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A Note on Explicit Formulas for Bernoulli Polynomials

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Abstract. For $r \in \left\{1, -1, \frac{1}{2}\right\}$, we prove several explicit formulas for the n-th Bernoulli polynomial $B_n(x)$, in which $B_n(x)$ is equal to a linear combination of the polynomials x^n, $(x+r)^n, \ldots, (x+rm)^n$, where m is any fixed positive integer greater than or equal to n.

Keywords: Appell polynomial, Bernoulli polynomial, binomial coefficients, combinatorial identities.

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1. Introduction and preliminaries

Let B_n and $B_n(x)$, $n = 0, 1, 2, \ldots$, be the Bernoulli numbers and polynomials defined by the generating functions

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

respectively. These numbers and polynomials play important roles in many different branches of mathematics [14]. Since their appearance in [4], many different explicit formulas for Bernoulli numbers and polynomials have been discovered throughout the years, see for example [5,8,11,12, 15,17]. Gould's article [8] is a remarkable retrospective concerning some old explicit formulas of Bernoulli numbers and contains a rich and very interesting bibliography. It emerges from Gould's study, that over time, it often happened that the same explicit formula concerning the n-th

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Bernoulli number B_n was rediscovered several times by different authors, each of these authors thinking to have discovered a new explicit formula. In [8], Gould reported that Munch [17] found an old result and published the formula in the form

$$B_n = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j j^n \frac{\binom{n+1}{k-j}}{\binom{n}{k}}. \tag{1}$$

In 2016, Komatsu and Pita Ruiz [12] generalized Munch's formula (1) to Bernoulli polynomials by proving that

$$B_n(x) = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} \frac{\binom{m+1}{j}}{\binom{m}{k}} (x+k-j)^n,$$

where $0 \leqslant n \leqslant m$. This formula can be written as

$$B_n(x) = \sum_{j=0}^{m} \lambda_j (x+j)^n, \tag{2}$$

where $\lambda_j = \lambda_j(m)$ depending only on m and j is given by

$$\lambda_j(m) = \frac{1}{m+1} \sum_{k=j}^{m} (-1)^j \frac{\binom{m+1}{k-j}}{\binom{m}{k}}.$$

More generally, it is easy to see that for any triplet (A, r, m) where $A = (A_n(x))$ is a sequence of Appell polynomials, $r \neq 0$ is a complex number and $m \geqslant 0$ is an any integer, there exists an unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that for $0 \leqslant n \leqslant m$, we have

$$A_n(x) = \sum_{j=0}^{m} \mu_j (x+rj)^n.$$

Thus $\mu_j(B, 1, m) = \lambda_j(m)$ with $B = (B_n(x))$.

The paper is organized as follows: In the second section we give a generalization of Theorem 2 given by Adell and Lekuona [1]. In the third section, we prove some lemmas. In the fourth and the last section, we determine different expressions of $\mu_j(B, r, m)$ for $r \in \left\{1, -1, \frac{1}{2}\right\}$ and we derive explicit formulas for Bernoulli polynomials among them the identity (2).

2. Generalization of a theorem of Adell and Lekuona

Let $A = (A_n(x))_{n \geqslant 0}$ be a sequence of polynomials and $a_n = A_n(0)$. A is called an Appell sequence [2] if $a_0 \neq 0$ and if in addition, one of the following equivalent conditions is satisfied

$$A'_n(x) = nA_{n-1}(x), \quad n \geqslant 1,$$

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k},$$

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) e^{xt},$$

$$A_n(x) = \Omega_A(x^n) \text{ where } \Omega_A = \sum_{k=0}^{\infty} a_k \frac{D^k}{k!}, \tag{3}$$

D being the usual differential operator. $B = (B_n(x))_{n\geqslant 0}$ is obviously an Appell sequence for which we have

$$\Omega_B = \frac{D}{e^D - 1} = \sum_{k=0}^{\infty} B_k \frac{D^k}{k!}. \tag{4}$$

We also have

$$\Omega_B = \frac{\log(1+\Delta)}{\Delta} = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k}{k+1}, \tag{5}$$

where Δ is the difference operator. More generally, if Δ_r is the difference operator defined by

$$\Delta_r(x^n) = (x+r)^n - x^n,$$

where $r \neq 0$ is a complex number, and Ω_B can be written as a series in Δ_r. To prove this, we need the use of Stirling numbers of the first and second kind, respectively denoted $s(n,k)$ and $S(n,k)$, $k = 0, 1, \ldots, n$. These numbers are defined by [6]

$$(x)_n = \sum_{k=0}^{\infty} s(n,k)\, x^k, \quad x^n = \sum_{k=0}^{\infty} S(n,k)\,(x)_k\,,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, or equivalently by their exponential generating functions

$$\frac{\log^k(t+1)}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!}, \quad \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}. \tag{6}$$

The following lemma proves that every operator written as a serie in D can be written as a serie in Δ_r and reciprocally.

