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Some Systems of Transcendental Equations

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Abstract. Several examples of transcendental systems of equations are considered. Since the number of roots of such systems, as a rule, is infinite, it is necessary to study power sums of the roots of negative degree. Formulas for finding residue integrals, their relation to power sums of a negative degree of roots and their relation to residue integrals (multidimensional analogs of Waring's formulas) are obtained. Calculations of multidimensional numerical series are given.

Keywords: transcendental systems of equations, power sums of roots, residue integral.

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Introduction

Based on the multidimensional logarithmic residue, for systems of non-linear algebraic equations in $\mathbb{C}^n$ formulas for finding power sums of the roots of a system without calculating the roots themselves were earlier obtained (see [1–3]). For different types of systems such formulas have different forms. Based on this, a new method for the study of systems of algebraic equations in $\mathbb{C}^n$ have been constructed. It arose in the work of L. A. Aizenberg [1], its development was continued in monographs [2–4]. The main idea is to find power sums of roots of systems (for positive powers) and then, to use one-dimensional or multidimensional recurrent Newton formulas (see [5]). Unlike the classical method of elimination, it is less labor-intensive and does not increase the multiplicity of roots. It is based on the formula (see [1]) for a sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations without finding the roots themselves.

For systems of transcendental equations, formulas for the sum of the values of the roots of the system, as a rule, cannot be obtained, since the number of roots of a system can be infinite and a series of coordinates of such roots can be diverging. Nevertheless, such transcendental systems of equations may very well arise, for example, in the problems of chemical kinetics [6, 7]. Thus, this is an important task to consider such systems.

In the works [8–21] power sums of roots in a negative power are considered for various systems of non-algebraic (transcendental) equations. To compute these power sums, a residue integral is used, the integration is carried out over skeletons of polycircles centered at the origin. Note that this residue integral is not, generally speaking, a multidimensional logarithmic residue or a Grothendieck residue. For various types of lower homogeneous systems of functions included in the system, formulas are given for finding residue integrals, their relationship with power sums of the roots of the system to a negative degree are established.

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The paper [12] investigated more complex systems in which the lower homogeneous parts are decomposed into linear factors and integration cycles in residue integrals are constructed from these factors. In [11], a system is studied that arises in the Zel'dovich–Semenov model (see [6,7]) in chemical kinetics.

The object of this study is some systems of transcendental equations in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations: formulas are found for calculating the residue integrals, power sums of roots for a negative power, their relationship with the residue integrals are established. See [21].

1. General systems of transcendental equations

In this section we follow the paper [22].

Let $f_1(z), \ldots, f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in the multidimensional complex space $\mathbb{C}^n$, $z = (z_1, \ldots, z_n)$.

We expand the functions $f_1(z), \ldots, f_n(z)$ in Taylor series in a neighborhood of the origin and consider a system of equations of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad i = 1, \ldots, n, \tag{1}$$

where P_j is the lowest homogeneous part of the Taylor expansion of the function $f_j(z)$. The degree of all monomials (with respect to the totality of variables) included in P_j, is equal to m_j, $j = 1, \ldots, n$. In the functions Q_jl, the degrees of all monomials are strictly greater than m_j.

The expansion of the functions Q_j, P_j, $j = 1, \ldots, n$ in a neighborhood of zero in Taylor series that converges absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\| > m_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\beta\| = m_j} b^j_\beta z^\beta, \tag{3}$$

$$j = 1, \ldots, n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indexes, i.e. α_j and β_j are non-negative integers, $j = 1, \ldots, n$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$, and monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z), \ldots, P_n(z)$ is *nondegenerate*, that is, its common zero is only the point 0, the origin.

Consider an open set (a special analytic polyhedron) of the form

$$D_P(r_1, \ldots, r_n) = \{z : \ |P_j(z)| < r_j, \ i = j, \ldots, n\},$$

where $r_1, \ldots, r_n$ are positive numbers. Its *skeleton* has the form

$$\Gamma_P(r_1, \ldots, r_n) = \Gamma_P(r) = \{z : \ |P_j(z)| = r_j, \ j = 1, \ldots, n\}.$$

Let us start with a statement.

