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Analytic Solvability of the Hörmander Problem and the Borel Transformation of Multiple Laurent Series

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Abstract. In this paper, the initial-boundary value problem of Hormander is formulated in the class of functions representable by Laurent series supported in rational cones. Using the Borel transformation of Laurent series we establish a connection between a differential and a difference problems and prove its global analytic solvability.

Keywords: Hörmander problem, polynomial differential operator, Borel transformation, difference operator.

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1. Introduction and preliminaries

Let $\omega = (\omega_1, \ldots, \omega_n)$ denote a multi-index, $||\omega|| = \omega_1 + \cdots + \omega_n$, $\partial = (\partial_1, \ldots, \partial_n)$, where ∂_j are derivatives with respect to the j-th variable and $c_\omega(z)$ are analytic functions of $z = (z_1, \ldots, z_n)$ in a neighborhood of zero in $\mathbb{C}^n$. Consider a polynomial differential operator of order d of the form $P(\partial, z) = \sum\limits_{||\omega|| \leqslant d} c_\omega(z)\partial^\omega$.

In the traditional formulation of the Cauchy-Kovalevskaya theorem, it is assumed that the equation is resolved with respect to the highest derivative, for example, ∂_n^d, where d is the order of the differential equation. For linear differential equations with analytic coefficients this means that $c_{(0,\ldots,0,d)}(0) \neq 0$ and the initial data are specified on the coordinate plane $z_n = 0$. In Hörmander's paper [1], a version of the Cauchy-Kovalevskaya theorem is given, where it is assumed that it is solvable with respect to an arbitrary derivative $\partial^m \mathcal{F}$, where $||m|| = d$. However, in this case, in addition to the constraint $c_m(0) \neq 0$, additional conditions must be required on the coefficients of higher-order derivatives d, and the initial data are specified on the union of the coordinate planes. Let us give an exact formulation of this result.

Let the condition

$$\sum_{||\omega||=||m||, \omega \neq m} |c_\omega(0)| < (2e)^{-||m||} |c_m(0)| \tag{1}$$

be satisfied for the equation

$$P(\partial, z)\mathcal{F} = \mathcal{G}, \tag{2}$$

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with analytic coefficients in the neighborhood of the point $z_0 = 0$.

Then equation (2) *with initial data*

$$\partial_j^k[\mathcal{F} - \Phi]|_{z_j=0} = 0, \;\; 0 \leqslant k < m_j, \;\; j = 1, \ldots, n, \tag{3}$$

has a unique analytic solution in a neighborhood of zero for any given analytic functions Φ *and* $\mathcal{G}$.

In this paper, we formulate a generalization of the problem (2)–(3) for polynomial differential operators of a special form, which were considered in [2] in connection with the study of the properties of generating functions of solutions of multidimensional difference equations. The most useful classes of generating functions in enumerative combinatorial analysis (see [3]), along with rational and algebraic ones, are D-finite ones. A power series is called D-finite if it satisfies a linear differential homogeneous equation of the form (2) with polynomial coefficients. In the case of multiple power series, various approaches to the definition of D-finiteness are possible (see [4,5]), one of which is that the power series satisfies a system of linear homogeneous differential equations with polynomial coefficients. To generalize the notion of D-finiteness to Laurent series, in [2] the derivations $D = (D_{a^1}, \ldots, D_{a^n})$ (see Sec. 2. below) in the ring of Laurent series supported in rational cones were defined and the corresponding definition of D-finiteness Laurent series was given.

The question naturally arises of describing the space of solutions of equations of the form (2), where the operators $P(D, z)$ are considered in a suitable way. One of the ways of such a description is to formulate an analogue of the initial-boundary value problem of Hörmander (2)–(3) instead of differential operators $P(\partial, z)$ and study its solvability. In the first section of the paper, the necessary notation and definitions are given, and sufficient conditions for global solvability of a polynomial difference operator with constant coefficients $P(D, z) = P(D)$, (i.e., the existence and uniqueness of the global solution) in the class of Laurent series supported in rational cones (Theorem 1) are proven.

