Journal of Siberian Federal University. Mathematics & Physics 2022, 15(2), 199–213

DOI: 10.17516/1997-1397-2022-15-2-199-213
УДК 517.946

On the Cauchy Problem for the Biharmonic Equation

Dilshod S. Shodiev*
Samarkand State University
Samarkand, Uzbekistan

Received 10.11.2021, received in revised form 30.12.2021, accepted 10.02.2021

Abstract. The work is devoted to the study of continuation and stability estimation of the solution of the Cauchy problem for the biharmonic equation in the domain G from its known values on the smooth part of the boundary ∂G. The problem under consideration belongs to the problems of mathematical physics in which there is no continuous dependence of solutions on the initial data. In this work, using the Carleman function, not only the biharmonic function itself, but also its derivatives are restored from the Cauchy data on a part of the boundary of the region. The stability estimates for the solution of the Cauchy problem in the classical sense are obtained.

Keywords: biharmonic equations, Cauchy problem, ill-posed problems, Carleman function, regularized solutions, regularization, continuation formulas.

Citation: D.S. Shodiev, On the Cauchy Problem for the Biharmonic Equation, J. Sib. Fed. Univ. Math. Phys., 2022, 15(2), 199–213. DOI: 10.17516/1997-1397-2022-15-2-199-213.

Introducnion

Let $x = (x_1, x_2), y = (y_1, y_2) \in R^2$ and G is a bounded simply connected domain in R^2 with boundary ∂G, consisting of compact part $T = \{y_1 \in R : \ a_1 \leqslant y_1 \leqslant b_1\}$ and a smooth arc of the curve $S : y_2 = h(y_1)$ lying in the half-plane $S : y_2 = h(y_1)$. $\bar{G} = G \cup \partial G$, $\partial G = S \cup T$.

In the domain G, consider the equation

$$\Delta^2 U(y) = 0, \ \ y \in G, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial y_1^2} + \dfrac{\partial^2}{\partial y_2^2}$ Laplace operator.

Problem definition. It is required to find the biharmonic function $U(y) = U(y_1, y_2) \in C^4(G) \cap C^3(\bar{G})$, for which the values on the part S of the boundary ∂G are known, i.e.

$$\begin{gathered} U(y_1, y_2)|_S = f_1(y), \quad \Delta U(y_1, y_2)|_S = f_2(y), \\ \left.\frac{\partial U(y_1, y_2)}{\partial n}\right|_S = f_3(y), \quad \left.\frac{\partial(\Delta U(y_1, y_2))}{\partial n}\right|_S = f_4(y), \end{gathered} \tag{2}$$

here $f_j(y) \in C^{4-j}(S), j = 1, 2, 3, 4$ are given functions, and $\dfrac{\partial}{\partial n}$ — operator of differentiation along the outward normal to ∂G.

The considered problem (1)–(2) refers to ill-posed problems of mathematical physics. The true nature of such problems was clarified for the first time in the work of A. N. Tikhonov [4],

*dilshod.shodiev.76@mail.ru

and he pointed out the practical importance of unstable problems, and also showed that if the class of possible solutions is reduced to a compact set, then the stability of the solution follows from the existence and uniqueness.

Formulas that make it possible to find a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman-type formulas. In [2] Carleman established a formula giving a solution to the Cauchy–Riemann equations in a domain of a special form. Developing his idea, G. M. Goluzin and V. I. Krylov [3] derived a formula for determining the values of analytic functions from data known only on the border on the border section, already for arbitrary domains. They found a formula for restoring a solution from its values on the boundary set of positive Lebesgue measure, and also proposed a new version of the extension formula. The monograph by L. A. Aizenberg [1] is devoted to one-dimensional and multidimensional generalizations of the Carleman formula. A formula of the Carleman type, which uses the fundamental solution of a differential equation with special properties (the Carleman function), was obtained by M. M. Lavrent'ev [7, 8]. In these works, the definition of the Carleman function is given for the case when the Cauchy data are given approximately, and the a scheme of regularization of the Cauchy problem for the Laplace equation is also proposed. Using this method, Sh.Ya.Yarmukhamedov [9, 10] constructed Carleman functions for a wide class of elliptic operators defined in spatial domains of a special form, when part of the boundary of the domain is a hypersurface or a conical surface. It should de noted that the Carleman function proposed by Sh.Yarmukhamedov was also studied by M. Ikehata [11].