Lemma 1. *The next equality is true*

$$J_\gamma = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n} =$$

$$= \frac{(-1)^n}{(2\pi i)^n} \int\limits_{\Gamma_{\tilde P}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\tilde f_1}{\tilde f_1} \wedge \frac{d\tilde f_2}{\tilde f_2} \wedge \ldots \wedge \frac{d\tilde f_n}{\tilde f_n} = (-1)^n \tilde J_\gamma.$$

For what follows, we need a generalized formula for transforming the Grothendieck residue (see [23]).

Theorem 1. *Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$ and $g_j(w)$, $j,k = 1,\ldots,n$, are related by*

$$g_j = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1,2,\ldots,n,$$

the matrix $A = \|a_{jk}\|_{j,k=1}^{n}$ consists of polynomials. Consider the cycles

$$\Gamma_f = \{w: \ |f_j(w)| = r_j, \ j = 1,\ldots,n\},$$

$$\Gamma_g = \{w: \ |g_j(z)| = r_j, \ j = 1,\ldots,n\},$$

where all $r_j > 0$. Then the equality

$$\int\limits_{\Gamma_f} h(w)\frac{dw}{f^{\alpha}} = \sum_{K,\sum\limits_{s=1}^{n} k_{sj}=\beta_s} \frac{\beta!}{\prod\limits_{s,j=1}^{n}(k_{sj})!} \int\limits_{\Gamma_g} h(w) \frac{\det A \prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}\, dw}{g^{\beta}}, \tag{4}$$

holds. Here $\beta! = \beta_1!\beta_2!\ldots\beta_n$, $\beta = (\beta_1,\beta_2,\ldots,\beta_n)$, the summation in the formula is over all non-negative integer matrices $K = \|k_{sj}\|_{s,j=1}^{n}$ with the conditions that the sum $\sum\limits_{s=1}^{n} k_{sj} = \alpha_j$, then $\beta_j = \sum\limits_{j=1}^{n} k_{js}$. Here$f^{\alpha} = f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^{\beta} = g_1^{\beta_1}\cdots g_n^{\beta_n}$.

Theorem 2. *The next formulas are valid*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}} =$$

$$= (2\pi i)^n) \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot \frac{d\tilde{f}_1}{\tilde{f}_1}\wedge\frac{d\tilde{f}_2}{\tilde{f}_2}\wedge\ldots\wedge\frac{d\tilde{f}_n}{\tilde{f}_n} =$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n} \frac{(-1)^{n+\|\alpha\|}}{(2\pi i)^n} \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\times$$

$$\times\frac{\tilde{\Delta}\cdot\tilde{Q}_1^{\alpha_1}\cdot\tilde{Q}_2^{\alpha_2}\cdots\tilde{Q}_n^{\alpha_n}\,dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\tilde{P}_1^{\alpha_1+1}\cdot\tilde{P}_2^{\alpha_2+1}\cdots\tilde{P}_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\|\leqslant\|\gamma\|+n} \frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n} k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\,\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\tilde{\Delta}\cdot\det A\cdot Q^{\alpha}\prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n} w_j^{\beta_j N_j+\beta_j+N_j}}\right],$$

where $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$, and the functional $\mathfrak{M}$ assigns its free term to the Laurent polynomial.

In fact, in Theorem 2, analogs of the classical Waring formulas for finding power sums of roots of a system of algebraic equations are obtained.

2. Examples

Example 1. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1,z_2) = z_1z_2 + b_1z_1 + b_2z_2 = 0, \\ f_2(z_1,z_2) = 1 + a_1z_1 + a_2z_2 = 0. \end{cases} \tag{5}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = 1 + b_2w_1 + b_1w_2 = 0, \\ \widetilde{f}_2 = w_1w_2 + a_2w_1 + a_1w_2 = 0. \end{cases} \tag{6}$$

The Jacobian of the system (6) $\widetilde{\Delta}$ is equal to

$$\widetilde{\Delta} = \begin{vmatrix} b_2 & b_1, \\ w_2 + a_2 & w_1 + a_1 \end{vmatrix} = b_2w_1 - b_1w_2 + (a_1b_2 - a_2b_1).$$