The main idea of the proof of Theorem 1 is to associate the differential initial-boundary value problem of Hörmander with its difference version, and the main role in this comparison is played by the Borel transformation of Laurent series, which is defined in the second section of the paper. With its help a connection between the analytic properties of a function and its Borel transformation is established (Proposition 1), which allows to prove the existence and uniqueness of a solution to a differential initial-boundary value problem in the class of functions representable by Laurent series with supports in the rational cones in the integer lattice (in Sec. 4.).

2. Notation, definitions and formulation of the main result

Let $a^j = (a_1^j, \ldots, a_n^j)$, $j = 1, \ldots, n$, be *linearly independent* vectors with integer coordinates. *The rational cone* spanned by the vectors $a^1, \ldots, a^n$ is the set

$$K = \big\{x \in \mathbb{R}^n : x = \lambda_1 a^1 + \cdots + \lambda_n a^n, \lambda_j \in \mathbb{R}_{\geqslant}, j = 1, \ldots, n\big\},$$

where $\mathbb{R}_{\geqslant}$ is the set of non-negative real numbers.

Let A be a matrix whose determinant is not equal to zero, and the columns consist of the coordinates of the vectors a^j

$$A = \begin{pmatrix} a_1^1 & \ldots & a_1^n \\ .. & .. & .. \\ a_n^1 & \ldots & a_n^n \end{pmatrix}.$$

We consider only *unimodular cones*, i.e. cones for which the determinant of the matrix $\det A = 1$. Any element $x \in K \cap \mathbb{Z}^n$ can be represented as a linear combination of basis vectors $x = \lambda_1 a^1 + \dots + \lambda_n a^n$, where $\lambda_j \in \mathbb{Z}_{\geqslant}$ is a set of non-negative integers, or in matrix form as $x = A\lambda$, where λ is a column vector.

Let us denote the rows of the inverse matrix A^{-1} by $\alpha^1, \dots, \alpha^n$ and note that they form a *mutual basis* (see, for example, [6]) for the vectors $a^1, \dots, a^n$, i.e. $\langle \alpha^i, a^j \rangle = \delta_{ij}$, where $\langle \alpha^i, a^j \rangle = \alpha_1^i a_1^j + \dots + \alpha_n^i a_n^j$, and δ_{ij} is the Kronecker symbol. Also, for $x \in K$, always $\lambda_j = \langle \alpha^j, x \rangle \geqslant 0$, $j = 1, \dots, n$.

Between the points $u, v \in \mathbb{R}^n$ we define the partial order relation $\underset{K}{\geqslant}$ as follows: $u \underset{K}{\geqslant} v \Leftrightarrow u - v \in K$, and $u \underset{K}{\ngeqslant} v$ means that $u - v \notin K$.

The cone $K^* = \{k \in \mathbb{R}^n : \langle k, x \rangle \geqslant 0, x \in K\}$ is called *dual to the cone* K, and the set of its interior points is denoted by $\overset{\circ}{K}{}^*$ and we fix the vector $\nu \in \overset{\circ}{K}{}^* \cap \mathbb{Z}^n$. For all integer points of the rational cone $x \in K \cap \mathbb{Z}^n$, the *weighted-homogeneous degree with weight* ν *(ν-degree) of the monomial* z^x is a nonnegative integer $||x||_\nu = \langle \nu, x \rangle$, and the ν-degree of the Laurent polynomial $Q(z) = \sum\limits_{x \in X} q_x z^x$ is defined by the formula $\deg_\nu Q(z) = \max\limits_{x \in X} ||x||_\nu$, where $X \subset K \cap \mathbb{Z}^n$ is a finite set of points of an n-dimensional integer lattice.

We denote the ring of formal Laurent series of the form

$$\mathcal{F}(z) = \sum_{x \in K \cap \mathbb{Z}^n} f(x) z^x \tag{4}$$

by $\mathbb{C}_K[[z]]$ and note that an operator mapping a ring into itself is called differentiation if it is linear and satisfies the usual rule for the derivative of a product (see, for example, [7]). For the Laurent series (4), taking the usual partial derivative $\partial_j = \dfrac{\partial}{\partial z_j}$ is not necessarily a derivation in the ring $\mathbb{C}_K[[z]]$, since for $x \in K \cap \mathbb{Z}^n$ the point $x - e^j$, where e^j are the unit vectors, generally speaking, may not lie in $K \cap \mathbb{Z}^n$. Derivations of the ring of Laurent series $\mathbb{C}_K[[z]]$ were defined in [2, 8], which made it possible to transfer the notion of D-finiteness of power series to Laurent series. Let us give this definition.