Lemma 1.2. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences of elements of $\mathbb{C}$. Then, the following equivalences hold*

$$\sum_{k=0}^{\infty} a_k \frac{D^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{\Delta_r^k}{k!} \Leftrightarrow b_n = \sum_{k=0}^{n} s(n,k)\frac{a_k}{r^k}, \quad n \geqslant 0$$
$$\Leftrightarrow a_n = \sum_{k=0}^{n} r^n S(n,k)\, b_k, \quad n \geqslant 0.$$

Proof. We have

$$\Delta_r = e^{rD} - 1 \ \text{ and } \ D = \frac{1}{r}\log(1+\Delta_r).$$

The lemma results from the following two relations deduced from (6):

$$\frac{D^k}{k!} = \frac{1}{r^k}\frac{\log^k(1+\Delta_r)}{k!} = \frac{1}{r^k}\sum_{n=0}^{\infty} s(n,k)\frac{\Delta_r^n}{n!}, \tag{7}$$

$$\frac{\Delta_r^k}{k!} = \frac{\left(e^{rD}-1\right)^k}{k!} = \sum_{n=0}^{\infty} r^n S(n,k)\frac{D^n}{n!}. \tag{8}$$

□

Before stating our theorem, first we prove Lemma 1.2 which is an important consequence of the relation (7). We consider the translations τ_r of $\mathbb{C}[x]$ which are the operators defined by [18, p. 195]

$$\tau_r(x^n) = (x+r)^n, n \geqslant 0.$$

Then, we have

$$\tau_r = e^{rD}$$

and

$$\Delta_r^k = (\tau_r - 1)^k = \left(e^{rD} - 1\right)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{rj}.$$

We deduce the useful following relation

$$\Delta_r^k(x^n) = (\tau_r - 1)^k(x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+rj)^n. \tag{9}$$

Lemma 2.2. *For $r \in \mathbb{C}^*$ and $m \in \mathbb{N}$, the family of polynomials*

$$F_m = \{x^m, (x+r)^m, (x+2r)^m, \ldots, (x+mr)^m\}$$

forms a base of the $\mathbb{C}$-vectorial space

$$\mathbb{C}_m[x] := \{P(x) \in \mathbb{C}[x] : \deg P(x) \leqslant m\}.$$

Proof. Indeed, the family F_m consists of $m+1$ vectors of vector space $\mathbb{C}_m[x]$ which is of dimension $m+1$. It is therefore sufficient to prove that x^k belongs to the subspace generated by F_m for $0 \leqslant k \leqslant m$. Using the relation (7), for $0 \leqslant k \leqslant m$, we have

$$\frac{D^{m-k}}{(m-k)!}(x^m) = \frac{1}{r^{m-k}} \sum_{\ell=0}^{m} s(\ell, m-k) \frac{\Delta_r^\ell}{\ell!}(x^m).$$

Using the relation (9), we deduce the identity verified for $0 \leqslant k \leqslant m$

$$\binom{m}{k} x^k = \sum_{j=0}^{m} \left(\sum_{\ell=j}^{m} s(\ell, m-k) \frac{(-1)^{\ell-j}}{r^{m-k}\ell!} \binom{\ell}{j} \right) (x+rj)^m,$$

through this demonstration we prove that x^k belongs to the vectorial subspace generated by F_m for $0 \leqslant k \leqslant m$ and this completes the proof. □

From Lemma 2.2, we can deduce that if $A = (A_n(x))$ is a sequence of Appell polynomials, for any complex $r \neq 0$ and for any integer $m \geqslant 0$, there exists unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that for $0 \leqslant n \leqslant m$

$$A_n(x) = \sum_{j=0}^{m} \mu_j (x+rj)^n. \tag{10}$$

Indeed, from the fact that $\deg A_m(x) = m$, there exists unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that

$$A_m(x) = \sum_{j=0}^{m} \mu_j (x+rj)^m. \tag{11}$$

Deriving $m-n$ times each of the two members of (11), we deduce that we have also (10).
The following theorem is a generalization of Theorem 2 given in [1]. The desired theorem follows from Lemma 1.2 and the characterization (3) of Appell sequence.

