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Convolutional Integro-Differential Equations in Banach Spaces With a Noetherian Operator in the Main Part

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Abstract. An initial-value problem for an integro-differential equation of convolution type with a finite index operator for the higher order derivative in Banach spaces is considered. The equations under consideration model the evolution of the processes with "memory" when the current state of the system is influenced not only by the entire history of observations but also by the factors that have formed it and that remain relevant to the current moment of observation. Solutions are constructed in the class of generalized functions with a left bounded support with the use of the theory of fundamental operator functions of degenerate integro-differential operators in Banach spaces. A fundamental operator function that corresponds to the equation under consideration is constructed. Using this function the generalized solution is restored. The relationship between the generalized solution and the classical solution of the original initial-value problem is studied. Two examples of initial-boundary value problems for the integro-differential equations with partial derivatives are considered.

Keywords: Banach space, generalized function, Jordan set, Noetherian operator, fundamental operator-function.

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Introduction

Let us consider the following initial-value problem

$$Bu^{(N)}(t) = Au(t) + \int_0^t \big(\alpha(t-s)A + \beta(t-s)B\big)u(s)ds + f(t), \tag{1}$$

$$u(0) = u_0,\ u'(0) = u_1,\ \dots\ , u^{(N-1)}(0) = u_{N-1}, \tag{2}$$

where A, B are closed linear operators with compact domains of definition which act from the Banach space E_1 into the Banach space E_2, operator B is Noetherian operator [1,2], $\alpha(t)$, $\beta(t)$ are sufficiently smooth numerical functions, $f(t)$ is a sufficiently smooth function with the values in E_2.

Some initial-boundary value problems of mathematical physics can be reduced to problem (1)–(2), for example, vibration of plates in visco-elastic media [3]. It is known that in the class of functions $C^N(t \geqslant 0; E_1)$ the initial-value problem (1)–(2) can be solved only if initial conditions (2) and function $f(t) are compatible$. Meanwhile, in the case of constructing solutions

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in $K'_+(E_1)$, in a space of distributions with a left bounded support, there are no requirements of such compatibility. Some results of corresponding studies in the case of differential equations, i.e., when $\alpha(t) = \beta(t) \equiv 0$, can be found in [4] and [5] for the space $C^N(t \geqslant 0; E_1)$ and in [6] for $K'_+(E_1)$. Below integro-differential equations of form (1) are studied.

1. Auxiliary data and designations

1.1. Jordan sets of Noetherian operators

Let us assume that the following condition is satisfied for operators A and B:
(A) $D(B) \subset D(A)$, $\overline{D(A)} = \overline{D(B)} = E_1$, $\dim N(B) = n$, $\dim N(B^*) = m$, $n \neq m$, operator B is normally solvable, i.e., $\overline{R(B)} = R(B)$.

The following designations are used: $\{\varphi_i\}_{i=1}^n \in E_1$ is the core basis $N(B)$ of operator B, $\{\phi_j\}_{j=1}^m \in E_2^*$ is the core basis $N(B^*)$ of the conjugate operator B^*, $\{z_j\}_{j=1}^m \in E_2$ and $\{\gamma\}_{i=1}^n \in E_1^*$ are the systems of elements and functionals that are bi-orthogonal to these bases, i.e., $\langle \varphi_i, \gamma_k \rangle = \delta_{ik}$, $i, k = 1, \ldots, n$ and $\langle z_k, \phi_j \rangle = \delta_{kj}$, $k, j = 1, \ldots, m$ (see [2], p. 168, Lemma 4). Let us now construct the projectors with the use of these systems of elements and functionals

$$P = \sum_{i=1}^n P_i = \sum_{i=1}^n \langle \cdot, \gamma_i \rangle \varphi_i : E_1 \to E_1, \quad Q = \sum_{j=1}^m Q_i = \sum_{j=1}^m \langle \cdot, \phi_j \rangle z_j : E_2 \to E_2.$$

It has been proved [7] that there is only one bounded pseudo-inverse operator $B^+ \in \mathcal{L}(E_2, E_1)$ which has the following properties

$$D(B^+) = R(B) \oplus \{z_1, \ldots, z_m\} \equiv E_2, \quad R(B^+) = N(P) \cap D(B),$$

$$BB^+ = I - Q \text{ on } D(B^+), \quad B^+B = I - P \text{ on } D(B).$$

Furthermore, $N(B^+) = \{z_1, \ldots, z_m\}$, the following operator equalities are valid $BB^+B = B$, $B^+BB^+ = B^+$, and operator AB^+ is bounded due to condition **(A)**.