On the monomials z^x, $x \in K \cap \mathbb{Z}^n$ we define the operator D_{a^j} as follows

$$D_{a^j} z^x = \langle x, \alpha^j \rangle z^{x - a^j},$$

where α^j are the vectors of the mutual basis, $j = 1, \dots, n$.

It is directly verified that in the case of a unimodular cone the operators D_{a^j}, $j = 1, \dots, n$, are derivations of the ring $\mathbb{C}_K[[z]]$. For $\omega \in K \cap \mathbb{Z}^n$, $\omega = \lambda_1 a^1 + \dots + \lambda_n a^n$ we define the D^ω operator as follows:

$$D^\omega = D_{a^1}^{\lambda_1} \dots D_{a^n}^{\lambda_n},$$

where $\lambda_j = \langle \omega, \alpha^j \rangle$ and $D_{a^j}^k = \underbrace{D_{a^j} \dots D_{a^j}}_{k \text{ times}}$. Note that for any $\omega', \omega'' \in K \cap \mathbb{Z}^n$, $D^{\omega'} D^{\omega''} = D^{\omega' + \omega''}$ is true and for $\omega = a^j$ we have $D^{a^j} z^x = \langle x, \alpha^j \rangle z^{x - a^j} = D_{a^j} z^x$, $j = 1, \dots, n$.

Thus, the operators D^ω for $\omega \in K \cap \mathbb{Z}^n$ are derivations of the ring of series $\mathbb{C}_K[[z]]$ and their action on the monomials z^x, $x \in K \cap \mathbb{Z}^n$ is conveniently given by the following formula:

$$D^\omega z^x = \begin{cases} 0, & \text{if } x \underset{K}{\ngeqslant} \omega, x \neq \omega, \\ \dfrac{\langle x, \alpha \rangle!}{\langle x - \omega, \alpha \rangle!} z^{x - \omega}, & \text{if } x \underset{K}{\geqslant} \omega, \end{cases} \tag{5}$$

where $\langle x,\alpha\rangle! = \langle x,\alpha^1\rangle!\dots\langle x,\alpha^n\rangle!$.

We consider polynomial differential operators of the form $P(D,z) = \sum\limits_{\omega\in\Omega} c_\omega(z)D^\omega$, where $\Omega \subset K\cap\mathbb{Z}^n$ is a finite set of points of the n-dimensional integer lattice and the coefficients $c_\omega(z) \in \mathbb{C}_K[[z]]$. *The characteristic polynomial* of this operator is the Laurent polynomial $P(\zeta,z) = \sum\limits_{\omega\in\Omega} c_\omega(z)\zeta^\omega$, and its support is denoted by $supp\, P = \{\omega \in \Omega : c_\omega(z) \neq 0\}$. *The order d_ν of the differential operator* $P(D,z)$ is the ν-degree $\deg_\nu P(\zeta,z)$ of the characteristic polynomial, that is $d_\nu = \max\limits_{\omega\in\Omega} ||\omega||_\nu$. In what follows, the subscript ν for d will be omitted, since $\nu \in \mathring{K}^*\cap\mathbb{Z}^n$ is fixed. Thus, the operator $P(D,z)$ of order d can be written as

$$P(D,z) = \sum_{||\omega||_\nu\leqslant d} c_\omega(z)D^\omega. \tag{6}$$

We denote by Γ_j the face of the cone K spanned by the vectors a^i, $i = 1,\dots,j-1,j+1,\dots,n$, $\Gamma_j = \{x : x = \lambda_1 a^1 + \dots + \lambda_{j-1}a^{j-1} + \lambda_{j+1}a^{j+1} + \dots + \lambda_n a^n, \lambda \in \mathbb{R}_\geqslant\}$ and by $\mathcal{F}(z)|_{z^{a^j}=0}$ the Laurent series supported by the face Γ_j of the rational cone K

$$\mathcal{F}(z)|_{z^{a^j}=0} = \sum_{x\in\Gamma_j\cap\mathbb{Z}^n} f(x)z^x. \tag{7}$$

Let the coefficients $c_\omega(z)$ of operator (6) lie in some subring $\mathcal{L}_K$ of the ring $\mathbb{C}_K[[z]]$. For m such that $||m||_\nu = d$ and $c_m(z) \neq 0$, we formulate the following analogue of the Hörmander problem (2)–(3).

