(y,x))\big) \leqslant \\ \leqslant\ &\alpha \max\Bigg\{\Bigg[\big(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\dots,,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\big)\times \\ &\times\frac{(D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))))}{(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\dots,(x_{2n+1},y_{2n+1}),(x,y)))}\Bigg], \\ &D\big((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\dots,(x_{2n+1},y_{2n+1}),(x,y)\big), \\ &\big(D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\dots,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\ &+D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))\big),\big(D((x,y),(x,y),\dots,(x,y),(x_{2n+2},y_{2n+2}))+ \\ &+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\dots,(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))\big)\Bigg\} \end{aligned}$$

Taking the limit as $n \to \infty$ in (2.9), we obtain

$$D\big((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))\big) \leqslant b^2\alpha\ D\big((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))\big).$$

Since $b^2\alpha < 1$, we have $D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))) = 0$.
That is, $f(x,y) = x$ and $f(y,x) = y$. Therefore (x,y) is a coupled fixed point of f.
Similarly we can show that $g(x,y) = x$ and $g(y,x) = y$. Hence $f(x,y) = x = g(x,y)$ and $f(y,x) = y = g(y,x)$.
Thus (x,y) is a coupled common fixed point of f and g. □

Theorem 2.3. *Suppose Theorem 2.1 or Theorem 2.2 satisfied, if further $\{x_n\}$ is an increasing sequence with $x_n \to x$ and $x_n \leqslant u$ for each n, then $x \leqslant u$. Then f and g have a unique coupled common fixed points. Further more, any fixed point of f is a fixed point of g, and conversely.*

Proof. Suppose the given condition holds. Let (x,y) and $(u,v) \in X \times X$, there exist $(x^*,y^*) \in$

$X \times X$, that is, comparable to (x,y) and (u,v).

$$D((x,y),(x,y),\dots,(x,y),(u,v)) =$$
$$= A(x,x,\dots,x,u) + A(y,y,\dots,y,u) =$$
$$= A(f(x,y),f(x,y),\dots,f(x,y),g(u,v)) + A(f(y,x),f(y,x),\dots,f(y,x),g(v,u)) \leqslant$$
$$\leqslant \alpha \max\Bigg\{\Bigg[(1 + D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))))\times$$
$$\times \frac{(D((u,v),(u,v),\dots,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),\dots,(x,y),(u,v)))}\Bigg], D((x,y),(x,y),\dots,(x,y),(u,v)),$$
$$(D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))) + D((u,v),(u,v),\dots,(u,v),(g(u,v),g(v,u)))),$$
$$(D((u,v),(u,v),\dots,(u,v),(f(x,y),f(y,x))) + D((x,y),(x,y),\dots,(x,y),(g(u,v),g(v,u))))\Bigg\} \leqslant$$
$$\leqslant \alpha\ (b+1)D((x,y),(x,y),\dots,(x,y),(u,v)).$$

Since $\alpha(b+1) < 1$, so that
$D((x,y),(x,y),\dots,(x,y),(u,v)) = 0$
$\implies (x,y) = (u,v) \implies x = u \textit{ and } y = v$
Suppose (x,y) and (x^*,y^*) are Coupled common fixed points such that $x \leqslant x^*$ and $y \geqslant y^*$, then $x = x^*$ and $y = y^*$.
Now

$$\begin{aligned} D((x,y),(x,y),\dots,(x,y),(x^*,y^*)) &= A(x,x,\dots,x,x^*) + A(y,y,\dots,y,y^*) = \\ &= A(f(x,y),f(x,y),\dots,f(x,y),g(x^*,y^*)) + \\ &\quad + A(f(y,x),f(y,x),\dots,f(y,x),g(y^*,x^*)) \leqslant \\ &\leqslant \alpha(b+1)D((x,y),(x,y),\dots,(x,y),(x^*,y^*)). \end{aligned}$$

Since $\alpha(b+1) < 1$, so that
$D((x,y),(x,y),\dots,(x,y),(x^*,y^*)) = 0$
$\implies (x,y) = (x^*,y^*)$
$\implies x = x^* \textit{ and } y = y^*$
we show that any fixed point of f is a fixed point of g, and conversely.
That is, to show that (x,y) is a fixed point of $f \iff (x,y)$ is a fixed point of g.
Suppose that (x,y) is a coupled fixed point of f

$$\begin{aligned} & D((x,y),(x,y),\dots,(x,y),(g(x,y),g(y,x))) = \\ = \ & A(f(x,y),f(x,y),\dots,f(x,y),g(x,y)) + A(f(y,x),f(y,x),\dots,f(y,x),g(y,x)) \leqslant \\ \leqslant \ & \alpha b\ D((g(x,y),g(y,x)),(g(x,y),g(y,x)),\dots,(g(x,y),g(y,x)),(x,y)). \end{aligned}$$