The conjugate operator $B^{+*} \in \mathcal{L}(E_1^*, E_2^*)$ has similar set of properties, i.e.,

$$N(B^{+*}) = \{\gamma_1, \ldots, \gamma_n\}, \quad B^*B^{+*}B^* = B^*, \quad B^{+*}B^*B^{+*} = B^{+*}, \quad B^{+*} = B^{*+},$$

$$D(B^{+*}) = R(B^*) \oplus \{\gamma_1, \ldots, \gamma_n\} \equiv E_1^*, \quad R(B^{+*}) = N(Q^*) \cap D(B^*),$$

$$B^*B^{*+} = I - P^* \text{ on } D(B^{*+}), \quad B^{*+}B^* = I - Q^* \text{ on } D(B^*),$$

here

$$P^* = \sum_{i=1}^n P_i^* = \sum_{i=1}^n \langle \varphi_i, \cdot \rangle \gamma_i : E_1^* \to E_1^*, \quad Q^* = \sum_{j=1}^m Q_i^* = \sum_{j=1}^m \langle z_j, \cdot \rangle \phi_j : E_2^* \to E_2^*.$$

Following [5, 8], let us introduce the systems of adjoined elements and functionals:

$$\varphi_i^{(j)} = (B^+A)^{j-1}\varphi_i^{(1)}, \ i = 1, \ldots, n, \ j \geqslant 2, \quad \varphi_i^{(1)} = \varphi_i,$$

$$\phi_i^{(j)} = (B^{+*}A^*)^{j-1}\phi_i^{(1)}, \ i = 1, \ldots, m, \ j \geqslant 2, \quad \phi_i^{(1)} = \phi_i.$$

The following inclusions $\varphi_i^{(j)} \in N(P)$ and $\phi_i^{(j)} \in N(Q^*)$ are valid for the adjoined elements and functionals. Due to their constructions and properties of operators B^+ and B^{+*} the following equalities are satisfied $\langle\varphi_i^{(j)},\gamma_k\rangle=0,\ i,k=1,\dots,n,\ j\geqslant 2$ and $\langle z_k,\phi_i^{(j)}\rangle=0,\ i,k=1,\dots,m,\ j\geqslant 2$.

Next, let us introduce condition [5,8]

(B) Elements $\varphi_i^{(j)}$ satisfy the system of equations and inequalities

$$B\varphi_i^{(j)}=A\varphi_i^{(j-1)},\ i=1,\dots,n,\ j=1,\dots,p_i,$$

$$B\varphi_i^{(p_i+1)}\neq A\varphi_i^{(p_i)},\ i=1,\dots,n,$$

$$rang\left\|\left\langle A\varphi_i^{(p_i)},\phi_k^{(1)}\right\rangle\right\|_{i=1,\dots,n,\,k=1,\dots,m}=\min(n,m)=l.$$

Condition **(B)** means that the system of elements $\left\{\varphi_i^{(j)},\ i=1,\dots,n,\ j=1,\dots,p_i\right\}$ forms a complete A-Jordan set of operator B [1,4]. It has been shown in [5,8] that bases $\{\varphi_i\}_{i=1}^n$, $\{\phi_j\}_{j=1}^m$ and the corresponding to them systems $\{z_j\}_{j=1}^m$, $\{\gamma_i\}_{i=1}^n$ can be chosen such that the following equalities are satisfied

$$\left\langle A\varphi_i^{(j)},\phi_k\right\rangle=\begin{cases}0, & i=1,\dots,n,\ j=1,\dots,p_i-1,\ k=1,\dots,m,\\ \delta_{ik}, & j=p_i,\ i,k=1,\dots,l,\end{cases}$$

$$\left\langle \varphi_i,A^*\phi_k^{(j)}\right\rangle=\begin{cases}0, & k=1,\dots,m,\ j=1,\dots,p_k-1,\ i=1,\dots,n,\\ \delta_{ik}, & j=p_k,\ i,k=1,\dots,l.\end{cases}$$