Theorem 1.2. *Let $r \in \mathbb{C}^*$. $A = (A_n(x))_{n\in\mathbb{N}}$ is an Appell sequence if, and only if, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{C}$ such that $b_0 \neq 0$ and*

$$A_n(x) = \Omega(x^n),$$

where

$$\Omega = \sum_{k=0}^{\infty} \frac{b_k}{k!} \Delta_r^k. \tag{12}$$

Such sequence is defined by one of the following equivalence relations

$$b_n = \sum_{k=0}^{n} s(n,k) \frac{a_k}{r^k}, \tag{13}$$

$$\frac{a_n}{r^n} = \sum_{k=0}^{n} S(n,k) b_k, \tag{14}$$

where $a_k = A_k(0)$.

Corollary 1.2. *Let $r \in \mathbb{C}^*$ et $m \in \mathbb{N}$ and let $(A_n(x))_{n\in\mathbb{N}}$ be a sequence of Appell polynomials of $\mathbb{C}[x]$. Then, there exists a unique sequence $(\mu_j)_{0\leqslant j\leqslant m}$ of $\mathbb{C}$ such that*

$$\forall n \in \{0,1,\dots,m\}, \quad A_n(x) = \sum_{j=0}^{m} \mu_j (x+jr)^n. \tag{15}$$

Furthermore, we have

$$\mu_j = \sum_{k=j}^{m} \frac{(-1)^{k-j}}{k!} \binom{k}{j} b_k, \ 0 \leqslant j \leqslant m \quad \text{with} \quad b_k = \sum_{\ell=0}^{k} s(k,\ell) \frac{A_\ell(0)}{r^\ell}. \tag{16}$$

Proof. The existence and uniqueness of the sequence $(\mu_j)_{0\leqslant j\leqslant m}$ come from Lemma 2.2, (15) is therefore proven. We deduce (16) from Theorem 1.2 and (9) noticing that

$$A_n(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \Delta_r^k (x^n) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} b_k (x+rj)^n.$$

□

To obtain the desired expressions and explicit formulas, we give some lemmas which will be used later.

3. Lemmas

In the following lemma, we express the operator Ω_B defined by (4) in function of the operators Δ, Δ_{-1} and $\Delta_{\frac{1}{2}}$.

Lemma 1.3. *We have*

$$\Omega_B = \frac{\log(1+\Delta)}{\Delta} = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k}{k+1}, \tag{17}$$

$$\Omega_B = \frac{\log(1+\Delta_{-1})}{\Delta_{-1}}(1+\Delta_{-1}) = 1 - \sum_{k=1}^{\infty} (-1)^k \frac{\Delta_{-1}^k}{k(k+1)}, \tag{18}$$

$$\Omega_B = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \frac{(-1)^k}{(s+1)\, 2^{k-s}} \right) \Delta_{\frac{1}{2}}^k, \tag{19}$$

$$\Omega_B = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \frac{(-1)^k}{k+1} \binom{k}{s}^{-1} \right) \Delta_{\frac{1}{2}}^k. \tag{20}$$

Proof. (17) is the well know relations (5) and (17) is obtained by substituting Δ by $-\dfrac{\Delta_{-1}}{1+\Delta_{-1}}$ in relation (17). We obtain (19) noticing that $D = 2\log\left(1+\Delta_{\frac{1}{2}}\right)$ and thus

$$\Omega_B = \frac{D}{e^D - 1} = \frac{1}{1+\frac{1}{2}\Delta_{\frac{1}{2}}} \frac{\log\left(1+\Delta_{\frac{1}{2}}\right)}{\Delta_{\frac{1}{2}}} = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \frac{(-1)^k}{(s+1)\, 2^{k-s}} \right) \Delta_{\frac{1}{2}}^k.$$

The relation (20) follows from (19) using the following identity [9, p. 21] or [16, Theorem 1]

$$\sum_{s=0}^{k} \frac{1}{(s+1)\, 2^{-s}} = \frac{2^k}{k+1} \sum_{s=0}^{k} \binom{k}{s}^{-1}.$$

□

The following lemmas will be useful to give explicit expressions for the coefficients $\mu_j(B, r, m)$ for $r \in \left\{1, -1, \frac{1}{2}\right\}$.