Since $\alpha b < 1$, we have
$D((g(x,y),g(y,x)),(g(x,y),g(y,x)),\dots,(g(x,y),g(y,x)),(x,y)) = 0$
$\implies (g(x,y),g(y,x)) = (x,y)$
$\implies x = g(x,y) \textit{ and } y = g(y,x)$
Therefore (x,y) is a coupled fixed point of g, and conversely. □

Taking $M = D((x,y),(x,y),\dots,(x,y),(u,v))$ and $g = f$ in Theorem 2.1, we get the following

Corollary 2.4. *Let $(X,\leqslant,A)$ be a partially ordered, complete A_b-metric space and let $f : X \times X \to X$ be the mapping such that*

(i) f has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant f(x_0, y_0), f(y_0, x_0) \leqslant y_0$,
(ii) there is an α such that $\alpha < 1$ and

$$\begin{aligned} &A(f(x,y), f(x,y), \ldots, f(x,y), f(u,v)) + A(f(y,x), f(y,x), \ldots, f(y,x), f(v,u)) \leqslant \\ &\leqslant \alpha\, D((x,y),(x,y), \ldots, (x,y),(u,v)), \end{aligned} \tag{2.10}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,
(iii) if f is continuous.
Then f has a coupled fixed point in X.

We give an example to demonstrate the validity of the result 2.1.

Example 2.5. *Let $(\mathbb{R}, \leqslant, A)$ be a partially ordered complete A_b-metric space with A_b-metric defined as $X = [-\infty, +\infty]$ by $A_b : X^n \to [-\infty, +\infty]$ by*
$A_b(x_1, x_2, \ldots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$, for all $x_i \in X$, $i = 1, 2, \ldots, n$. Then (X, A_b) is an A_b-metric space with b=2.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be two maps defined by $f(x,y) = \frac{4x-2y+48n-2}{48n}$ and $g(x,y) = \frac{6x-3y+72n-3}{72n}$.
Then the pair (f, g) has mixed weakly monotone property on $\mathbb{R}$

$$\begin{aligned} &A(f(x,y), f(x,y), \ldots, f(x,y), g(u,v)) + A(f(y,x), f(y,x), \ldots, f(y,x), g(v,u)) = \\ &= (n-1)(|f(x,y) - g(u,v)|) + (n-1)(|f(y,x) - g(v,u)|) = \\ &= (n-1)\left(\left|\frac{4x-2y+48n-2}{48n} - \frac{6u-3v+72n-3}{72n}\right|\right) + \\ &\quad + (n-1)\left(\left|\frac{4y-2x+48n-2}{48n} - \frac{6v-3u+72n-3}{72n}\right|\right) = \\ &= \frac{(n-1)}{24n}(|2(x-u) - (y-v)| + |2(y-v) - (x-u)|) \leqslant \\ &\leqslant \frac{(n-1)}{24n}(3|x-u| + 3|y-v|) \leqslant \\ &\leqslant \frac{(n-1)}{8n}(|x-u| + |y-v|) = \\ &= \frac{(n-1)}{8n}\, D((x,y),(x,y), \ldots, (x,y),(u,v)). \end{aligned}$$

For $n = 2$ and b=2, since $\alpha b^2((n-1)b+1) < 1 \implies \alpha < \frac{1}{12}$.
Then the contractive condition (2.1) is satisfied with $\alpha = \frac{1}{16} < \frac{1}{12}$ and also $(1,1)$ is the unique coupled common fixed point of f and g.

3. Application

The following type system of integral equations:

$$\begin{aligned} u(t) &= q(t) + \int_a^b \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \\ v(t) &= q(t) + \int_a^b \lambda(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds, \end{aligned} \tag{3.1}$$

where the space $X = C([a,b],\mathbb{R})$ of continuous functions defined in $[a,b]$. Obviously, the space with the metric is given by

$$A(u,v) = \max_{t\in[a,b]} |u(t) - v(t)|\,,\ \ u,v \in C([a,b],\mathbb{R})$$

is a complete metric space.
Let $X = C([a,b],\mathbb{R})$ the natural partial order relation, that is,
$u,v \in C([a,b],\mathbb{R}),\ u \leqslant v \iff u(t) \leqslant v(t),\ t \in [a,b]$.

Theorem 3.1. *Consider the corollary 2.4 and assume that the following conditions are hold:*

(i) $f_1, f_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) $q : [a,b] \to \mathbb{R}$ *is continuous;*

(iii) $\lambda : [a,b] \times \mathbb{R} \to [0,\infty)$ *is continuous;*

(iv) there exist $c > 0$ *and* $0 \leqslant \alpha < 1$, *such that for all* $u,v \in \mathbb{R},\ v \geqslant u$,
$0 \leqslant f_1(s,v) - f_1(s,u) \leqslant c\alpha(v-u)$
$0 \leqslant f_2(s,v) - f_2(s,u) \leqslant c\alpha(v-u)$;

(v) assume that $c \max\limits_{t\in[a,b]} \int\limits_a^b \lambda(t,s)ds \leqslant 1$;

(vi) there exist $x_0, y_0 \in X$ *such that*

$$x_0(t) \geqslant q(t) + \int_a^b \lambda(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds$$
$$y_0(t) \leqslant q(t) + \int_a^b \lambda(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds.$$

Then the system of Volterra type integral equation (3.1) has a unique solution in $X \times X$ *with* $X = C([a,b],\mathbb{R})$.