Therefore, if condition **(B)** is satisfied, then (without any restriction of the generality) one can state that $z_i=A\varphi_i^{(p_i)}$, $\gamma_k=A^*\phi_k^{(p_k)},\ i,k=1,\dots,l$. In this case, matrices $\left\|\left\langle A\varphi_i^{(p_i)},\phi_k^{(1)}\right\rangle\right\|$ and $\left\|\left\langle \varphi_i^{(1)},A^*\phi_k^{(p_k)}\right\rangle\right\|$, where $i=1,\dots,n,\ k=1,\dots,m$, have the same full rank l. Both matrices are equivalent to a rectangular matrix with the unit main rank minor of order l and with zeros at the rest of the places. Next, let us suppose that all such transformations of the bases are fulfilled. Along with projector Q of space E_2, we introduce another projector

$$\tilde{Q}=\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle\cdot,\phi_i^{(j)}\right\rangle A\varphi_i^{(p_i+1-j)}. \tag{3}$$

Furthermore, if $n>m$, then we assume that $\phi_i^{(1)}=0$ when $i=m+1,\dots,n$, and other functionals $\phi_i^{(j)}\in E_2^*$, $i=m+1,\dots,n$, $j=2,\dots,p_i$ are arbitrary ("free parameters"). Hence, for $n>m$

$$Q\tilde{Q}=Q,\quad Q(I-\tilde{Q})=0\quad\text{and}\quad Q(AB^+)^k(I-\tilde{Q})=0,\ \forall k\in N. \tag{4}$$

Respectively, when $n<m$, we have the following relations $\forall k\in N$

$$\begin{cases}Q_j\tilde{Q}=Q_j, & Q_j(I-\tilde{Q})=0, & Q_j(AB^+)^k(I-\tilde{Q})=0, & j=1,\dots,n,\\ Q_j\tilde{Q}=0, & Q_j(I-\tilde{Q})=Q_j, & Q_j(AB^+)^k(I-\tilde{Q})=Q_j(AB^+)^k, & j=n+1,\dots,m.\end{cases} \tag{5}$$

Hence, the following auxiliary lemma is valid.

Lemma 1. *If $n<m$ then*

$$AB^+\left[I-\tilde{Q}\right]B-\left[I-\tilde{Q}\right]A\equiv 0, \tag{6}$$

$$B^+\left[I-\tilde{Q}\right]B+\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle\cdot,A^*\phi_i^{(j)}\right\rangle\varphi_i^{(p_i+1-j)}=I. \tag{7}$$

Proof. Actually we have

$$AB^+\left[I-\tilde{Q}\right]B-\left[I-\tilde{Q}\right]A = AB^+B - AB^+\tilde{Q}B - A + \tilde{Q}A =$$
$$= A(I-P) - AB^+\tilde{Q}B - A + \tilde{Q}A = -AP - AB^+\tilde{Q}B + \tilde{Q}A \equiv 0.$$

Due to the choice of the bases we have

$$AP + AB^+\tilde{Q}B = \sum_{i=1}^{n}\left\langle \cdot, A^*\phi_i^{(p_i)}\right\rangle A\varphi_i^{(1)} + \sum_{i=1}^{n}\sum_{j=2}^{p_i}\left\langle \cdot, B^*\phi_i^{(j)}\right\rangle AB^+A\varphi_i^{(p_i+1-j)} =$$
$$= \sum_{i=1}^{n}\left\langle \cdot, A^*\phi_i^{(p_i)}\right\rangle A\varphi_i^{(1)} + \sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle AB^+A\varphi_i^{(p_i+1-j)} = \tilde{Q}A.$$

Then equality (6) is proved.

Another relation is proved similarly:

$$B^+\left[I-\tilde{Q}\right]B + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)} =$$
$$= I - P - \sum_{i=1}^{n}\sum_{j=2}^{p_i}\left\langle \cdot, B^*\phi_i^{(j)}\right\rangle B^+A\varphi_i^{(p_i+1-j)} + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)} =$$
$$= I - \sum_{i=1}^{n}\left\langle \cdot, A^*\phi_i^{(p_i)}\right\rangle \varphi_i^{(1)} - \sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)} + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)} = I.$$

□

1.2. Generalized functions in Banach spaces

Let E be a Banach space and E^* be the conjugate Banach space. A set of finite infinitely differentiable functions $s(t)$ with the values in E^* will be called the main space $K(E^*)$. Set $K(E^*)$ is provided with some topology by introducing the concept of convergence.

Definition. *Sequence $s_n(t) \in K(E^*)$ converges to zero in $K(E^*)$ when*
a) $\exists R > 0$ such that $\forall n \in N$ inclusion $supp s_n(t) \subset [-R; R] is valid$;
b) $\forall k \in N$ the sequence of functions $\|s_n^{(k)}(t)\| \to 0$ for $n \to \infty$ uniformly in $[-R; R]$.