Lemma 2.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m} \binom{k}{j} x^k = \sum_{k=j}^{m} \binom{m+1}{k-j} x^k (1-x)^{m-k}. \tag{21}$$

Proof. We have

$$\begin{aligned}
\sum_{k=j}^{\infty} \binom{k}{j} x^k &= \frac{x^j}{(1-x)^{j+1}} = x^j (1-x)^{-j-1} (x + (1-x))^{m+1} = \\
&= \sum_{k=0}^{m-j} \binom{m+1}{k} x^{k+j} (1-x)^{m-k-j} + \sum_{k=m+1-j}^{m+1} \binom{m+1}{k} x^{k+j} (1-x)^{m-k-j} = \\
&= \sum_{k=j}^{m} \binom{m+1}{k-j} x^k (1-x)^{m-k} + x^{m+1} \sum_{s=0}^{j} \binom{m+1}{j-s} \frac{x^s}{(1-x)^{s+1}}.
\end{aligned} \tag{22}$$

We obtain (21) by writing the equality of the polynomial parts constituted by the terms of degree less than or equal to m in each of the two members of (22). □

Lemma 3.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m} \frac{(-1)^j}{k+1} \binom{k}{j} = \sum_{k=j}^{m} \frac{(-1)^j}{m+1} \frac{\binom{m+1}{k-j}}{\binom{m}{k}}. \tag{23}$$

Proof. By Lemma 2.3 and the identity $\int\limits_0^1 t^k (1-t)^{m-k} \, dt = \frac{1}{m+1}\binom{m}{k}^{-1}$ it follows

$$\begin{aligned}
\sum_{k=j}^{m} \frac{(-1)^j}{k+1}\binom{k}{j} &= \sum_{k=j}^{m} (-1)^j \binom{k}{j} \int_0^1 t^k dt = \\
&= \sum_{k=j}^{m} (-1)^j \binom{m+1}{k-j} \int_0^1 t^k (1-t)^{m-k} \, dt = \\
&= \sum_{k=j}^{m} \frac{(-1)^j}{m+1} \frac{\binom{m+1}{k-j}}{\binom{m}{k}}.
\end{aligned}$$

□

Lemma 4.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m} \frac{(-1)^j}{k+1}\binom{k}{j} = \sum_{k=j}^{m} \frac{(-1)^k}{k+1}\binom{m+1}{k+1}. \tag{24}$$

Proof. Let $[x^j]\,(P(x))$ be the coefficient of x^j in the polynomial $P(x) = \sum\limits_{k=0}^{m} \frac{(1-x)^k}{k+1}$. On the one hand, one have

$$[x^j]\,(P(x)) = \sum_{k=j}^{m} \frac{(-1)^j}{k+1}\binom{k}{j}, \ 0 \leqslant j \leqslant m. \tag{25}$$

On the other hand, we have

$$\begin{aligned}
[x^j]\, P(x) &= [x^j]\, \frac{1}{x-1} \sum_{k=0}^{m} \int_1^x (1-t)^k \, dt = \\
&= [x^j]\, \frac{-1}{x-1} \int_1^x \frac{(1-t)^{m+1}-1}{t} dt = \\
&= [x^j]\, \frac{1}{x-1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1} \int_1^x t^k dt = \\
&= [x^j] \sum_{k=0}^{m} \frac{(-1)^k}{k+1}\binom{m+1}{k+1} \frac{x^{k+1}-1}{x-1} = \\
&= [x^j] \sum_{k=0}^{m} \frac{(-1)^k}{k+1}\binom{m+1}{k+1} \sum_{\ell=0}^{k} x^\ell = \\
&= \sum_{k=j}^{m} \frac{(-1)^k}{k+1}\binom{m+1}{k+1}.
\end{aligned} \tag{26}$$

By writing the equality of the two expressions (25) and (26) of $[x^j]\,(P(x))$, we obtain (24). □

4. Explicit expressions for Bernoulli polynomials

In this section, we determine different expressions of $\mu_j(B, r, m)$ for $r \in \left\{1, -1, \frac{1}{2}\right\}$.

Theorem 1.4. (Case $r=\pm 1$**).** *For all integers m, n such that $0\leqslant n\leqslant m$, we have*

$$B_n(x)=\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j}\right)(x+j)^n, \tag{27}$$

$$B_n(x)=\sum_{j=0}^{m}\left(\frac{1}{m+1}\sum_{k=j}^{m}(-1)^j\frac{\binom{m+1}{k-j}}{\binom{m}{k}}\right)(x+j)^n, \tag{28}$$

$$B_n(x)=\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1}\right)(x+j)^n, \tag{29}$$

$$B_n(x)=\frac{x^n}{m+1}+\sum_{j=1}^{m}\left(\sum_{k=j}^{m}\frac{(-1)^{j+1}}{k(k+1)}\binom{k}{j}\right)(x-j)^n. \tag{30}$$