Note that

$$\begin{cases} \widetilde{Q}_1 = 1, \\ \widetilde{Q}_2 = a_1w_2 + a_2w_1. \end{cases} \tag{7}$$

$$\begin{cases} \widetilde{P}_1 = b_1w_2 + b_2w_1, \\ \widetilde{P}_2 = w_1w_2. \end{cases} \tag{8}$$

Let us calculate $\det A$:
Since

$$w_1^2 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$
$$w_2^2 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

where $\widetilde{P}_1 = b_1w_2 + b_2w_1$, $\widetilde{P}_2 = w_1w_2$.

Therefore, the elements of a_{ii} are equal

$$a_{11} = \frac{w_1}{b_2}, a_{12} = -\frac{b_1}{b_2},$$

$$a_{21} = \frac{w_2}{b_1}, a_{22} = -\frac{b_2}{b_1}.$$

Therefore,

$$\det A = \frac{w_2}{b_2} - \frac{w_1}{b_1} = \frac{w_2b_1 - w_1b_2}{b_1b_2}.$$

By Theorem 2

$$J_{(0,0)} = \sum_{\|K\| = k_{11}+k_{12}+k_{21}+k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times \mathfrak{M}\left[\frac{\widetilde{\Delta} \cdot \det A \cdot \widetilde{Q}_1^{k_{11}+k_{21}} \cdot \widetilde{Q}_2^{k_{12}+k_{22}} \cdot a_{11}^{k_{11}} \cdot a_{12}^{k_{12}} \cdot a_{21}^{k_{21}} \cdot a_{22}^{k_{22}}}{w_1^{2(k_{11}+k_{12})} \cdot w_2^{2(k_{21}+k_{22})}} \right],$$

$$J_{(0,0)} = \sum_{\|K\| = k_{11}+k_{12}+k_{21}+k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times\mathfrak{M}\left[\frac{(-1)^{k_{12}+k_{22}}(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^{1+k_{21}+k_{22}-k_{12}}\cdot b_2^{1+k_{11}+k_{12}-k_{22}}}\right]\left[\frac{(a_1w_2+a_2w_1)^{k_{12}+k_{22}}}{w_1^{k_{11}+2k_{12}}\cdot w_2^{k_{21}+2k_{22}}}\right].$$

Calculate the value of the sums

$(0,0,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2}\right]=0,$$

$(1,0,0,0)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2^2\cdot w_1}\right]=\frac{a_1b_2-a_2b_1}{b_1b_2},$$

$(0,1,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_2^2\cdot w_1^2}\right]=$$

$$=\frac{-a_2(a_1b_2-a_2b_1)}{b_2}=-a_1a_2+\frac{a_2^2b_1}{b_2},$$

$(0,0,1,0)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^2b_2\cdot w_2}\right]=-\frac{a_1b_2-a_2b_1}{b_1b_2},$$

$(0,0,0,1)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^2\cdot w_2^2}\right]=$$

$$=\frac{a_1(a_1b_2-a_2b_1)}{b_1}=-a_1a_2+\frac{a_1^2b_2}{b_1},$$

$(2,0,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2^3\cdot w_1^2}\right]=-\frac{1}{b_1b_2},$$

$(0,2,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2\cdot b_1}{b_2^3\cdot w_1^4}\right]=-\frac{a_2^2b_1}{b_2},$$

$(0,0,2,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^3b_2\cdot w_2^2}\right]=-\frac{1}{b_1b_2},$$

$(0,0,0,2)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2\cdot b_2}{b_1^3\cdot w_2^4}\right]=-\frac{a_1^2b_2}{b_1},$$

$(1,1,0,0)$:

$$\mathfrak{M}\left[\frac{-2(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_2^3\cdot w_1^3}\right]=\frac{2a_2}{b_2},$$

$(1,0,1,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^2b_2^2\cdot w_1w_2}\right]=\frac{2}{b_1b_2},$$

$(1,0,0,1)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^2b_2\cdot w_1w_2^2}\right]=\frac{a_2}{b_2}-\frac{2a_1}{b_1},$$