For any $\Phi(z)$, $\mathcal{G}(z) \in \mathcal{L}_K$ find $\mathcal{F} \in \mathcal{L}_K$ satisfying the differential equation

$$P(D,z)\mathcal{F} = \mathcal{G}, \tag{8}$$

and the initial conditions:

$$D^{a^j k}[\mathcal{F} - \Phi]|_{z^{a^j}=0} = 0, \quad 0 \leqslant k < \langle m,\alpha^j\rangle, \quad j = 1,\dots,n. \tag{9}$$

For constant coefficients, the case $\mathcal{L}_K = \mathbb{C}_K[[z]]$ was studied in [8], and the case $K = \mathbb{R}^n_\geqslant$ was considered in [9], and the global solvability of the Cauchy–Kovalevskaya problem was proved in the class of entire functions of exponential type.

We define a subring of the ring $\mathbb{C}_K[[z]]$ of Laurent series, in which we will prove the solvability of problem (8)–(9).

Let $Exp(\mathbb{C}^n)$ be the space of entire functions $U(\xi) : \mathbb{C}^n_\xi \to \mathbb{C}$ of exponential type, that is, of entire functions satisfying the inequality $|U(\xi)| \leqslant Ce^{\langle\tau,|\xi|\rangle}$, where $\tau = (\tau_1,\dots,\tau_n)$, τ_j, $C \geqslant 0$ are constants, $|\xi| = (|\xi_1|,\dots,|\xi_n|)$ (see, for example, [10]). Note that the set $\sigma_U = \left\{\tau \in \mathbb{R}^n_> : |U(\xi)| \leqslant Ce^{\langle\tau,|\xi|\rangle}\right\}$ has the following property: together with each point τ_0, all points τ for which $\tau \underset{\mathbb{R}^n_\geqslant}{>} \tau_0$ also belong to it.

We denote by $\mathcal{A}$ a mapping from $\mathbb{Z}^n_\geqslant$ to $K\cap\mathbb{Z}^n$ with matrix $A = (a_i^j)_{n\times n}$. This mapping induces a mapping of rings $\mathcal{A}_* : \mathbb{C}_K[[z]] \to \mathbb{C}[[\xi]]$, which associates the Laurent series $\sum\limits_{x\in K\cap\mathbb{Z}^n} f(x)z^x$ with a power series $\sum\limits_{\lambda\in\mathbb{Z}^n_\geqslant} f(A\lambda)\xi^\lambda$, where $\xi = z^A$ and $z^A = (z^{a^1},\dots,z^{a^n})$.

Since the mapping $\mathcal{A}_*$ is invertible in the case $\det A = 1$, we see that $\mathcal{A}_*^{-1}(Exp(\mathbb{C}^n)) = \mathcal{L}_K$ is a subring of the ring $\mathbb{C}_K[[z]]$. *The functions $\mathcal{F}$ representable by Laurent series from the subring $\mathcal{L}_K$ will be called exponential in the class of Laurent series ($\mathcal{L}_K$-exponential).*

Theorem 1. *Let the coefficients of the polynomial differential operator* (6) *be constant and* $m \in suppP \subset \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} m\}$, *then for any* $\mathcal{L}_K$*-exponential functions* $\mathcal{G}$, Φ *problem* (8)–(9) *has a unique* $\mathcal{L}_K$*-exponential solution* $\mathcal{F}$.