The Carleman matrix for the Cauchy–Riemann equation in the case when S is an arbitrary set of positive measure was constructed in [13]. In [14] in classical domains, Carleman's formulas are given that restore the values of a function inside a domain from its values given on a set of positive measure on the skeleton.

The Cauchy problem for linear elliptic differential operators has numerous applications in physics, electrodynamics, fluid mechanics (see [8, 12, 15]). It is known that if the Carleman function is constructed, then using Green's formula one can write the regularized solution explicitly. This implies that the efficiency of constructing the Carleman function is equivalent to constructing a regularized solution to the Cauchy problem.

In [16], a method is proposed for the regularization of the solution of the Cauchy problem for the Laplace equation by introducing a biharmonic operator with a small parameter, and it is shown that if a solution to the original problem exists, then the difference between the spectral expansions of the solutions of the original and regularized equations tends to zero as the parameter tends regularization to zero in the space of square-summable functions. In recent years, many numerical methods have been presented solving the Cauchy problem for elliptic equations. In the paper [18] L. Marin investigated the iterative method of fundamental solutions algorithms together with the Tikhonov regularization method.

An estimate for the conditional stability of a boundary value problem for a fourth-order elliptic type equation in rectangular domains was obtained in [19].

In [20], using the Carleman function, not only the harmonic function itself, but also its derivatives for the Laplace equation are reconstructed from the Cauchy data on a part of the boundary of the domain.

Note that when solving applied problems, one should find the approximate values of the solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_i},\ x \in G,\ i = 1, 2.$

In this paper, we construct a family of functions $U(x,\sigma,f_{k\delta}) = U_{\sigma\delta}(x)$ and $\dfrac{\partial U(x,\sigma,f_{k\delta})}{\partial x_i} = \dfrac{\partial U_{\sigma\delta}(x)}{\partial x_i}$, $k=1,2,3,4;\ i=1,2$ depending on a parameter σ and prove that with a special choice of parameter $\sigma=\sigma(\delta)$ the family $U_{\sigma\delta}(x)$ and $\dfrac{\partial U_{\sigma\delta}(x)}{\partial x_i}$ at $\delta\to 0$ converges at each point $x\in G$ to the solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_i}$, respectively. The family of functions $U(x,\sigma,f_{k\delta})$ and $\dfrac{\partial U(x,\sigma,f_{k\delta})}{\partial x_i}$, $i=1,2$ with indicated properties is said to be a regularized solution by M. M. Lavrent'ev [7]. If, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given deviation in the uniform metric are given, then an explicit regularization formula is proposed. In this case, it is assumed that the solution is bounded on the part T of the boundary.

The proof of these results is based on the construction in an explicit form of the fundamental solution of the biharmonic equation depending on a positive parameter, disappearing along with its derivatives as the parameter tends to infinity on T when the pole of the fundamental solution lies in the half-plane $y_2>0$.

1. Construction of the Carleman function

Let us define the function $\Phi_\sigma(x,y)$ (from [10]) as follows

$$-2\pi e^{\sigma x_2^2}\Phi_\sigma(x,y) = \int_0^\infty Im\left[\frac{e^{\sigma w^2}}{w-x_2}\right]\frac{udu}{\sqrt{u^2+\alpha^2}}. \tag{3}$$

Separating the imaginary part of the function $\Phi_\sigma(x,y)$, we have

$$\begin{aligned}\Phi_\sigma(x,y) = \frac{1}{2\pi}e^{-\sigma(\alpha^2+x_2^2-y_2^2)}\Bigg[\int_0^\infty \frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2} - \\ -\int_0^\infty \frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\Bigg],\end{aligned} \tag{4}$$

where $y'=(y_1,0)$, $x'=(x_1,0)$, $r=|y-x|$, $\alpha=|y'-x'|$, $\alpha>0$, $\sigma>0$, $w=i\sqrt{u^2+\alpha^2}+y_2$, $u\geqslant 0$.

It the paper [10], one has proved that the function $\Phi_\sigma(x,y)$ defined by the equalities (3) with $\sigma>0$ is presentable in the from

$$\Phi_\sigma(x,y) = F(r) + G_\sigma(x,y), \tag{5}$$

where $F(r)=\dfrac{1}{2\pi}\ln\dfrac{1}{r}$, $G_\sigma(x,y)$ is harmonic function with respect to y in R^2, including $y=x$. It follows that the function $\Phi_\sigma(x,y)$ for any $\sigma>0$ in y is a fundamental solution of the Laplace equation. The fundamental solution $\Phi_\sigma(x,y)$ with the indicated property is said to be the Carleman function for the half-space [7].

