DOI: 10.17516/1997-1397-2022-15-2-253-260
УДК 517.55

Morera's Boundary Theorem in Siegel Domain of the First Kind

Bukharbay T. Kurbanov*
Karakalpak State University
Nukus, Uzbekistan

Received 13.10.2021, received in revised form 27.01.2022, accepted 02.02.2022

Abstract. The paper considers the realization of the matrix unit polydisk in the form of a Siegel domain of the first kind and proves the boundary analogue of Morera's theorem.

Keywords: matrix unit polydisk, automorphism, Poisson kernel, holomorphic function, holomorphic extension.

Citation: B. T. Kurbanov, Morera's Boundary Theorem in Siegel Domain of the First Kind, J. Sib. Fed. Univ. Math. Phys., 2022, 15(2), 253–260. DOI: 10.17516/1997-1397-2022-15-2-253-260.

Introduction

In multidimensional complex analysis, the theory of integral representations has numerous applications in the study of functions with the one-dimensional holomorphic continuation property and in the proof of boundary analogues of Morera's theorem. Here it is worth noting the works of A. M. Kytmanov, S. G. Myslivets, S. Kosbergenov ([1–3]).The monograph [4] provides a detailed overview of the results in this direction obtained by various authors in recent years. In this article, using the properties of the Poisson integral, we prove a boundary version of Morer's theorem for one Siegel domain of the 1st kind ([5]). It asserts the possibility of a holomorphic continuation of the functions f from the boundary ∂D of the domain $D \subset \mathbb{C}^n$, provided that the integrals of f are equal to zero along the boundaries of analytical disks lying on ∂D.

The unit disc and its various multidimensional generalizations (unit n-dimensional ball, polydisk, matrix unit disc, classical domains of types according to Cartan's classification, matrix ball) are well studied: by now, many important questions of multidimensional complex analysis have been solved, such as description of automorphism groups, obtaining integral formulas of Cauchy–Szego, Bergman, Poisson type, proving necessary and sufficient conditions for holomorphic extendability of functions from the boundary, etc. Extensive results obtained in these areas are presented in the monographs [4] and [6].

Quite often, problems posed for a unit disc on a plane are transferred to the upper half-plane using the Cayley transform

$$w = \frac{i(1+z)}{1-z}.$$

In this regard, it is urgent to find multidimensional analogs of the formula for the realization of the "unit disc – upper half-plane" type ([7]). We consider the realization of the matrix unit polydisk in the form of the Siegel domain of the first kind and the transformation of the invariant Poisson kernel for such a realization.

*bukharbay@inbox.ru https://orcid.org/0000-0003-3272-5789

1. Realization of a matrix unit polydisk in the form of a Siegel domain of the 1st kind

Let be

$$D = \{U = (U_1, \dots, U_n) \in \mathbb{C}^n[m \times m] : \operatorname{Im} U_j > 0,\ j = \overline{1,n}\},$$

U_j is a matrix of order $[m \times m]$ with elements from $\mathbb{C}$. The skeleton of this domain is denoted

$$\Gamma = \{\operatorname{Im} Uj = 0,\ j = \overline{1,n}\} = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$$

Domain D is a Siegel domain of the 1st kind ([5]). In particular, for $m = n = 1$ the domain D is reduced to the upper half-plane, and the skeleton Γ coincides with the real axis.

Domain T of space $\mathbb{C}^n[m \times m]$:

$$T = \{Z = (Z_1, \dots, Z_n) :\ Z_j Z_j^* < E,\ j = \overline{1,n}\},$$

where Z_j is a matrix of order $[m \times m]$ with elements from $\mathbb{C}$, and E is the identity matrix of order m, is called *matrix unit polydisk.*

The boundary ∂T is a union of surfaces

$$\gamma^\nu = \{ Z = (Z_1, \dots, Z_n) : Z_\nu Z_\nu^* = E,\ Z_\mu Z_\mu^* \leqslant E,\ \nu \neq \mu \},$$

each of which is a $(2nm^2 - 1)$-dimensional surface (since the $2nm^2$ coordinates of the point Z are related by one real relation $\det(E - Z_\nu Z_\nu^*) = 0$). Therefore, the entire boundary

$$\partial T = \bigcup_{\nu=1}^{n} \gamma^\nu$$

is $(2nm^2 - 1)$-dimensional. For the disc $\tau_j = Z_j Z_j^* < E$, the skeleton is the set of all unitary matrices $S_j = Z_j Z_j^* = E$ ([8, page 10]). Then the skeleton $S(T)$ of the domain T, defined as the Cartesian product of the discs τ_j, is the Cartesian product of all S_j that is,

$$S(T) = \{Z_j Z_j^* = E,\ j = \overline{1,n}\} = S_1 \times S_2 \times \dots \times S_n \subset \partial T.$$

The dimension of the skeleton is nm^2.