Let any linear continuous functional in $K(E^*)$ be called the generalized function. The set of generalized functions $K'(E)$ is a linear space, and it is provided with some topology by introducing weak ("pointwise") convergence. The concepts of support, derivative, linear replacement of the variables in space $K'(E)$ are introduced as in the classical Sobolev-Schwarz theory [9,10]. The Sobolev-Schwarz space distributions is traditionally denoted by $\mathcal{D}'(R^1)$ [9,10]. The space of main functions is denoted by $\mathcal{D}(R^1)$. The space of generalized functions (distributions) with the left bounded support will be denoted by $K'_+(E)$, similarly to $\mathcal{D}'_+(R^1)$ (see [9,10]).

Assume that E_1, E_2 are Banach spaces, $\mathcal{K}(t) \in \mathcal{L}(E_1, E_2)$ are strongly continuous operator function of bounded operators, $g(t) \in \mathcal{D}'(R^1)$. Hence the expression of the form $\mathcal{K}(t)g(t)$ will be called the generalized operator function. The operation of convolution and the fundamental operator function are the key concepts for further consideration.

Definition. *If $v(t) \in K'_+(E_1)$ then the generalized function of class $K'_+(E_2)$ which acts according to the rule*

$$(\mathcal{K}(t)g(t) * v(t), s(t)) = \big(g(t), \big(v(y), \mathcal{K}^*(t)s(t+y)\big)\big), \quad \forall s(t) \in K(E_2),$$

under the assumption that $\mathcal{K}^*(t) \in \mathcal{L}(E_2^*, E_1^*)$ *exists almost everywhere in* R^1, *is called the convolution of the generalized operator function* $\mathcal{K}(t)g(t)$ *and the distribution* $v(t) \in K'_+(E_1)$.

Due to restrictions imposed on the supports, the operation of convolution introduced above always exists. Furthermore, it has the property of transitivity. In particular, the equality

$$A\delta^{(i)}(t) * \mathcal{K}(t)g(t) * v(t) = (A\mathcal{K}(t)g(t))^{(i)} * v(t),$$

is valid, where $A \in \mathcal{L}(E_2, E_3)$, $\delta(t)$ is the Dirac delta-function, $R(\mathcal{K}(t)) \subset D(A)\, \forall t \in R^1$. This equality is satisfied by definition for the closed operator A.

For $t < 0$ we set $\tilde{u}(t) = u(t)\theta(t)$, where $\theta(t)$ is the Heaviside function. Then solution of initial-value problem (1)–(2) $u(t) \in C^N(t \geqslant 0, E_1)$ satisfies the convolution equation

$$\begin{gathered}\big(B\delta^{(N)}(t) - A\delta(t) - \big(\alpha(t)A + \beta(t)B\big)\theta(t)\big) * \tilde{u}(t) = \\ = f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t).\end{gathered} \tag{8}$$

Then relation (8) is initial-value problem (1)–(2) written in terms of generalized functions.

Definition. *The fundamental operator function for the integro-differential operator*

$$\mathcal{L}_N(\delta(t)) = B\delta^{(N)}(t) - A\delta(t) - \big(\alpha(t)A + \beta(t)B\big)\theta(t) =$$

$$= \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)B - \big(\delta(t) + \alpha(t)\theta(t)\big)A$$

defined in class $K'_+(E_2)$ *is a generalized operator function* $\mathcal{E}_N(t)$ *which satisfies the following relations*

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) * v(t) = v(t), \quad \forall v(t) \in K'_+(E_2), \tag{9}$$

$$\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) * u(t) = u(t), \quad \forall u(t) \in K'_+(E_1). \tag{10}$$

Relation (9) implies that the following function of class $K'_+(E_1)$

$$\tilde{\tilde{u}}(t) = \mathcal{E}_N(t) * \Big(f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_0\delta^{(N-1)}(t)\Big) \tag{11}$$

is a solution of equation (8). Accordingly, equality (10) guarantees uniqueness of solution (11) in class $K'_+(E_1)$. Indeed, if function $h(t) \in K'_+(E_1)$ is a solution of equation (8) then due to (10) we have

$$h(t) = \mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) * h(t) = \mathcal{E}_N(t) * (\mathcal{L}_N(\delta(t)) * h(t)) =$$

$$= \mathcal{E}_N(t) * \Big(f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t)\Big) = \tilde{\tilde{u}}(t).$$

Therefore, fundamental operator function allows one to solve the problem of existence and uniqueness of the solution of original initial-value problem (1)-(2), firstly, in class $K'_+(E_1)$. Then using analysis of representation (11) for the generalized solution, we obtain theorems on solvability of initial-value problem (1)–(2) in the class of functions $C^N(t \geqslant 0, E_1)$, i.e., we obtain classical solutions and conditions of their existence in the form of simple corollaries.