Proof. From the relations (17) and (9) and noticing that $\Delta=\Delta_1$, we deduce that

$$B_n(x)=\Omega_B(x^n)=\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\Delta^k(x^n)=\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j}\right)(x+j)^n.$$

This proves (27) and also gives

$$\mu_j(B,1,m)=\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j}. \tag{31}$$

From Lemma 3.3 and (31), we have

$$\mu_j(B,1,m)=\sum_{k=j}^{m}\frac{(-1)^j}{m+1}\frac{\binom{m+1}{k-j}}{\binom{m}{k}},$$

and we obtain (28). From Lemma 4.3 and (31), we have

$$\mu_j(B,1,m)=\sum_{k=j}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1},$$

and (29) follows. By the relations (18) and (9) we have

$$\begin{aligned}B_n(x)=\Omega_B(x^n)&=\left(1-\sum_{k=1}^{m}\frac{(-1)^k\Delta_{-1}^k}{k(k+1)}\right)(x^n)=x^n-\sum_{k=1}^{m}\sum_{j=0}^{k}\frac{(-1)^j}{k(k+1)}\binom{k}{j}(x-j)^n=\\&=\left(1-\sum_{k=1}^{m}\frac{1}{k(k+1)}\right)x^n-\sum_{j=1}^{m}\left(\sum_{k=j}^{m}\frac{(-1)^j}{k(k+1)}\binom{k}{j}\right)(x-j)^n,\end{aligned}$$

from which the identity (30) follows. $\square$

Theorem 2.4. *(Case $r=\frac{1}{2}$). For all integers m, n such that $0\leqslant n\leqslant m$, we have*

$$B_n(x)=\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\sum_{s=0}^{k}\frac{(-1)^j}{(s+1)2^{k-s}}\binom{k}{j}\right)\left(x+\frac{j}{2}\right)^n, \tag{32}$$

$$B_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \sum_{s=0}^{k} \frac{(-1)^j \binom{k}{j}}{(k+1)\binom{k}{s}} \right) \left(x + \frac{j}{2} \right)^n . \tag{33}$$

Proof. From the relations (19) and (9), we have

$$B_n(x) = \Omega_B(x^n) = \sum_{k=0}^{m} \sum_{s=0}^{k} \frac{(-1)^k}{(s+1)\, 2^{k-s}} \Delta_{\frac{1}{2}}^{k}(x^n) = \sum_{k=0}^{m} \sum_{s=0}^{k} \frac{(-1)^k}{(s+1)\, 2^{k-s}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{\frac{j}{2}}(x^n) =$$

$$= \sum_{k=0}^{m} \sum_{s=0}^{k} \sum_{j=0}^{k} \frac{(-1)^j}{(s+1)\, 2^{k-s}} \binom{k}{j} \left(x + \frac{j}{2} \right)^n .$$

We deduce (32). By the relations (20) and (9), we have

$$B_n(x) = \Omega_B(x^n) = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \frac{(-1)^k}{k+1} \binom{k}{s}^{-1} \right) \Delta_{\frac{1}{2}}^{k}(x^n) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \sum_{j=0}^{k} \frac{(-1)^j \binom{k}{j}}{(k+1)\binom{k}{s}} \left(x + \frac{j}{2} \right)^n .$$

We deduce (33). □

The identity (28) was discovered in 2016 by Komatsu and Pita Ruiz and represents a generalization to Bernoulli polynomials of identity (1) that Munch [17] proved in 1959. Their proof is based on many combinatorial identities extracted from Gould tables [9, 10]. The identity (29) represents a generalization to Bernoulli polynomials of the following identity

$$B_n = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{j=0}^{k-1} j^n ,$$

which is an identity proved by Kronecker [13] in 1883, rediscovered by Bergmann and Gould [3,8] and generalized later by Funkuhara et al. [7] in 2018.

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Заметка о явных формулах для многочленов Бернулли

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Аннотация. При $r \in \left\{1, -1, \frac{1}{2}\right\}$ доказаны несколько явных формул для n-го многочлена Бернулли $B_n(x)$, в котором $B_n(x)$ равно линейной комбинации многочленов x^n, $(x+r)^n, \ldots, (x+rm)^n$, где m — любое фиксированное натуральное число, большее или равное n.

Ключевые слова: многочлен Аппеля, многочлен Бернулли, биномиальные коэффициенты, комбинаторные тождества.