For $m = n = 1$, the domain of T is reduced to a unit disc, and the skeleton of $S(T)$ is a unit circle from $\mathbb{C}$.

By $\Phi = (\Phi^1, \dots, \Phi^n)$ we denote the transformation, where

$$U_j = \Phi^j(Z) = i(E + Z_j)(E - Z_j)^{-1}, \quad j = \overline{1,n}, \tag{1}$$

which is a biholomorphic map of T onto D, with $S(T)$ being mapped to Γ.

It is known that

$$\Phi_{A_j}(Z_j) = R_j^{-1}(E - Z_j A_j^*)^{-1}(Z_j - A_j)Q_j$$

is an automorphism of the generalized circle $Z_j Z_j^* < E$ taking the point A_j to 0 ([8]). Then the mapping

$$\Phi_A(Z) = (\Phi_A^1(Z), \dots, \Phi_A^n(Z)), \quad \Phi_A^j(Z) = \Phi_{A_j}(Z_j)$$

is an automorphism of the domain T that maps the point $A = (A_1, \dots, A_n)$ to $0 = (0, \dots, 0)$.

Using the transformations Φ and Φ_A, we define the following transformation

$$\Psi_B = \Phi \circ \Phi_A \circ \Phi^{-1}, \quad B = \Phi(A)$$

which is an automorphism of the domain D taking the point B to the point $(iE, \dots, iE)$.

2. On the Poisson kernel in the Siegel domain of the 1st kind

Let $\dot{Z}$ be the volume element in $S(T)$, and $\dot{U}$ the volume element in Γ.

We have

$$\dot{Z} = \prod_{j=1}^{n} \dot{Z}_j, \quad \dot{U} = \prod_{j=1}^{n} \dot{U}_j,$$

where $\dot{Z}_j$ is the volume element in $Z_j Z_j^* = E$, and $\dot{U}_j$ is volume element in $\mathrm{Im} U_j = 0$.

Because

$$\dot{Z}_j = 2^{m^2} |\det(U_j + iE)|^{-2m} \dot{U}_j, ([8])$$

then $\dot{Z}$ can be represented as

$$\dot{Z} = \dot{Z}_1 \wedge \dots \wedge \dot{Z}_n = 2^{nm^2} \prod_{j=1}^{n} |\det(U_j + iE)|^{-2m} \dot{U}.$$

So the following is true

Lemma 1. *The following relation holds*

$$\dot{Z} = 2^{nm^2} \prod_{j=1}^{n} |\det(U_j + iE)|^{-2m} \dot{U}.$$

The Poisson kernel for the generalized upper half-plane $\mathrm{Im} U_j > 0$ has the form (see [8])

$$P(U_j, B_j) = c \cdot \frac{\det^m (B_j - B_j^*)}{|\det(U_j - B_j^*)|^{2m}}, \quad \mathrm{Im} B_j > 0, \quad \mathrm{Im} U_j = 0,$$

here c is some constant.

Since the domain D is defined as the Cartesian product of the generalized upper half-planes, then using the formula of repeated integration (Fubini's theorem) we obtain that the Poisson kernel for the domain D will have the form:

$$P_D(U, B) = \prod_{j=1}^{n} P(U_j, B_j) = c^n \prod_{j=1}^{n} \frac{\det^m (B_j - B_j^*)}{|\det(U_j - B_j^*)|^{2m}},$$

where $B \in D$, $U \in \Gamma$.

The invariant Poisson kernel for the domain T is defined in the same way:

$$P(W, A) = \prod_{j=1}^{n} P(W_j, A_j) = \prod_{j=1}^{n} \frac{\det^m (E - A_j A_j^*)}{|\det(E - A_j W_j^*)|^{2m}},$$

where $A \in T$, $W \in S(T)$.