$(0,1,1,0)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1b_2^2\cdot w_1^2w_2}\right]=\frac{a_1}{b_1}-\frac{2a_2}{b_2},$$

$(0,1,0,1)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2}{b_1b_2\cdot w_1^2w_2^2}\right]=-\frac{a_1^2b_2}{b_1}-\frac{a_2^2b_1}{b_2}+2a_1a_2,$$

$(0,0,1,1)$:

$$\mathfrak{M}\left[\frac{-2(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^3\cdot w_2^3}\right]=\frac{2a_1}{b_1}.$$

Therefore

$$J_{(0,0)}=2a_1a_2+\frac{a_2}{b_2}+\frac{a_1}{b_1}-\frac{a_1^2b_2}{b_1}-\frac{a_2^2b_1}{b_2}.$$

Example 2. Consider a system of equations in two complex variables

$$\begin{cases}\widetilde{f}_1=1+b_1w_2+b_2w_1=0,\\ \widetilde{f}_2=w_1w_2+a_1w_2+a_2w_1=0.\end{cases}\tag{9}$$

Let $u=w_1w_2$, then $w_1=\dfrac{u}{w_2}$, substitute in our system

$$\begin{cases}1+b_1w_2+\dfrac{b_2u}{w_2}=0,\\ u+a_1w_2+\dfrac{a_2u}{w_2}=0.\end{cases}\tag{10}$$

multiply each equation of the system by w_2

$$\begin{cases}w_2+b_1w_2^2+b_2u=0,\\ w_2u+a_1w_2^2+a_2u=0.\end{cases}\tag{11}$$

Now we multiply the first equation of the system by a_1, and the second by b_1 and subtract one from the other

$$w_2(b_1u-a_1)-u(a_1b_2-a_2b_1)=0,$$

so

$$w_2=\frac{u(a_1b_2-a_2b_1)}{b_1u-a_1}.$$

Let us substitute this into the first equation of the system and get rid of the denominator and the second variable

$$b_2u(b_1u-a_1)^2+u(a_1b_2-a_2b_1)(b_1u-a_1)+b_1^2u^2(a_1b_2-a_2b_1)^2=0.$$

We get

$$b_1^2 b_2 u^2 + (b_1(a_1b_2 - a_2b_1)^2 + b_1(a_1b_2 - a_2b_1) - 2a_1b_1b_2)u + a_1^2 b_2 - a_1(a_1b_2 - a_2b_1) = 0.$$

By the generalized Vieta theorem

$$J_{(0,0)} = \sum_{j=1}^{p} w_{j1} \cdot w_{j2} = -\frac{(a_1b_2 - a_2b_1)^2}{b_1b_2} + \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

Example 3. Consider a system (13) of equations in two complex variables (example 1).

Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{k^2\pi^2}\Big) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2k+1)!},$$

which uniformly and absolutely converge on the complex plane and have the order of growth 1/2.

Consider the system of equations

$$\begin{cases} f_1(z_1, z_2) = z_1z_2 + b_1z_1 + b_2z_2 = 0, \\ f_2(z_1, z_2) = \dfrac{\sin\sqrt{a_1z_1 + a_2z_2}}{\sqrt{a_1z_1 + a_2z_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{a_1z_1 + a_2z_2}{s^2\pi^2}\right) = 0. \end{cases} \tag{12}$$

Using the formula obtained above and the known expansion of the series, we obtain that the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = -\sum_{s=1}^{\infty}\frac{a_1^2b_2}{\pi^4s^4b_1} - \sum_{s=1}^{\infty}\frac{a_2^2b_1}{\pi^4s^4b_2} + \sum_{s=1}^{\infty}\frac{a_2}{\pi^2s^2b_2} + \sum_{s=1}^{\infty}\frac{a_1}{\pi^2s^2b_1} + \sum_{s=1}^{\infty}\frac{2a_1a_2}{\pi^4s^4} =$$

$$= -\frac{(a_1b_2 - a_2b_1)^2}{90b_1b_2} + \frac{a_1b_2 + a_2b_1}{6b_1b_2}.$$

Example 4. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = a_1z_1 - a_2z_2 + z_1^2z_2 = 0, \\ f_2(z_1, z_2) = b_1z_1 + b_2z_2 + z_1z_2^2 = 0. \end{cases} \tag{13}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = -a_2w_1^2 + a_1w_1w_2 + 1 = 0, \\ \widetilde{f}_2 = b_2w_1w_2 + b_1w_2^2 + 1 = 0. \end{cases} \tag{14}$$