Proof. Define the mapping $F : X \times X \to X$ by

$$F(u,v)(t) = q(t) + \int_a^b \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \tag{3.2}$$

for all $u,v \in X$ *and* $t \in [a,b]$.

Now we have to show that all the conditions of Corollary 2.4 are satisfied.

From (iv) of the Theorem 3.1, clearly F has mixed monotone property.

For $x,y,u,v \in X$ with $x \geqslant u$ and $y \leqslant v$, we have

$$\begin{aligned}
&A(F(x,y), F(x,y), \ldots, F(x,y), F(u,v)) + A(F(y,x), F(y,x), \ldots, F(y,x), F(v,u)) = \\
=\ &(n-1) \max_{t\in[a,b]} (|F(x,y)(t) - F(u,v)(t)| + |F(y,x)(t) - F(v,u)(t)|) = \\
=\ &(n-1) \max_{t\in[a,b]} \left| \int_a^b \lambda(t,s)(f_1(s,x(s)) + f_2(s,y(s)))ds - \int_a^b \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \right| + \\
&+(n-1) \max_{t\in[a,b]} \left| \int_a^b \lambda(t,s)(f_1(s,y(s)) + f_2(s,x(s)))ds - \int_a^b \lambda(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds \right| \leqslant \\
\leqslant\ &(n-1) \max_{t\in[a,b]} \bigg(\int_a^b |f_1(s,x(s)) - f_1(s,u(s))|\, |\lambda(t,s)|\, ds +
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b |f_2(s,y(s))-f_2(s,v(s))|\,|\lambda(t,s)|\,ds+\\
&+\int_a^b |f_1(s,y(s))-f_1(s,v(s))|\,|\lambda(t,s)|\,ds+\int_a^b |f_2(s,x(s))-f_2(s,u(s))|\,|\lambda(t,s)|\,ds\Big)\leqslant\\
\leqslant\;& (n-1)\max_{t\in[a,b]} c\alpha\Big(\int_a^b |x(s)-u(s)|\,|\lambda(t,s)|\,ds+\int_a^b |y(s)-v(s)|\,|\lambda(t,s)|\,ds+\\
&+\int_a^b |y(s)-v(s)|\,|\lambda(t,s)|\,ds+\int_a^b |x(s)-v(s)|\,|\lambda(t,s)|\,ds\Big)\leqslant\\
\leqslant\;& (n-1)\left(\max_{t\in[a,b]}|x(t)-u(t)|+\max_{t\in[a,b]}|y(t)-v(t)|+\right.\\
&\left.+\max_{t\in[a,b]}|y(t)-v(t)|+\max_{t\in[a,b]}|x(t)-u(t)|\right)c\alpha\int_a^b|\lambda(t,s)|\,ds\leqslant\\
\leqslant\;& 2(n-1)\left(\max_{t\in[a,b]}|x(t)-u(t)|+\max_{t\in[a,b]}|y(t)-v(t)|\right)c\alpha\int_a^b|\lambda(t,s)|\,ds\leqslant\\
\leqslant\;& 2(n-1)\;\alpha\;(A(x,x,\dots,x,u)+A(y,y,\dots,y,v))=\\
=\;& 2(n-1)\;\alpha\;D((x,y),(x,y),\dots,(x,y),(u,v)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&A(F(x,y),F(x,y),\dots,F(x,y),F(u,v))+A(F(y,x),F(y,x),\dots,F(y,x),F(v,u))\leqslant\\
&\leqslant 2(n-1)\;\alpha D((x,y),(x,y),\dots,(x,y),(u,v)).
\end{aligned}
$$

For n=2, $\alpha<\dfrac{1}{2}<1$. Which is the contractive condition in Corollary 2.4.

Thus, F has a coupled fixed point in X.

That is, the system of integral equations has a solution. □

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Связанные теоремы о неподвижной точке через свойство смешанной монотонности в A_b-метрических пространствах и приложения к интегральным уравнениям

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Аннотация. В этой статье мы устанавливаем некоторые результаты о существовании и единственности связанных теорем об общей неподвижной точке в частично упорядоченных A_b-метрических пространствах. Приведены примеры для обоснования актуальности результатов, полученных в результате анализа существующей теоремы. Кроме того, мы также находим приложение к интегральным уравнениям через теоремы о неподвижной точке в A_b-метрических пространствах.

Ключевые слова: связанная неподвижная точка, смешанная слабомонотонность, A_b-метрическое пространство, интегральное уравнение.