Therefore, for the function $U(y)=U(y_1,y_2)\in C^4(G)\cap C^3(\bar G)$ and any $x\in G$ the following integral Green formula holds true [17]:

$$U(x)=\int_{\partial G}\left[U(y)\frac{\partial\left(\Delta L(x,y)\right)}{\partial n}-\Delta L(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+ \\ +\int_{\partial G}\left[\Delta U(y)\frac{\partial L(x,y)}{\partial n}-L(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y,\ x\in G, \tag{6}$$

where $L(x,y)=r^2\ln\dfrac{1}{r}$ is the fundamental solution to the equation (1).

Since $\Phi_\sigma(x,y)$ is represented in the form (5), then in the integral representation (6) $L(x,y)$ replacing the function $L_\sigma(x,y)=r^2\Phi_\sigma(x,y)$, we have

$$U(x)=\int_{\partial G}\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+ \\ +\int_{\partial G}\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y,\ \ x\in G. \tag{7}$$

2. The formula of continuation and regularization by M. M. Lavrent'ev

We denote

$$U_\sigma(x)=\int_S\left[f_1(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-f_3(y)\Delta L_\sigma(x,y)\right]dS_y+ \\ +\int_S\left[f_2(y)\frac{\partial L_\sigma(x,y)}{\partial n}-f_4(y)L_\sigma(x,y)\right]dS_y,\ \ x\in G. \tag{8}$$

The main result of this paper is contained in the following theorem.

Theorem 1. *Let the function $U(y)=U(y_1,y_2)\in C^4(G)\cap C^3(\bar G)$ on the part S of boundary ∂G satisfy the conditions* (2), *and on the part T of boundary ∂G the inequality be fulfilled*

$$|U(y)|+\left|\frac{\partial U(y)}{\partial n}\right|+|\Delta U(y)|+\left|\frac{\partial\Delta U(y)}{\partial n}\right|\leqslant M,\quad y\in T,\quad M>0. \tag{9}$$

Then, for any $x\in G$ and $\sigma>0$, the estimates hold true

$$|U(x)-U_\sigma(x)|\leqslant\varphi(\sigma,x_2)Me^{-\sigma x_2^2}, \tag{10}$$

$$\left|\frac{\partial U(x)}{\partial x_i}-\frac{\partial U_\sigma(x)}{\partial x_i}\right|\leqslant\varphi_i(\sigma,x_2)Me^{-\sigma x_2^2},\ i=1,2, \tag{11}$$

where

$$\varphi(\sigma,x_2)=\frac{23\sqrt{\sigma\pi}}{4\sigma}+\left(\frac{3\sqrt{\sigma\pi}}{4\sigma}+20\sqrt{\sigma\pi}+8\sqrt{\sigma\pi}\sigma\right)x_2+ \\ +\left(2\sqrt{\sigma\pi}+4\sqrt{\sigma\pi}\sigma\right)x_2^2+\frac{9\sqrt{\sigma\pi}}{2}x_2^3+\frac{9\sqrt{\sigma\pi}}{\sigma x_2}, \tag{12}$$

$$\varphi_1(\sigma,x_2)=10+\frac{1}{\sigma}+\frac{13\sqrt{\pi}}{\sqrt{\sigma}}+\frac{165\sqrt{\pi\sigma}}{2}+\frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}+\frac{4\sqrt{\pi}\sigma^3}{\sqrt{\sigma}}+ \\ +\left(44\sigma+\frac{\sqrt{\pi}}{4\sqrt{\sigma}}+3+\frac{2\sqrt{\pi}\sigma}{\sqrt{\sigma}}\right)x_2+\left(\frac{17}{2}+\frac{9\sqrt{\pi}\sigma}{2\sqrt{\sigma}}+\frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}+4\sigma\right)x_2^2+ \\ +9\sigma x_2^3+\left(\frac{66\sqrt{\pi}}{\sqrt{\sigma}}+\frac{4\sqrt{\pi}\sigma}{\sqrt{\sigma}}+\frac{1}{2\sigma}+8\right)\frac{1}{x_2}+\frac{20\sqrt{\pi}}{\sqrt{\sigma}x_2^2}+16\sigma, \tag{13}$$