The Jacobian of the system (6) $\widetilde{\Delta}$ is equal to

$$\widetilde{\Delta} = \begin{vmatrix} -2a_2w_1 + a_1w_2 & a_1w_1 \\ b_2w_2 & 2b_1w_2 + b_2w_1 \end{vmatrix} = -2a_2b_2w_1^2 - 4a_2b_1w_1w_2 + 2a_1b_1w_2^2.$$

Note that

$$\begin{cases} \widetilde{Q}_1 = 1, \\ \widetilde{Q}_2 = 1. \end{cases} \tag{15}$$

$$\begin{cases}\widetilde{P}_1 = -a_2 w_1^2 + a_1 w_1 w_2 = 0,\\ \widetilde{P}_2 = b_2 w_1 w_2 + b_1 w_2^2 = 0.\end{cases} \tag{16}$$

Calculate $\det A$:

$$Res = \begin{vmatrix} -a_2 & a_1 & 0 & 0\\ 0 & -a_2 & a_1 & 0\\ 0 & b_2 & b_1 & 0\\ 0 & 0 & b_2 & b_1 \end{vmatrix}.$$

$\Delta = a_2 b_1 (a_2 b_1 + a_1 b_2)/.$

Let us determine the minors:

$\widetilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0\\ b_2 & b_1 & 0\\ 0 & b_2 & b_1 \end{vmatrix} = -a_2 b_1^2 - a_1 b_1 b_2,$ $\widetilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0\\ b_2 & b_1 & 0\\ 0 & b_2 & b_1 \end{vmatrix} = -a_1 b_1^2,$

$\widetilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0\\ -a_2 & a_1 & 0\\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2 b_1,$ $\widetilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0\\ -a_2 & a_1 & 0\\ b_2 & b_1 & 0 \end{vmatrix} = 0.$

$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1\\ 0 & b_2 & b_1\\ 0 & 0 & b_2 \end{vmatrix} = 0,$ $\Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0\\ 0 & b_2 & b_1\\ 0 & 0 & b_2 \end{vmatrix} = -a_2 b_2^2,$

$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0\\ 0 & -a_2 & a_1\\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2 b_2,$ $\Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0\\ 0 & -a_2 & a_1\\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2 b_1 + a_1 a_2 b_2.$

Therefore, the elements of a_{ii} are equal

$$a_{11} = \frac{1}{\Delta}\left(\widetilde{\Delta}_1 w_1 + \widetilde{\Delta}_2 w_2\right) = \frac{1}{\Delta}\left((-a_2 b_1^2 - a_1 b_1 b_2) w_1 - a_1 b_1^2 w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\widetilde{\Delta}_3 w_1 + \widetilde{\Delta}_4 w_2\right) = \frac{a_1^2 b_1 w_1}{\Delta},$$

$$a_{21} = \frac{1}{\Delta}\left(\Delta_1 w_1 + \Delta_2 w_2\right) = \frac{-a_2 b_2^2 w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3 w_1 + \Delta_4 w_2\right) = \frac{1}{\Delta}\left(-a_2^2 b_2 w_1 + (a_2^2 b_1 + a_1 a_2 b_2) w_2\right).$$

Then

$$w_1^3 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$
$$w_2^3 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

were $\widetilde{P}_1 = -a_2 w_1^2 + a_1 w_1 w_2$, $\widetilde{P}_2 = b_2 w_1 w_2 + b_1 w_2^2$.