1.3. Auxiliary convolution-operator equalities

Let us introduce the following designations:

$\Lambda(t)$ is the resolvent of core $(-\alpha(t)\theta(t))$;

$\mathcal{R}(t)$ is the resolvent of core $k(t)\theta(t) = \dfrac{t^{N-1}}{(N-1)!}\theta(t) * \beta(t)\theta(t)$;

$$\mathcal{U}_N(AB^+t) = \sum_{k=1}^{\infty} \frac{t^{kN-1}}{(kN-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k * \big(\delta(t) + \alpha(t)\theta(t)\big)^{k-1}(AB^+)^{k-1}; \quad (12)$$

$$\mathcal{V}_N(t) =$$
$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \left\langle \cdot, \phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^k * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} \right], \quad (13)$$

here degree k of the generalized function is understood as its k-tuple convolution with itself, zero degree of the generalized function is the Dirac delta-function $\delta(t)$.

The following two statements are valid.

Lemma 2.

$$\mathcal{L}_N(\delta(t)) * B^+\mathcal{U}_N(AB^+t) = (I - Q)\delta(t) - Q\mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+, \quad (14)$$

$$\mathcal{L}_N(\delta(t)) * \mathcal{V}_N(t) = -\tilde{Q}\delta(t). \quad (15)$$

Proof. Since

$$\mathcal{L}_N(\delta(t)) * B^+\mathcal{U}_N(AB^+t) = \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) B * B^+\mathcal{U}_N(AB^+t) -$$
$$-\Big(\delta(t) + \alpha(t)\theta(t)\Big)A * B^+\mathcal{U}_N(AB^+t),$$

we can sequentially find each of the terms. We have

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) B * B^+\mathcal{U}_N(AB^+t) = (I - Q)\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * \mathcal{U}_N(AB^+t),$$

but $\forall k \in N$

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * \frac{t^{kN-1}}{(kN-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k =$$
$$= (\delta(t) - k(t)\theta(t)) * \frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k =$$
$$= \frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^{k-1}.$$

Therefore,

$$\begin{gathered}\big(\delta^{(N)}(t) - \beta(t)\theta(t)\big) B * B^+\mathcal{U}_N(AB^+t) = \\ = (I - Q)\left(I\delta(t) + \mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+\right).\end{gathered} \quad (16)$$

One can similarly obtain

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big)A * B^+\mathcal{U}_N(AB^+t) = \mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+. \quad (17)$$

Subtracting equality (17) from equation (16), we obtain desired equality (14).

Equality (15) is proved in a similar way

$$\mathcal{L}_N(\delta(t)) * \mathcal{V}_N(t) = \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)\, B * \mathcal{V}_N(t) - \Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t),$$

and because $B\varphi_i^{(1)} = 0$ we have

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)\, B * \mathcal{V}_N(t) = \tag{18}$$

$$= \sum_{i=1}^{n}\left[\sum_{k=0}^{p_i-2}\left\{\sum_{j=1}^{p_i-k-1}\left\langle \cdot, \phi_i^{(j)}\right\rangle B\varphi_i^{(p_i-k+1-j)}\right\}\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k+1} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1}\right].$$

Since

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} = \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k},$$

we obtain

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t) = \tilde{Q}\delta(t) +$$

$$+ \sum_{i=1}^{n}\left[\sum_{k=1}^{p_i-1}\left\{\sum_{j=1}^{p_i-k}\left\langle \cdot, \phi_i^{(j)}\right\rangle A\varphi_i^{(p_i-k+1-j)}\right\}\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k}\right]$$

Then we obtain the following relation

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t) = \tilde{Q}\delta(t) + \tag{19}$$

$$+ \sum_{i=1}^{n}\left[\sum_{k=0}^{p_i-2}\left\{\sum_{j=1}^{p_i-k-1}\left\langle \cdot, \phi_i^{(j)}\right\rangle A\varphi_i^{(p_i-k-j)}\right\}\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k+1} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1}\right].$$

Subtracting equation (19) from equation (18) and taking equality $B\varphi_i^{(p_i-k+1-j)} = A\varphi_i^{(p_i-k-j)}$ into account, we obtain relation (15). □