Пусть $B = \Phi(A)$, $U = \Phi(W)$.

Lemma 2. *Mapping Φ transforms the Poisson kernel as follows:*

$$P(\Phi^{-1}(U), \Phi^{-1}(B)) = \frac{(-2i)^{nm}}{c^n} \prod_{j=1}^{n} |\det(U_j + iE)|^{2m} P_D(U, B).$$

Proof. For each component of the mapping Φ with fixed j, we prove

$$P(\Phi_j^{-1}(U_j), \Phi_j^{-1}(B_j)) = \frac{(-2i)^m}{c} P(U_j, B_j) |\det(U_j + iE)|^{2m},$$

whence passing to the product over $j = \overline{1, n}$ we obtain the required equality. Finding the converse from (1), we have

$$W_j = (U_j + iE)^{-1}(U_j - iE), \;\; A_j = (B_j + iE)^{-1}(B_j - iE).$$

Then

$$E - A_j W_j^* = E - (B_j + iE)^{-1}(B_j - iE)(U_j + iE)(U_j - iE)^{-1} =$$
$$= (B_j + iE)^{-1}\big((B_j + iE)(U_j - iE) - (B_j - iE)(U_j + iE)\big)(U_j - iE)^{-1} =$$
$$= 2i(B_j + iE)^{-1}(U - B)(U_j - iE)^{-1}.$$

Similar

$$E - W_j A_j^* = -2i(U_j + iE)^{-1}(U - B^*)(B_j^* - iE)^{-1},$$
$$E - A_j A_j^* = -2i(B_j + iE)^{-1}(B - B^*)(B_j^* - iE)^{-1}.$$

Substituting these found into the expression for $P(W, A)$ and using the properties of the determinants of the matrices, we obtain the required equality. Lemma 2 is proved. □

3. Morera's boundary theorem

Consider the following embedding of the circle $\triangle = \{t \in \mathbb{C} : \; |t| < 1\}$ into D:

$$\{\zeta \in \mathbb{C}^n[m \times m] : \zeta_j = t\Lambda_j, \, j = 1, \dots, n, \, t \in \triangle\}, \tag{2}$$

where $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \Gamma$. If Ψ is an arbitrary automorphism of the domain D, then set (2) under the action of this automorphism goes to some analytic disc with boundary on Γ.

Theorem 1. *Let f be a continuous bounded function on Γ. If the function f satisfies the condition*

$$\int_{\partial\triangle} f(\Psi(\Lambda^0 t)) \, dt = 0 \tag{3}$$

for all automorphisms Ψ of the domain $\mathcal{D}$ and fixed $\Lambda^0 \in \Gamma$, then the function f extends holomorphically in D to a function of class $\mathcal{H}^\infty(D)$ continuous up to Γ.

Proof. Since Γ is invariant under unitary transformations, condition (3) will hold for arbitrary $\Lambda \in \Gamma$.

Consider an automorphism Ψ_B that takes the point B from D to the point $(iE, \dots, iE)$:

$$\Psi_B = \Phi \circ \Phi_A^{-1} \circ \Phi^{-1}.$$

Then substituting the automorphism Ψ_B into condition (3) instead of Ψ, we obtain

$$\int_{\partial\triangle} f(\Phi \circ \Phi_A^{-1} \circ \Phi^{-1}(\Lambda t)) \, dt = 0. \tag{4}$$

We denote $\nu = \Phi^{-1}(\Lambda)$. Then (4) can be written as

$$\int_{\partial\triangle} f(\Phi \circ \Phi_A^{-1}(\nu t)) \, dt = 0. \tag{5}$$

We parametrize the manifold S_j as follows:

$$\zeta_j = t\nu_j, \quad t = e^{i\phi}, \quad 0 \leqslant \phi \leqslant 2\pi, \ \nu_j \in S'_j$$

if $\zeta_j \in S_j$. Here S'_j is the group of special unitary matrices, that is, $\det \nu_j = 1$. Volume element $d\mu_j$ on the manifold S_j can be written in the form

$$d\mu_j = \frac{d\phi}{2\pi} \wedge d\mu_0(\nu_j) = \frac{1}{2\pi i}\frac{dt}{t} \wedge d\mu_0(\nu_j),$$

where $d\mu_0(\nu_j)$ is a differential form that defines a positive measure on S'_j.