It is not difficult to make sure that

$\Delta(1,3) = -\begin{vmatrix} -a_2 & a_1\\ 0 & b_2 \end{vmatrix} = a_2 b_2,$

$\Delta(1,4) = \begin{vmatrix} -a_2 & a_1\\ b_2 & b_1 \end{vmatrix} = -a_2 b_1 - a_1 b_2,$

$\Delta(2,4) = -\begin{vmatrix} a_1 & 0\\ b_2 & b_1 \end{vmatrix} = -a_1 b_1,$

$\Delta(2,3) = \begin{vmatrix} a_1 & 0\\ 0 & b_2 \end{vmatrix} = a_1 b_2.$

Let us calculate now $\det A$:

$$\det A = \frac{1}{\Delta}\left(\Delta(1,3)w_1^2 + \Delta(2,4)w_2^2 + (\Delta(2,3)+\Delta(1,4))w_1w_2\right) = \\ = \frac{1}{\Delta}\left(a_2b_2w_1^2 - a_2b_1w_1w_2 - a_1b_1w_2^2\right).$$

By Theorem 2

$$J_{(0,0)} = \sum_{\|K\|=k_{11}+k_{12}+k_{21}+k_{22}\leqslant 2} \frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times \\ \times\mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1}\cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

We calculate the value of the sums by denoting $\bar{\Delta} = a_2b_1 + a_1b_2$,
$(0,0,0,0)$:

$$\mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2}\right] = 0,$$

$(1,0,0,0)$:

$$-\mathfrak{M}\left[\frac{a_{11}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2}\right] = \frac{-2a_1a_2^2b_1^2b_2^2 - 2a_2^2b_1b_2(a_2b_1^2 + a_1b_1b_2)}{\Delta^2} = -\frac{2a_1b_2^2}{\bar{\Delta}^2} - \frac{2b_2}{\bar{\Delta}},$$

$(0,1,0,0)$:

$$-\mathfrak{M}\left[\frac{a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2}\right] = -\frac{a_1^2b_1(2a_2^2b_1b_2 - 4a_2^2b_1b_2)}{\Delta^2} = \frac{2a_1^2b_2}{\bar{\Delta}^2},$$

$(0,0,1,0)$:

$$-\mathfrak{M}\left[\frac{a_{21}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^4}\right] = \frac{a_2b_2^2(4a_1a_2b_1^2 - 2a_1a_2b_1^2)}{\Delta^2} = \frac{2a_1b_2^2}{\bar{\Delta}^2},$$

$(0,0,0,1)$:

$$-\mathfrak{M}\left[\frac{a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^4}\right] = -\frac{2a_1^2a_2^2b_1^2b_2 + 2a_1a_2b_1^2(a_2^2b_1 + a_1a_2b_2)}{\Delta^2} = -\frac{2a_1^2b_2}{\bar{\Delta}^2} - \frac{2a_1}{\bar{\Delta}},$$

$(2,0,0,0)$:

$$\mathfrak{M}\left[\frac{a_{11}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right] = 0,$$

$(0,2,0,0)$:

$$\mathfrak{M}\left[\frac{a_{12}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right] = 0,$$

$(0,0,2,0)$:

$$\mathfrak{M}\left[\frac{a_{21}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right] = 0,$$

$(0,0,0,2)$:
$$\mathfrak{M}\left[\frac{a_{22}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right]=0,$$
$(1,1,0,0)$:
$$2\mathfrak{M}\left[\frac{a_{11}a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right]=0,$$
$(1,0,1,0)$:
$$-\mathfrak{M}\left[\frac{a_{11}a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$
$(0,1,1,0)$:
$$-\mathfrak{M}\left[\frac{a_{12}a_{21}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$
$(0,1,0,1)$:
$$\mathfrak{M}\left[\frac{a_{12}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$
$(0,0,1,1)$:
$$2\mathfrak{M}\left[\frac{a_{21}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right]=0,$$
$(1,0,0,1)$:
$$\mathfrak{M}\left[\frac{a_{11}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0.$$

Therefore
$$J_{(0,0)}=-\frac{2a_1b_2^2}{\bar{\Delta}^2}-\frac{2a_1^2b_2}{\bar{\Delta}^2}-\frac{2b_2}{\bar{\Delta}}-\frac{2a_1}{\bar{\Delta}}+\frac{2a_1^2b_2}{\bar{\Delta}^2}+\frac{2a_1b_2^2}{\bar{\Delta}^2}=$$
$$=-\frac{2a_1b_2(a_1+b_2)}{\bar{\Delta}^2}-\frac{2(a_1+b_2)}{\bar{\Delta}}+\frac{2a_1b_2(a_1+b_2)}{\bar{\Delta}^2},$$
$$J_{(0,0)}=-\frac{2(a_1+b_2)}{\bar{\Delta}}. \tag{17}$$