$$\begin{aligned}\varphi_2(\sigma,x_2)=\frac{21\sqrt{\pi}}{2\sqrt{\sigma}}+\frac{78\sqrt{\pi}\sigma}{\sqrt{\sigma}}+\left(\sqrt{\sigma\pi}+\frac{1}{2}+\frac{3\sqrt{\pi}}{4\sqrt{\sigma}}+\frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}\right)x_2+\\+\left(29\sqrt{\sigma\pi}+58\sqrt{\sigma\pi}\sigma\right)x_2^2+\frac{\sqrt{\sigma\pi}}{2}x_2^3+10\sqrt{\sigma\pi}\sigma x_2^4+\frac{60\sqrt{\pi}}{\sqrt{\sigma}x_2^2}.\end{aligned}\tag{14}$$

Proof. Let us prove the inequality (10). Denote by $I_\sigma(x)$ the difference

$$I_\sigma(x)=U(x)-U_\sigma(x).$$

Further, from (7) and (8), we have

$$\begin{aligned}I_\sigma(x)=\int_T\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+\\+\int_T\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y,\ \ x\in G.\end{aligned}\tag{15}$$

From this and the inequality (9) we obtain

$$\begin{aligned}|I_\sigma(x)|=\left|\int_T\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+\right.\\\left.+\int_T\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y\right|\leqslant MN_\sigma(x),\end{aligned}$$

where

$$N_\sigma(x)=\int_T\left[\left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right|+|\Delta L_\sigma(x,y)|+\left|\frac{\partial L_\sigma(x,y)}{\partial n}\right|+|L_\sigma(x,y)|\right]dS_y=J_1+J_2+J_3+J_4.$$

To show that the estimate (10) is valid, we prove the following

$$N_\sigma(x)\leqslant\varphi(\sigma,x_2)e^{-\sigma x_2^2},\ \ \sigma>0.\tag{16}$$

According to (4), we have

$$\begin{aligned}L_\sigma(x,y)=r^2\Phi_\sigma(x,y)=r^2\left\{\frac{1}{2\pi}e^{-\sigma(\alpha^2+x_2^2-y_2^2)}\left[\int_0^\infty\frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2}-\right.\right.\\\left.\left.-\int_0^\infty\frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\right]\right\}.\end{aligned}$$

Hence, setting $y_2=0$ we get

$$L_\sigma(x,y)=\left((y_1-x_1)^2+x_2^2\right)\left\{\frac{1}{2\pi}e^{-\sigma x_2^2}\int_0^\infty\frac{e^{-\sigma(u^2+\alpha^2)}udu}{u^2+(y_1-x_1)^2+x_2^2}\right\}.$$

Now we estimate the following integral

$$J_1=\int_T|L_\sigma(x,y)|\,dS_y\leqslant\int_{a_1}^{b_1}\left\{\frac{(y_1-x_1)^2+x_2^2}{2\pi}e^{-\sigma x_2^2}\int_0^\infty\frac{|u|\,e^{-\sigma(u^2+(y_1-x_1)^2)}du}{u^2+(y_1-x_1)^2+x_2^2}\right\}dy_1\leqslant\frac{\sqrt{\pi}}{2\sqrt{\sigma}}e^{-\sigma x_2^2}.$$

Here, in the estimation, we used the inequality

$$\frac{(y_1-x_1)^2+x_2^2}{u^2+(y_1-x_1)^2+x_2^2}<1$$

and introduced polar coordinate systems. Considering

$$\frac{\partial L_\sigma(x,y)}{\partial n} = \frac{\partial L_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2}\sin\gamma,$$

get

$$\frac{\partial\left(L_\sigma(x,y)\right)}{\partial y_2} = \frac{\partial}{\partial y_2}\left[r^2\Phi_\sigma(x,y)\right] = 2(y_2-x_2)\Phi_\sigma(x,y) + r^2\frac{\partial\Phi_\sigma(x,y)}{\partial y_2},$$

here $\cos\gamma,\ \sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G.