Multiplying (5) by $d\mu_0$ and integrating over S'_j, from (5) we obtain

$$\int_{S_j} f(\Phi \circ \Phi_A^{-1}(\zeta_j))\zeta_j^k \, d\mu_j(\zeta_j) = 0. \tag{6}$$

where ζ_j^k is a component of the vector ζ_j $(k = 1, \ldots, m^2)$.

By Fubini's theorem, we obtain from this that

$$\int\limits_{S=S_1\times\cdots\times S_n} f(\Phi \circ \Phi_A^{-1}(\zeta))\zeta^k \, d\mu(\zeta) = 0, \tag{7}$$

where ζ^k is the k-th component of the vector ζ.

Let's make the change of variables $W = \Phi_A^{-1}(\zeta)$. Then (7) becomes the condition

$$\int_S f(\Phi(W))\Phi_{A,l}^k(W) \, d\mu(\Phi_A(W)) = 0. \tag{8}$$

Since ([9, Lemma 3.4])

$$d\mu_j(\Phi_A^j(W)) = P(W_j, A_j) \, d\mu_j(W_j),$$

then

$$d\mu(\Phi_A(W)) = P(W, A) \, d\mu(W).$$

Then from (8) we obtain

$$\int_S f(\Phi(W))\Phi_{A,l}^k(W) \, P(W, A) \, d\mu(W) = 0 \quad (k = 1, \ldots, m^2). \tag{9}$$

As you know ([8])

$$\Phi_{A,l}(W) = \Phi_A^l(W) = R_l^{-1}(E - W_l A_l^*)^{-1}(W_l - A_l)Q_l,$$

where R_l, Q_l are nonsingular and depend only on A_l. Therefore, if condition (9) is satisfied for the components of the mapping $\Phi_{A,l}$, then the same condition will be satisfied for the components of the mapping

$$\varphi_{A,l}(W) = (E - W_l A_l^*)^{-1}(W_l - A_l) \quad (l = \overline{1,n}).$$

Denoting the components of the mapping $\varphi_{A,l}(W)$ by $\varphi_{A,l}^{s,k}$ $(s, k = \overline{1,m})$, from (9) we get

$$\int_S f(\Phi(W))\varphi_{A,l}^{sk}(W) \, P(W, A) \, d\mu(W) = 0. \tag{10}$$

Now let's make the change of variables $U = \Phi(W)$. Then, taking into account Lemma 1 and Lemma 2, we obtain

$$\int_\Gamma f(U)\psi_{0,l}^{sk}(U)\, P_D(U,B)\, d\mu_\Gamma(U) = 0, \tag{11}$$

here $d\mu_\Gamma$ is the volume element on Γ, and

$$\psi_0 = \Phi_A \circ \Phi,$$

$$\psi_{0,l} = (iE - B_l^*)(U_l - B_l^*)^{-1}(U_l - B_l)(B_l + iE)^{-1}.$$

Since $B_l - B_l^*$ is non-singular, then condition (11) will also hold for the components of the mapping

$$\Phi_{B,l} = (U_l - B_l^*)^{-1}(U_l - B_l)(B_l - B_l^*)^{-1} \quad (l = \overline{1,n}).$$

So

$$\int_\Gamma f(U)\Phi_{B,l}^{sk}(U)\, P_D(U,B)\, d\mu_\Gamma(U) = 0 \quad (s,k = \overline{1,m},\ l = \overline{1,n}). \tag{12}$$

In [10] (see also [6, p.141]) it was proved that

$$\sum_{s,k=1}^{m} \overline{b}_{sk}^j \frac{\partial P(U_j,B_j)}{\partial \overline{b}_{sk}^j} + i\sum_{s=1}^{m} \frac{\partial P(U_j,B_j)}{\partial \overline{b}_{ss}^j} = mP(U_j,B_j)\left(\sum_{s,k=1}^{m} \overline{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^{m} \Phi_{B,j}^{ss}\right),$$

where

$$\prod_{j=1}^{n} P(U_j,B_j) = P_D(U,B),$$

and $\Phi_{B,j}^{sk}$ is the sk-component of the mapping $\Phi_{B,j}$, where $\Phi_B = (\Phi_{B,1},\ldots,\Phi_{B,n})$. From here