Example 5. Consider a system of equations in two complex variables
$$\begin{cases}\widetilde{f}_1=-a_2w_1^2+a_1w_1w_2+1=0,\\ \widetilde{f}_2=b_2w_1w_2+b_1w_2^2+1=0.\end{cases} \tag{18}$$

Let $u=w_1w_2$, then $w_2=\dfrac{u}{w_1}$, substitute in our system
$$\begin{cases}-a_2w_1^2+a_1u+1=0,\\ b_2u+b_1\frac{u^2}{w_1^2}+1=0.\end{cases} \tag{19}$$

We get rid of the denominators
$$\begin{cases}-a_2w_1^2+a_1u+1=0,\\ b_2uw_1^2+b_1u^2+w_1^2=0.\end{cases} \tag{20}$$

Calculate $\det\Delta$:

$$\Delta = \begin{vmatrix} -a_2 & 0 & a_1u+1 & 0 \\ 0 & -a_2 & 0 & a_1u+1 \\ b_2u+1 & 0 & b_1u^2 & 0 \\ 0 & b_2u+1 & 0 & b_1u^2 \end{vmatrix},$$

$$\Delta = u^4(a_1^2b_2^2 + 2a_1a_2b_1b_2 + a_2^2b_1^2)+$$

$$+u^3(2a_1a_2b_1 + 2a_2b_1b_2 + 2a_1^2b_2 + 2a_1b_2^2) + u^2(2a_2b_1 + a_1^2 + 4a_1b_2 + b_2^2) + u(2a_1 + 2b_2) + 1.$$

By the generalized Vieta theorem

$$\begin{aligned} J_{(0,0)} &= \sum_{j=1}^{p} w_{j1}^{\gamma_1+1} \cdot w_{j2}^{\gamma_2+1} = -\frac{a_3}{a_4} = \frac{2a_1a_2b_1 + 2a_2b_1b_2 + 2a_1^2b_2 + 2a_1b_2^2}{a_1^2b_2^2 + 2a_1a_2b_1b_2 + a_2^2b_1^2} = \\ &= -2\frac{a_2b_1(a_1+b_2) + a_1b_2(a_1+b_2)}{\bar{\Delta}^2} = -2\frac{(a_1+b_2)(a_1b_2+a_2b_1)}{\bar{\Delta}^2} = -\frac{2(a_1+b_2)}{\bar{\Delta}}. \end{aligned} \tag{21}$$

Example 6. Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2\pi^2}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2k+1)!},$$

which uniformly and absolutely converge on any compact from the complex plane and have the order of growth $1/2$.

Consider the system of equations

$$\begin{cases} f_1(w_1,w_2) = \dfrac{\sin\sqrt{-a_1w_1w_2 + a_2w_1^2}}{\sqrt{-a_1w_1w_2 + a_2w_1^2}} = \prod\limits_{k=1}^{\infty}\left(1 - \dfrac{-a_1w_1w_2 + a_2w_1^2}{k^2\pi^2}\right) = 0, \\ f_2(w_1,w_2) = \dfrac{\sin\sqrt{b_1w_2^2 + b_2w_1w_2}}{\sqrt{b_1w_2^2 + b_2w_1w_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{b_1w_2^2 + b_2w_1w_2}{s^2\pi^2}\right) = 0. \end{cases} \tag{22}$$

Each of the functions of this system decomposes into an infinite product of functions from the system (18).

Therefore, the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = \sum_{k,s=1}^{\infty}\frac{2\pi^2(a_1s^2 - b_2k^2)}{a_1b_2 + a_2b_1}.$$

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Некоторые системы трансцендентных уравнений

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Аннотация. Рассмотрены различные примеры систем трансцендентных уравнений. Так как число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Получены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга. Вычислены суммы многомерных числовых рядов.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.