Further, setting $y_2 = 0$, we estimate the following integral

$$J_2 = \int_T \left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y \leqslant \int_{a_1}^{b_1}\left\{\frac{x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(y^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+\right.$$
$$+\frac{x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\left|(y_1-x_1)^2+x_2^2\right| e^{-\sigma(y^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+$$
$$\left.+\frac{\sigma x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\left|(y_1-x_1)^2+x_2^2\right| e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du\right\}dy_1$$
$$\leqslant \left(\frac{3\sqrt{\pi}x_2}{4\sqrt{\sigma}} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} + \frac{\sqrt{\sigma\pi}}{2}x_2^3\right)e^{-\sigma x_2^2}.$$

In what follows, we need the following expressions

$$\Delta L_\sigma(x,y) = \Delta\left(r^2\Phi_\sigma(x,y)\right) = \frac{\partial}{\partial y_1^2}\left[r^2\Phi_\sigma(x,y)\right] + \frac{\partial}{\partial y_2^2}\left[r^2\Phi_\sigma(x,y)\right] =$$
$$= 4\Phi_\sigma(x,y) + 4(y_1-x_1)\frac{\partial\Phi_\sigma(x,y)}{\partial y_1} + 4(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial y_2}.$$

$$\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} = \frac{\partial}{\partial y_2}\left[\Delta\left(r^2\Phi_\sigma(x,y)\right)\right] =$$
$$= 8\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + 4(y_1-x_1)\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_1\partial y_2} + 4(y_2-x_2)\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_2^2},$$

where

$$\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} = \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_1}\cos\gamma + \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_2}\sin\gamma.$$

In these expressions, flat $y_2 = 0$, estimating, we get

$$J_3 = \int_T \left|\Delta L_\sigma(x,y)\right| dS_y \leqslant \int_{a_1}^{b_1}\left\{\frac{2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+\right.$$
$$+\frac{4\sigma(y_1-x_1)^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+$$
$$+\frac{4(y_1-x_1)^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du + \frac{4x_2^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{\left(u^2+(y_1-x_1)^2+x_2^2\right)^2}du+$$
$$\left.+\frac{4\sigma x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du\right\}dy_1 \leqslant \left(\frac{5\sqrt{\pi}}{\sqrt{\sigma}} + 2\sqrt{\sigma\pi}x_2^2\right)e^{-\sigma x_2^2}.$$

$$J_4 = \int_T \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y \leqslant \left[\left(20\sqrt{\sigma\pi} + 8\sqrt{\sigma\pi}\sigma\right) x_2 + 4\sqrt{\sigma\pi}\sigma x_2^2 + 4\sqrt{\sigma\pi} x_2^3 + \frac{9\sqrt{\sigma\pi}}{\sigma x_2} \right] e^{-\sigma x_2^2}.$$

In estimating the integrals, we used the inequality

$$\frac{|u|\,|y_1 - x_1|}{u^2 + (y_1 - x_1)^2 + x_2^2} < 1.$$

Taking into account the obtained estimates, we have

$$\begin{aligned} \int_T \left[\left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| + |\Delta L_\sigma(x,y)| + \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| + |L_\sigma(x,y)| \right] dS_y \leqslant \\ \leqslant \left[\frac{23\sqrt{\sigma\pi}}{4\sigma} + \left(\frac{3\sqrt{\sigma\pi}}{4\sigma} + 20\sqrt{\sigma\pi} + 8\sqrt{\sigma\pi}\sigma \right) x_2 + \right. \\ \left. + \left(2\sqrt{\sigma\pi} + 4\sqrt{\sigma\pi}\sigma\right) x_2^2 + \frac{9\sqrt{\sigma\pi}}{2} x_2^3 + \frac{9\sqrt{\sigma\pi}}{\sigma x_2} \right] e^{-\sigma x_2^2}. \end{aligned} \tag{17}$$

From (17) follows the proof of the inequality (10).

Let us prove the inequality (11). Differentiating the equalities (7) and (8) by x_i, $i = 1, 2$ we get

$$\begin{aligned} \frac{\partial U(x)}{\partial x_i} = \int_{\partial G} \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y + \\ + \int_{\partial G} \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y, \end{aligned}$$

$$\begin{aligned} \frac{\partial U_\sigma(x)}{\partial x_i} = \int_S \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y + \\ + \int_S \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y. \end{aligned}$$

Denote by $I_{i\sigma}(x)$ the difference of the derivatives

$$\begin{aligned} I_{i\sigma}(x) = \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_\sigma(x)}{\partial x_i} = \int_T \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y + \\ + \int_T \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y. \end{aligned}$$

From this and the inequality (9) it follows

$$\begin{aligned} |I_{i\sigma}(x)| = \left| \int_T \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y + \right. \\ \left. + \int_T \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y \right| \leqslant M N_{i\sigma}(x), \end{aligned}$$

where

$$N_{i\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_i} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \right| + \left| \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_i} \right| \right] dS_y.$$