$$\sum_{s,k=1}^{m} \overline{b}_{sk}^j \frac{\partial P_D(U,B)}{\partial \overline{b}_{sk}^j} + i\sum_{s=1}^{m} \frac{\partial P_D(U,B)}{\partial \overline{b}_{ss}^j} =$$

$$= \sum_{s,k=1}^{m} \overline{b}_{sk}^j \frac{\partial \prod_{l=1}^{n} P(U_l,B_l)}{\partial \overline{b}_{sk}^j} + i\sum_{s=1}^{m} \frac{\partial \prod_{l=1}^{n} P(U_l,B_l)}{\partial \overline{b}_{ss}^j} =$$

$$= \prod_{l\neq j} P(U_l,B_l)\left[\sum_{s,k=1}^{m} \overline{b}_{sk}^j \frac{\partial P(U_j,B_j)}{\partial \overline{b}_{sk}^j} + i\sum_{s=1}^{m} \frac{\partial P(U_j,B_j)}{\partial \overline{b}_{ss}^j}\right] =$$

$$= \prod_{l\neq j} P(U_l,B_l)\cdot m\cdot P(U_j,B_j)\left(\sum_{s,k=1}^{m} \overline{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^{m} \Phi_{B,j}^{ss}\right) =$$

$$= m\, P_D(U,B)\left(\sum_{s,k=1}^{m} \overline{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^{m} \Phi_{B,j}^{ss}\right).$$

Hence,

$$\sum_{j=1}^{n}\sum_{s,k=1}^{m} \overline{b}_{sk}^j \frac{\partial P_D(U,B)}{\partial \overline{b}_{sk}^j} + i\sum_{j=1}^{n}\sum_{s=1}^{m} \frac{\partial P_D(U,B)}{\partial \overline{b}_{ss}^j} = m\, P_D(U,B)\left(\sum_{j=1}^{n}\sum_{s,k=1}^{m} \overline{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{j=1}^{n}\sum_{s=1}^{m} \Phi_{B,j}^{ss}\right).$$

Taking this into account, we obtain from (12)

$$\mathfrak{d}F(B) = 0, \tag{13}$$

where

$$\mathfrak{d} = \sum_{j=1}^{n}\sum_{s,k=1}^{m} \bar{b}_{sk}^{j}\frac{\partial}{\partial \bar{b}_{sk}^{j}} + i\sum_{j=1}^{n}\sum_{s=1}^{m}\frac{\partial}{\partial \bar{b}_{ss}^{j}},$$

and

$$F(B) = \int_{\Gamma} f(U)\, P_D(U,B)\, d\mu_\Gamma(U)$$

is the Poisson integral of the function f. The function $F(B)$ is real-analytic in the domain D. We expand it into a Taylor series in a neighborhood of the point $I = (iE, \dots, iE)$:

$$F(B) = \sum_{|\alpha|,|\beta|} c_{\alpha,\beta}\,(B-I)^{\alpha}\,\overline{(B-I)}^{\beta},$$

where $|\alpha| = \|\alpha_{sk}^{l}\|$, $|\beta| = \|\beta_{sk}^{l}\|$ are matrices with integer elements,

$$|\alpha| = \sum_{l=1}^{n}\sum_{s,k=1}^{m}\alpha_{sk}^{l},\ \ B^{\alpha} = \prod_{l=1}^{n}\prod_{s,k=1}^{m} b_{sk}^{l,\alpha_{sk}^{l}}.$$

Then condition (13) implies

$$\mathfrak{d}F(B) = \sum_{|\alpha|,|\beta|} |\beta|\cdot c_{\alpha,\beta}\,(B-I)^{\alpha}\,\overline{(B-I)}^{\beta},$$

hence all the coefficients $c_{\alpha,\beta}$. Hence $F(B)$ is holomorphic in D and belongs to $\mathcal{H}^{\infty}(D)$. □

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Граничная теорема Морера в области Зигеля первого рода

Бухарбай Т. Курбанов
Каракалпакский государственный университет
Нукус, Узбекистан

Аннотация. В работе рассмотрена реализация матричного единичного поликруга в виде области Зигеля первого рода и доказывается граничный вариант теоремы Морера.

Ключевые слова: матричный единичный поликруг, автоморфизм, ядро Пуассона, голоморфная функция, голоморфное продолжение.