The conditions of Theorem 1 mean, in particular, that m is the vertex of the Newton polytope of the characteristic polynomial $P(\zeta)$ and c_m is the only nonzero coefficient at the «highest derivative». In [8], problem (8)–(9) was studied under a weaker than in Theorem 1 restriction on the operator $P(D)$, namely, the condition $supp\, P \subset \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} m\}$ was not required, but solutions were sought in the class of formal Laurent series.

Let us give an example of an operator satisfying the conditions of Theorem 1. Let the cone K be spanned by the vectors $a^1 = (1, -1)$, $a^2 = (-1, 2)$. Let us fix $\nu \in \mathring{K}^* \cap \mathbb{Z}^n$, for example, $\nu = (3, 2)$, $m = (0, 1)$, the set $\Omega = \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} (0, 1)\} = \{(0, 0), (1, -1), (0, 1), (-1, 2)\}$. Consider the operator

$$P(D) = D^{(0,1)} + D^{(1,-1)} + D^{(-1,2)} + 1 \tag{10}$$

of the Hörmander problem (8)–(9) for operator (10) with $\mathcal{L}_K$-exponential initial data and the right-hand side will be a $\mathcal{L}_K$-exponential function, i.e. function represented by *Laurent series.*

Operator $D = (D_{a^1}, \ldots, D_{a^n})$ is related to partial derivatives $\partial = (\partial_1, \ldots, \partial_n)$ by formulas $z^{a^j} D_{a^j} = \langle \alpha^j, z\partial \rangle$, $j = 1, \ldots, n$, where $z\partial = (z_1\partial_1, \ldots, z_n\partial_n)$, therefore the polynomial differential operator (10) is expressed through $\partial = (\partial_1, \partial_2)$ as follows

$$\mathcal{P}(\partial_1, \partial_2, z) = 2z_1^2 z_2^{-1} \partial_1^2 + 3z_1\partial_1\partial_2 + z_2\partial_2^2 +$$

$$+(2z_2 + z_1^2 z_2^{-2} + 2z_1 z_2^{-1})\partial_1 + (z_1^{-1} z_2^2 + z_1 z_2^{-1} + 1)\partial_2 + 1.$$

Note that this operator does not satisfy condition (1), which ensures the existence of an analytic solution, at any point $z = z_0$.

3. The Borel transformation of Laurent series and the connection between a differential and difference problems

In this section, we define the Borel transformation of Laurent series and prove an analogue of the Borel theorem on the connection between the analytic properties of a function and its Borel transformation (Proposition 1) in the class of $\mathcal{L}_K$-exponential functions.

For a function $f(x) : K \cap \mathbb{Z}^n \to \mathbb{C}$ of a discrete argument $x \in K \cap \mathbb{Z}^n$ we define two types of generating series (functions):

$$\mathcal{F}(z) = \sum_{x \in K \cap \mathbb{Z}^n} \frac{f(x) z^x}{\langle x, \alpha \rangle !}, \tag{11}$$

and

$$F(z) = \sum_{x \in K \cap \mathbb{Z}^n} \frac{f(x)}{z^x}. \tag{12}$$

Series (12) is called the *Borel transformation of series* (11) and in the one-dimensional case it is the classical Borel transformation of power series (see [11]). If $K = \mathbb{R}^n_{\geqslant}$, then we obtain the definition of the Borel transformation of multiple power series from [12].

Functions (11) and (12) are the *upper and lower functions of the Borel transformation*, respectively.

The definition of $\mathcal{L}_K$-exponentiality implies that $\mathcal{F}(z) \in \mathcal{L}_K$ if and only if for some $\sigma \in \mathbb{R}^n_>$ the inequality

$$|\mathcal{F}(z)| \leqslant Ce^{\langle \sigma^A, |z^A| \rangle}$$

holds, where $C \geqslant 0$ is a constant.

Let $\sigma_{\mathcal{F}}$ denote the set

$$\sigma_{\mathcal{F}} = \{\sigma \in \mathbb{R}^n_> : |\mathcal{F}(z)| \leqslant Ce^{\langle \sigma^A, |z^A| \rangle}\}$$

and call it the *type-set of the function $\mathcal{F}$.*

Note that in the case $A = E$, where E is the identity matrix, the set $\sigma_{\mathcal{F}} = \sigma_U$ and the related concept of conjugate types of entire functions were used in [12–14] to study the growth of entire functions.