To show that the estimate (11) is valid, we prove the following inequality

$$N_{i\sigma}(x) \leqslant \varphi_i(\sigma, x_2) e^{-\sigma x_2^2}, \ \ \sigma > 0. \tag{18}$$

For $i = 1$, we have

$$N_{1\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_1} \right| + \left| \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_1} \right| \right] dS_y.$$

We denote

$$Z_1 = \frac{\partial L_\sigma(x,y)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[r^2 \Phi_\sigma(x,y) \right] = r^2 \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 2(y_1 - x_1)\Phi_\sigma(x,y), \tag{19}$$

$$Z_2 = \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial y_1} \cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2} \sin\gamma \right],$$

$$\begin{gathered} \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial}{\partial y_2} \left[r^2 \Phi_\sigma\left(x,y\right) \right] \right] = \\ = 2(y_2 - x_2) \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 2(y_1 - x_1) \frac{\partial \Phi_\sigma(x,y)}{\partial y_2} + r^2 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2}. \end{gathered} \tag{20}$$

$$\begin{gathered} Z_3 = \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\Delta \left(r^2 \Phi_\sigma(x,y) \right) \right] = \\ = 4 \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 4 \frac{\partial \Phi_\sigma(x,y)}{\partial y_1} + 4(y_1 - x_1) \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_1} + 4(y_2 - x_2) \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2}. \end{gathered} \tag{21}$$

$$Z_4 = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_1} \cos\gamma + \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_2} \sin\gamma \right],$$

$$\begin{gathered} \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_2} \right] = 8 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2} - 4 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial y_1 \partial y_2} + \\ + 4(y_1 - x_1) \frac{\partial^3 \Phi_\sigma(x,y)}{\partial x_1 \partial y_1 \partial y_2} + 4(y_2 - x_2) \frac{\partial^3 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2^2}. \end{gathered} \tag{22}$$

When $\frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right]$ and $\frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right]$, $\cos\gamma$, $\sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G.

In (19), (20), (21), (22), setting $y_2 = 0$ and estimating the resulting integrals, we have:

$$\begin{gathered} N_{1\sigma}(x) \leqslant \Bigg(10 + \frac{1}{\sigma} + \frac{13\sqrt{\pi}}{\sqrt{\sigma}} + \frac{165\sqrt{\sigma\pi}}{2} + 16\sigma + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^3}{\sqrt{\sigma}} + \\ + \left(44\sigma + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} + 2\sqrt{\sigma\pi} + 3 \right) x_2 + \left(\frac{17}{2} + \frac{9\sqrt{\sigma\pi}}{2} + 4\sqrt{\sigma\pi}\sigma + 4\sigma \right) x_2^2 + \\ + 9\sigma x_2^3 + \left(\frac{66\sqrt{\pi}}{\sqrt{\sigma}} + 4\sqrt{\sigma\pi} + \frac{1}{2\sigma} + 8 \right) \frac{1}{x_2} + \frac{20\sqrt{\pi}}{\sqrt{\sigma} x_2^2} \Bigg) e^{-\sigma x_2^2}. \end{gathered} \tag{23}$$

The inequality (18) is proved for $i = 1$. Now let us prove the inequality (18) for $i = 2$.

Taking into account (15) we have

$$N_{2\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_2} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_2} \right| + \left| \frac{\partial}{\partial x_2} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_2} \right| \right] dS_y.$$

We introduce the following designation

$$\Phi_1 = \frac{\partial L_\sigma(x,y)}{\partial x_2} = \frac{\partial}{\partial x_2} \left[r^2 \Phi_\sigma(x,y) \right] = r^2 \frac{\partial \Phi_\sigma(x,y)}{\partial x_2} - 2(y_2 - x_2)\Phi_\sigma(x,y), \tag{24}$$

$$\Phi_2 = \frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2}\sin\gamma\right],$$

$$\begin{gathered}\frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial y_2}\left[r^2\Phi_\sigma(x,y)\right]\right] = \\ = -2\Phi_\sigma(x,y) + 2(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial x_2} - 2(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + r^2\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2}.\end{gathered} \tag{25}$$