Lemma 3. *When $n < m$ the following equalities are satisfied:*

$$B^+\mathcal{U}_N(AB^+t)\left[I - \tilde{Q}\right] * \mathcal{L}_N(\delta(t)) = B^+\left[I - \tilde{Q}\right] B\delta(t), \tag{20}$$

$$\mathcal{V}_N(t) * \mathcal{L}_N(\delta(t)) = -\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}\delta(t). \tag{21}$$

Proof. Now we can sequentially find

$$B^+\mathcal{U}_N(AB^+t)\left[I - \tilde{Q}\right] * \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) B = B^+\left[I - \tilde{Q}\right] B\delta(t) +$$

$$+ B^+\sum_{k=2}^{\infty}\frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^{k-1} * \big(\delta(t) + \alpha(t)\theta(t)\big)^{k-1}(AB^+)^{k-1}\left[I - \tilde{Q}\right] B =$$

$$= B^+\left[I - \tilde{Q}\right] B\delta(t) + B^+\mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big) AB^+\left[I - \tilde{Q}\right] B,$$

$$B^+\mathcal{U}_N(AB^+t)\left[I - \tilde{Q}\right] * \Big(\delta(t) + \alpha(t)\theta(t)\Big) A = B^+\mathcal{U}_N(AB^+t) * \Big(\delta(t) + \alpha(t)\theta(t)\Big)\left[I - \tilde{Q}\right] A.$$

Subtracting the second equality from the first one and taking into account (6) from Lemma 1, we obtain (20).

The second equality of this Lemma is proved similarly:

$$\mathcal{V}_N(t) * \mathcal{L}_N(\delta(t)) =$$

$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=2}^{p_i-k} \left\langle \cdot, B^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k+1} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k+1} \right] -$$

$$- \sum_{i=1}^{n} \left[\sum_{k=1}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k} \right] -$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i+1-j)} \delta(t) =$$

$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=1}^{p_i-k-1} \left\langle \cdot, B^*\phi_i^{(j+1)} - A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k-j)} \right\} \cdot \right.$$

$$\left. \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k+1} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k+1} \right] - \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i+1-j)} \delta(t).$$

Because $B^*\phi_i^{(j+1)} = A^*\phi_i^{(j)}$, we obtain equality (21). □

2. Theorems on fundamental operator functions and their applications

Theorem 1. *Assume that conditions* **(A)** *and* **(B)**, $n > m$ *are satisfied. Hence the integro-differential operator* $\mathcal{L}_N(\delta(t))$ *has the fundamental operator function*

$$\mathcal{E}_N(t) = B^+ \mathcal{U}_N(AB^+t) \left[I - \tilde{Q}\right] - \mathcal{V}_N(t), \tag{22}$$

in class $K'_+(E_2)$ *(components* $\tilde{Q}, \mathcal{U}_N(AB^+t), \mathcal{V}_N(t)$ *are defined in (3), (12), (13)).*

Proof. Let us demonstrate the validity of identity (9) from the definition of the fundamental operator function. Taking into account relations (14), (15) and (4), we have

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) = \left(I - Q\right) \left[I - \tilde{Q}\right] \delta(t) - Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] + \tilde{Q}\delta(t) =$$

$$= I\delta(t) - Q\left[I - \tilde{Q}\right]\delta(t) - Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] = I\delta(t),$$

because

$$Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] =$$

$$= \sum_{k=1}^{\infty} \frac{t^{kN-1}}{(kN-1)!} \theta(t) * \left(\delta(t) + \mathcal{R}(t)\theta(t)\right)^k * \left(\delta(t) + \alpha(t)\theta(t)\right)^k Q(AB^+)^k \left[I - \tilde{Q}\right] = 0.$$

Therefore, it has been proved that function (11) is a solution of convolution equation (8). The existence of free functionals in the projector $\tilde{Q}$ (see (3)) shows that solution (11) is represented by a multi-parametric function, therefore, the solution is not unique. So, in the given case, there is no sense to verify identity (10) from the definition of the fundamental operator function, the identity is not satisfied. □

Theorem 2. *Assume that conditions* **(A)** *and* **(B)**, $n < m$ *are satisfied. Hence operator function (22) is fundamental operator function for the integro-differential operator* $\mathcal{L}_N(\delta(t))$ *in the subclass of generalized functions from* $K'_+(E_2)$ *such that*

$$Q_J \mathcal{U}_N(AB^+t) * v(t) \equiv 0, \quad j = n+1, \dots, m. \tag{23}$$

Proof. Repeating the process of reasoning in the proof of Theorem 1 and using relations (5), we obtain