The domain of convergence $\mathcal{D}_F$ of the Laurent series (12) is the open kernel of the set of those points z at which this series converges absolutely. We denote the image of the convergence domain under the projection

$$z = (z_1, \ldots, z_n) \to |z| = (|z_1|, \ldots, |z_n|) \tag{13}$$

by $|\mathcal{D}_F|$. Note that it follows from Lemma 7 in [15] that if $R = (R_1, \ldots, R_n) \in |\mathcal{D}_F|$, then series (12) also converges for all points of the set $T_K(R) = \{z \in \mathbb{C}^n : |z^{a^j}| > R^{a^j}, j = 1, \ldots, n\}$. It follows that after logarithmic projection

$$Log\ :\ z = (z_1, \ldots, z_n) \to (\log(|z_1|), \ldots, \log(|z_n|)) = Log|z|, \tag{14}$$

the set $Log|\mathcal{D}_F|$, together with each point $LogR$, also contains the affine cone $LogR + \mathring{K}^*$, where K^* is the cone dual to the cone K.

Let us give an analogue of Borel's theorem for $\mathcal{L}_K$-exponential functions.

Proposition 1. *If $\mathcal{F}(z)$ is a $\mathcal{L}_K$-exponential function with type set $\sigma_{\mathcal{F}}$ and $\mathcal{D}_F$ is the domain of convergence of its Borel transformation $F(z)$, then $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$.*

Proof. When transform $\mathcal{A}_* : \mathbb{C}_K[[z]] \to \mathbb{C}[[\xi]]$ of the function $\mathcal{F}(z)$

$$\mathcal{A}_*(\mathcal{F}(z)) = \sum_{\lambda \in \mathbb{Z}^n_\geqslant} \frac{f(A\lambda)}{\lambda!} \xi^\lambda,$$

the Borel transformation is the function

$$\mathcal{A}_*(F(z)) = \sum_{\lambda \in \mathbb{Z}^n_\geqslant} \frac{f(A\lambda)}{\xi^\lambda}.$$

It follows from Borel's theorem for multiple power series (see [12], Theorem 3.3.3) that

$$\sigma_{\mathcal{A}_*(\mathcal{F})} = |\mathcal{D}_{\mathcal{A}_*(F)}|, \tag{15}$$

where $\sigma_{\mathcal{A}_*(\mathcal{F})} = \{\tau \in \mathbb{R}^n_> : |\mathcal{A}_*(\mathcal{F})| \leqslant Ce^{\langle \tau, |\xi| \rangle}\}$, and $|\mathcal{D}_{\mathcal{A}_*(F)}|$ is the image of the convergence domain under projection (13) of the function $\mathcal{A}_*(F)$. The set $|\mathcal{D}_{\mathcal{A}_*(F)}|$ possesses the property that, together with each point r, the set $\{|\xi| : |\xi_j| > r_j, j = 1, \ldots, n\}$ also belongs to it. After monomial changes $\tau = \sigma^A,\ \xi = z^A,\ r = R^A$, from equality (15) we obtain $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$. □

Let us formulate a difference version of the problem (8)–(9), which we need in the proof of Theorem 1. On the complex-valued functions $f(x) = f(x_1, \dots, x_n)$ of integer variables $x_1, \dots, x_n$, we define the shift operators δ_j in the variables x_j :

$$\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n)$$

and polynomial difference operator of the form

$$P(\delta) = \sum_{||\omega||_\nu \leqslant d} c_\omega \delta^\omega,$$

where $\delta^\omega = \delta_1^{\omega_1} \dots \delta_n^{\omega_n}$ and coefficients c_ω are constant.

For m such that $||m||_\nu = d$ and $c_m \neq 0$ we formulate the following problem. For any functions g, φ of integer variables $x = (x_1, \dots, x_n)$, it is required to find a function $f(x)$ satisfying the difference equation

$$P(\delta) f(x) = g(x), \; x \in K \cap \mathbb{Z}^n \tag{16}$$

and the initial-boundary conditions

$$\delta^{a^j k}[f(x) - \varphi(x)]|_{x \in \Gamma_j \cap \mathbb{Z}^n} = 0, \quad 0 \leqslant k < \langle m, \alpha^j \rangle, \; j = 1, \dots, n, \tag{17}$$

where Γ_j is face of the cone K spanned by vectors $a^i, \; i = 1, \dots, j-1, j+1, \dots, n$.