$$\begin{gathered}\Phi_3 = \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_2} = \frac{\partial}{\partial x_2}\left[\Delta\left(r^2\Phi_\sigma(x,y)\right)\right] = 4\frac{\partial\Phi_\sigma(x,y)}{\partial x_2} + \\ 4(y_1-x_1)\frac{\partial\Phi_\sigma(x,y)}{\partial y_1\partial x_2} - 4\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + 4(y_2-x_2)\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2}.\end{gathered} \tag{26}$$

$$\Phi_4 = \frac{\partial}{\partial x_2}\left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_1}\cos\gamma + \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_2}\sin\gamma\right],$$

$$\begin{gathered}\frac{\partial}{\partial x_2}\left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_2}\right] = 8\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2} - 4\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_2^2} + \\ +4(y_1-x_1)\frac{\partial^3\Phi_\sigma(x,y)}{\partial x_2\partial y_1\partial y_2} + 4(y_2-x_2)\frac{\partial^3\Phi_\sigma(x,y)}{\partial x_2\partial y_2^2}.\end{gathered} \tag{27}$$

In (24), (25), (26), (27), setting $y_2 = 0$ and estimating the resulting integrals, we have

$$\begin{gathered}N_{2\sigma}(x) \leqslant \left(\frac{21\sqrt{\pi}}{2\sqrt{\sigma}} + \frac{78\sqrt{\pi}\sigma}{\sigma} + \left(\sqrt{\sigma\pi} + \frac{3\sqrt{\pi}}{4\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + \frac{1}{2}\right)x_2 + \right. \\ \left. + \left(29\sqrt{\sigma\pi} + \frac{58\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}\right)x_2^2 + \frac{\sqrt{\sigma\pi}}{2}x_2^3 + \frac{10\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}x_2^4 + \frac{60\sqrt{\pi}}{\sqrt{\sigma}x_2^2}\right)e^{-\sigma x_2^2}.\end{gathered} \tag{28}$$

The inequality (18) is proved for $i = 2$.

From (23) and (28) follows the proof of the inequality (11). Theorem 1 is proved. □

Corollary 1. *With each $x \in G$, the equality holds true*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \quad \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \quad i = 1,2.$$

Let us denote

$$\bar{G}_\varepsilon = \left\{(x_1,x_2) \in G,\ a > x_2 \geqslant \varepsilon,\ a = \max_T h(x_1),\ 0 < \varepsilon < a\right\}.$$

It is easy to see that the set $\bar{G}_\varepsilon \subset G$ is compact.

Corollary 2. *If $x \in \bar{G}_\varepsilon$, then the family of functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_i}\right\}$ converges uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \quad \frac{\partial U_\sigma(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \quad i = 1,2.$$

It should be noted that the sets $\Pi_\varepsilon = G\backslash\bar{G}_\varepsilon$ present the boundary lever of this problem, as in the theory of singular perturbations, where there is no uniform convergence.

3. An estimate of the stability of the solution to the Cauchy problem

Consider the set

$$E=\left\{U \in C^4(G)\cap C^3(\bar{G}) :\ |U(y)|+\left|\left\{\frac{\partial U(y)}{\partial n}\right\}\right|+|\Delta U(y)|+\left|\left\{\frac{\partial \Delta U(y)}{\partial n}\right\}\right|\leqslant M,\ M>0,\ y\in T\right\}.$$

We put

$$\max_T h(y_1) = a, \quad \max_T \sqrt{1+\left(\frac{dh}{dy_1}\right)^2} = b.$$

Theorem 2. *Let the function $U(y) \in E$, satisfy the equations (1) and on the part S of the boundary of the domain G the inequality*

$$|U(y)| + \left|\left\{\frac{\partial U(y)}{\partial n}\right\}\right| + |\Delta U(y)| + \left|\left\{\frac{\partial \Delta U(y)}{\partial n}\right\}\right| \leqslant \delta, \quad y \in S. \tag{29}$$