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) = I\delta(t) - \sum_{j=n+1}^{m} Q_j \delta(t) * \Big(I\delta(t) + \mathcal{U}_N(AB^+t) * \Big(\delta(t) + \alpha(t)\theta(t)\Big)AB^+\Big) =$$

$$= I\delta(t) - (\delta(t) - k(t)\theta(t)) * \frac{d^N}{dt^N}\left(\sum_{j=n+1}^{m} Q_j \mathcal{U}_N(AB^+t)\right).$$

According to condition (23), we obtain that relation (9) is satisfied.

On the other hand, taking into account relations (20) and (21), we have

$$\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) = B^+\left[I - \tilde{Q}\right] B\delta(t) + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}\delta(t).$$

Now, using equality (7), we obtain $\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) = I\delta(t)$, i.e., from the definition of the fundamental operator function equality (10) is satisfied under given conditions. □

If $n = m$ in condition (**A**), i.e., operator B is a Fredholm operator [1, 2] then Theorem 1 assumes the following form.

Theorem 3. *If* $n = m$ *in conditions* **(A)** *and* **(B)** *then integro-differential operator* $\mathcal{L}_N(\delta(t))$ *has the fundamental operator function of the form*

$$\mathcal{E}_N(t) = \Gamma \mathcal{U}_N(A\Gamma t)\left[I - \tilde{Q}\right] - \mathcal{V}_N(t), \tag{24}$$

in class $K'_+(E_2)$, *where* $\Gamma = \left(B + \sum\limits_{i=1}^{n}\langle \cdot, \gamma_i\rangle z_i\right)^{-1} \in \mathcal{L}(E_2, E_1)$ *is the Trenogin-Schmidt operator (see in [1, 2]).*

Relation (24) assumes the most compact form when $p_1 = \ldots = p_n = 1$. In this case

$$\mathcal{E}_N(t) = \Gamma \mathcal{U}_N(A\Gamma t)\left[I - \sum_{i=1}^{n}\langle \cdot, \phi_i^{(1)}\rangle A\varphi_i^{(1)}\right] - \sum_{i=1}^{n}\langle \cdot, \phi_i^{(1)}\rangle \varphi_i^{(1)}\Big(\delta(t) + \Lambda(t)\theta(t)\Big).$$

Generalized solution (11) of initial-value problem (1)–(2) turns out to be a regular generalized function which transforms equation (1) into an identity. Conditions wherein a regular function satisfies initial conditions (2) are the conditions of solvability for initial-value problem (1)–(2) in the classical sense. So, we can postulate the following statement.

Theorem 4. *If the lengths of all the A-Jordan chains (see in [1, 2]) are equal to 1 under the conditions of Theorem 3 then initial-value problem (1)–(2) has unique solution (11) in class* $C^N(t \geqslant 0, E_1)$ *if and only if the following conditions are satisfied*

$$\left\langle Au_j + f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \cdots + \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0), \psi_i^{(1)}\right\rangle = 0,$$

$$j = 0, 1, \dots, N-1, \quad i = 1, \dots, n.$$

Let us use Theorem 4 in order to study the following two initial-boundary value problems from the theory of vibrations in visco-elastic media [3].

Example 1. Consider the following equation

$$(\lambda - \Delta)\, u_t - (\mu - \Delta)\, u - \int_0^t g(t-\tau)\,(\gamma - \Delta)\, u(\tau, \bar{x}) d\tau = f(t, \bar{x}), \tag{25}$$

where $g(t),\ f(t,\bar{x})$ are given functions, $u = u(t,\bar{x})$ is the required function, $\bar{x} \in \Omega \subset R^m$ is the bounded domain with an infinitely smooth boundary $\partial\Omega$, Δ is the Laplace operator, $u = u(t,\bar{x})$ is defined on a cylinder $R_+ \times \Omega$ and satisfies the following initial and boundary conditions

$$u\Big|_{t=0} = u_0(\bar{x}),\ \ \bar{x} \in \Omega;\ \ u\Big|_{\bar{x}\in\partial\Omega} = 0,\ \ t \geqslant 0. \tag{26}$$