Various versions of the statement of the problem (16)–(17) and the study of the question of its solvability were considered, for example, in [16–19].

4. Proof of the main result

In this section, we present some information from the theory of amoebas of algebraic surfaces, in order to formulate the relation between the generating function of the solution of the Cauchy problem for an inhomogeneous multidimensional difference equation and the generating function of the initial data, and also prove the main result of the work (Theorem 1).

The *Newton polytope* N_P of a polynomial $P(z) = \sum_{\omega \in \Omega} c_\omega z^\omega$ is the convex hull in $\mathbb{R}^n$ of elements of the set Ω.

The *amoeba* $\mathcal{A}_V$ *of an algebraic surface* $V = \{z \in \mathbb{C}^n : P(z) = 0\}$ is a image of the set of zeros V of the polynomial $P(z)$ under the mapping (14).

To prove Theorem 1, we need series expansions of the function $\dfrac{1}{P(z)}$ (see [20]), where $P(z)$ is the characteristic polynomial of the operator $P(D)$.

Each vertex of the Newton polytope N_P of the Laurent polynomial $P(z)$ corresponds to a non-empty connected component E_m of the complement of the amoeba $\mathbb{R}^n \backslash \mathcal{A}_V$, and in the domain $Log^{-1} E_m$ the function $\dfrac{1}{P(z)}$ expands into the Laurent series

$$\frac{1}{P(z)} = \sum_{x \in m + \Lambda_m \cap \mathbb{Z}^n} \frac{\mathcal{P}_m(x)}{z^x}, \tag{18}$$

where Λ_m is the cone constructed on the vectors $m - \omega, \; \omega \in \Omega, \; \Lambda_m \subset K$.

If the point m is a vertex of the polytope N_P, then the coefficients $\mathcal{P}_m(x)$ of the expansion (18) can be obtained as follows: at the first step, we use the expansion in a series of geometric progression

$$\frac{1}{P(z)} = \frac{1}{c_m z^m + \sum\limits_{\alpha \neq m} c_\alpha z^\alpha} = \frac{1}{c_m z^m (1 - \sum\limits_{\alpha \neq m} \tilde{c}_\alpha z^{\alpha - m})} =$$

$$= \frac{1}{c_m z^m} \sum_{k=0}^{\infty} \left(\sum_{\alpha \neq m} \tilde{c}_\alpha z^{\alpha - m} \right)^k,$$

and then, after standard transformations and reduction of similar ones, we obtain an expansion of the form $\frac{1}{P(z)} = \sum\limits_{x \in m + \Lambda_m} \frac{\mathcal{P}_m(x)}{z^x}$, where the series converges in the domain $Log^{-1} E_m$.

The *dual cone* C_m to point m of N_P is defined as follows:

$$C_m = \{s \in \mathbb{R}^n : \max_{x \in N_P} \langle s, x \rangle = \langle s, m \rangle\}.$$

Note that it is *asymptotic*, i.e. together with each point $u \in E_m$, this component also belongs to the affine cone $u + C_m \subset E_m$. If $m \underset{K}{\geqslant} \omega$ and $\Lambda_m \subset K$, then $C_m \supset K^*$; therefore, the image of the convergence domain $|\mathcal{D}_{P^{-1}}|$ in the projection (13) of the series $\frac{1}{P(z)}$, together with each point z_0, also contains points z such that $|z^{a^j}| > |z_0^{a^j}|$, $j = 1, \ldots, n$.