Cauchy-Dirichlet problem (25)–(26) is reduced to initial-value problem (1)–(2) when the Banach spaces E_1 and E_2 are Sobolev spaces, i.e.,

$$E_1 \equiv \left\{ v(\bar{x}) \in W_2^2(\Omega) :\ v\Big|_{\partial\Omega} = 0 \right\},\ \ E_2 \equiv W_2(\Omega), \tag{27}$$

and operators A and B are defined as follows

$$B = \lambda - \Delta, \quad A = \mu - \Delta, \quad \lambda \in \sigma(\Delta), \quad \mu \neq \lambda. \tag{28}$$

In this case,

$$\alpha(t) = \frac{\gamma - \mu}{\lambda - \mu} g(t), \quad \beta(t) = \frac{\lambda - \gamma}{\lambda - \mu} g(t),$$

operator B is a Fredholm operator, the dimension n of the core of operator B is equal to the multiplicity of the eigenvalue $\lambda \in \sigma(\Delta)$ of the Laplace operator; lengths of all $A-$Jordan chains of the elements of core $\varphi_i \in B,\ i = 1, \ldots, n$ are equal to 1, here $\lambda\varphi_i = \Delta\varphi_i,\ \varphi_i\Big|_{\bar{x}\in\partial\Omega} = 0$. All conditions of Theorem 4 are satisfied. Hence, the following statement is true.

Theorem 5. *Assume that for Cauchy-Dirichlet problem (25)–(26) spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28) and $\lambda \in \sigma(\Delta)$. Cauchy-Dirichlet problem (25)–(26) is unequivocally solvable in class $C^1(t \geqslant 0, E_1)$ if and only if initial-boundary value conditions (26) and function $f(t,\bar{x})$ satisfy the following relations*

$$\Big\langle (\mu - \lambda) u_0(\bar{x}) + f(0,\bar{x}), \varphi_i(\bar{x}) \Big\rangle = 0, \quad i = 1, \ldots, n.$$

Example 2. Consider the following equation

$$(\lambda - \Delta)\, u_{tt} - (\mu - \Delta)\, u - \int_0^t g(t-\tau)\,(\gamma - \Delta)\, u(\tau, \bar{x}) d\tau = f(t, \bar{x}), \tag{29}$$

with initial and boundary conditions

$$u\Big|_{t=0} = u_0(\bar{x}), \quad u_t\Big|_{t=0} = u_1(\bar{x}), \quad \bar{x} \in \Omega; \quad u\Big|_{\bar{x}\in\partial\Omega} = 0,\ \ t \geqslant 0. \tag{30}$$

Spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28). Then the following statement is valid.

Theorem 6. *Assume that for Cauchy-Dirichlet problem (29)–(30) spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28) and $\lambda \in \sigma(\Delta)$. Hence Cauchy-Dirichlet problem (29)–(30) is unequivocally solvable in clsss $C^2(t \geqslant 0, E_1)$ if and only if initial-boundary value conditions (30) and function $f(t, \bar{x})$ satisfy the following relations*

$$\Big\langle (\mu-\lambda)^2 u_1(\bar{x}) + (\mu-\lambda) f_t'(0,\bar{x}) + (\gamma-\mu) g(0) f(0,\bar{x}), \varphi_i(\bar{x}) \Big\rangle = 0,$$

$$\Big\langle (\mu-\lambda) u_0(\bar{x}) + f(0,\bar{x}), \varphi_i(\bar{x}) \Big\rangle = 0, \quad i = 1, \dots, n.$$

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Сверточные интегро-дифференциальные уравнения в банаховых пространствах с нетеровым оператором в главной части

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Аннотация. В работе исследуется задача Коши для интегро-дифференциального уравнения сверточного типа с оператором конечного индекса при старшей производной в банаховых пространствах. Рассматриваемые уравнения моделируют эволюцию процессов с "памятью", когда на текущее состояние системы влияет не только вся история наблюдений, но и формировавшие ее факторы, остающиеся актуальными на текущий момент наблюдений. Решения строятся в классе обобщенных функций с ограниченным слева носителем методами теории фундаментальных оператор-функций вырожденных интегро-дифференциальных операторов в банаховых пространствах. Построена фундаментальная оператор-функция, соответствующая рассматриваемому уравнению, с помощью которой восстановлено обобщенное решение, исследована связь между обобщенным и классическим решениями исходной задачи Коши. Абстрактные результаты проиллюстрированы на примерах начально-краевых задач для интегро-дифференциальных уравнений в частных производных прикладного характера.

Ключевые слова: банахово пространство, обобщенная функция, жорданов набор, нетеров оператор, фундаментальная оператор-функция.