Proof of Theorem 1. Let $\mathcal{F}(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n} \frac{f(x) z^x}{\langle x, \alpha \rangle !}$ be the required solution to the problem (8)–(9) for the given initial data $\varPhi(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n, x \underset{K}{\ngeqslant} m} \frac{\varphi(x) z^x}{\langle x, \alpha \rangle !}$ and the right-hand side $\mathcal{G}(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n} \frac{g(x) z^x}{\langle x, \alpha \rangle !}$. Its solvability in the class of formal Laurent series $\mathbb{C}_K[[z]]$ was proved in [8] and the proof is based on the statement that $\mathcal{F}(z)$ is a solution to the problem (8)–(9) if and only if $f(x)$ is a solution to the corresponding difference problem (16)–(17), the solvability of which was proved in [15]. To prove the solvability of the differential problem (8)–(9) in the class of L_K-exponential series, we use the fact that the generating function of the solution $f(x)$ and the data $\varphi(x)$, $g(x)$ of the difference problem (16)–(17) $F(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n} \frac{f(x)}{z^x}$, $\Phi(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n, x \underset{K}{\ngeqslant} m} \frac{\varphi(x)}{z^x}$ and $G(z) = \sum\limits_{x \in K \cap \mathbb{Z}^n} \frac{g(x)}{z^x}$ are the Borel transformations of the series $\mathcal{F}(z)$, $\varPhi(z)$ and $\mathcal{G}(z)$, respectively. The formula connecting these generating functions for the homogeneous difference problem (16)–(17) is given in [2, 21], its modification for $g(x) \neq 0$ has the following form

$$F(z) = \sum_{\omega \in \Omega} c_\omega z^\omega \frac{1}{P(z)} \Phi_\omega(z) + G(z) \frac{1}{P(z)}. \tag{19}$$

By the condition of the theorem, $\varPhi(z)$ and $\mathcal{G}(z)$ are L_K-exponential; therefore, Proposition 1 implies that $\sigma_\varPhi = |\mathcal{D}_\Phi|$ and $\sigma_\mathcal{G} = |\mathcal{D}_G|$, where $\mathcal{D}_\Phi$, $\mathcal{D}_G$ are domains in which the series $\Phi(z)$ and $G(z)$ converge.

Let us prove that $\mathcal{F}(z)$ is an $\mathcal{L}_K$-exponential function. For the Borel transformation of the function $\mathcal{F}$, formula (19) is valid, which yields that the image under the map (13) of the convergence domain of the generating function $F(z)$ of the solution to problem (16)–(17) contains an intersection of the images of the domains of convergence of the generating functions $\Phi_\omega(z)$ of the initial data, the generating function $G(z)$ of the right-hand side and series $1/P(z)$:

$|\mathcal{D}_F| \supset \bigcap\limits_{\omega} |\mathcal{D}_{\Phi_\omega}| \cap |\mathcal{D}_G| \cap |\mathcal{D}_{P^{-1}}|$. For $\omega \in \Omega$ we have $supp\,\Phi_\omega \subset supp\,\Phi$, then the convergence domain of the series Φ_ω can only increase: $|\mathcal{D}_{\Phi_\omega}| \supseteq |\mathcal{D}_\Phi|$, therefore

$$|\mathcal{D}_F| \supset |\mathcal{D}_\Phi| \cap |\mathcal{D}_G| \cap |\mathcal{D}_{P^{-1}}|. \tag{20}$$

Since the cone K^* is asymptotic for both $Log|\mathcal{D}_\Phi|$, $Log|\mathcal{D}_G|$ and $Log|\mathcal{D}_{P^{-1}}|$, the intersection on the right-hand side of (20) is not empty.

Appling the inverse Borel transformation to $F(z)$, yields the function $\mathcal{F}(z)$, and by Proposition 1 for its image $\sigma_{\mathcal{F}}$, we have $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$, that is $\sigma_{\mathcal{F}} \neq \emptyset$ and, therefore, $\mathcal{F}$ is an $\mathcal{L}_K$-exponential function. $\square$

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Аналитическая разрешимость задачи Хермандера и преобразование Бореля кратных рядов Лорана

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Аннотация. В работе формулируется начально-краевая задача Хермандера в классе функций, представимых рядами Лорана с носителями в рациональных конусах. Преобразование Бореля рядов Лорана позволяет установить связь дифференциальной задачи с разностной и доказать теорему о ее глобальной аналитической разрешимости.

Ключевые слова: задача Хермандера, дифференциальный оператор, преобразование Бореля, разностный оператор.