Then, for any $x \in G$ and $\sigma > 0$, the following estimate holds

$$|U(x)| \leqslant \Psi(\sigma, x_2)\, M^{1-\frac{x_2^2}{a^2}}\, \delta^{\frac{x_2^2}{a^2}}, \tag{30}$$

where $\Psi(\sigma, x_2) = \max(\varphi(\sigma, x_2), \psi(\sigma, x_2))$,

$$\begin{gathered}\psi(\sigma, x_2) = \frac{3b}{\sigma} + \frac{19ab\sqrt{\sigma\pi}}{\sigma} + 30a^2b + \frac{97ab(a-x_2)}{2} + \frac{ab\sqrt{\sigma\pi}}{2} + \frac{3ab}{2\sigma} + 4a^2b\sqrt{\sigma\pi}+ \\ +\frac{b\sqrt{\sigma\pi}}{4\sigma}(a-x_2)+\frac{21b\sqrt{\sigma\pi}}{4\sigma} + 20ab + 8ab\sigma\sqrt{\sigma\pi}(a-x_2)+\frac{5ab\sqrt{\sigma\pi}(a-x_2)}{\sigma} + 8a^2b\sigma(a-x_2)+ \\ +2b(a-x_2) + 16ab\sigma + \frac{4ab\sqrt{\sigma\pi}}{\sigma(a-x_2)} + \frac{2b}{\sigma(a-x_2)^2} + 4a^3b\sigma\sqrt{\sigma\pi} + 207ab\sqrt{\sigma\pi} + \frac{16ab\sqrt{\sigma\pi}}{(a-x_2)^2}+ \\ +5b\sqrt{\sigma\pi} + 48a^2b\sigma(a-x_2)^2 + 16a^3b\sigma^2 + 2a^3b\sigma + 182a^2b\sigma + 8a^2b\sigma\sqrt{\sigma\pi} + 128ab\sigma(a-x_2)+ \\ +42b + \frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)} + 4b\sigma(a-x_2)^2 + 16a^2b\sigma^2(a-x_2)^2 + 16ab\sigma(a-x_2)^3 + 4ab\sqrt{\sigma\pi}(a-x_2)^2+ \\ +40ab(a-x_2)^3 + \frac{8ab\sqrt{\sigma\pi}}{a-x_2} + 16a^2b\sigma^2 + \frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2) + 4b\sqrt{\sigma\pi}(a-x_2)+ \\ +16a^2b\sigma\sqrt{\sigma\pi}(a-x_2) + \frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2 + 16a^3b\sigma^2(a-x_2)+ \\ +8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2 + 4b(a-x_2)^2,\end{gathered}$$

$\varphi(\sigma, x_2)$ is determined by the formula (12).

Proof. From Green's integral formula we have

$$\begin{aligned}U(x) = &\int_S \left[U(y)\frac{\partial(\Delta L_\sigma(x,y))}{\partial n} - \Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right] dS_y+ \\ &+\int_T \left[U(y)\frac{\partial(\Delta L_\sigma(x,y))}{\partial n} - \Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right] dS_y+ \\ &+\int_S \left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n} - L_\sigma(x,y)\frac{\partial \Delta U(y)}{\partial n}\right] dS_y+ \\ &+\int_T \left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n} - L_\sigma(x,y)\frac{\partial \Delta U(y)}{\partial n}\right] dS_y.\end{aligned} \tag{31}$$

From the condition (2) and the inequality (29) we obtain

$$\begin{gathered}|U(x)| \leqslant \left| \int_S \left[f_1(y) \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} + f_3(y)\Delta L_\sigma(x,y)\right] dS_y + \right.\\ + \int_T \left[U(y) \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} - \Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right] dS_y + \\ \int_S \left[f_2(y)\frac{\partial L_\sigma(x,y)}{\partial n} + f_4(y) L_\sigma(x,y)\right] dS_y + \\ \left. + \int_T \left[\Delta U(y) \frac{\partial L_\sigma(x,y)}{\partial n} - L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right] dS_y \right| \leqslant \delta\left|U_\sigma(x)\right| + \\ + M\left(\int_T \left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right| dS_y + \int_T \left|\Delta L_\sigma(x,y)\right| dS_y + \int_T \left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y + \right. \\ \left. + \int_T \left|L_\sigma(x,y)\right| dS_y \right) \leqslant \delta\left|U_\sigma(x)\right| + M\varphi(\sigma,x_2)e^{-\sigma x_2^2}.\end{gathered} \tag{32}$$

The estimate used here

$$\begin{gathered}M\left(\int_T \left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right| dS_y + \int_T \left|\Delta L_\sigma(x,y)\right| dS_y + \right. \\ \left. + \int_T \left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y + \int_T \left|L_\sigma(x,y)\right| dS_y \right) \leqslant M\varphi(\sigma,x_2)e^{-\sigma x_2^2},\end{gathered}$$

proved in Theorem 1.

Next, estimate $|U_\sigma(x)|$

$$\begin{gathered}|U_\sigma(x)| \leqslant \int_S \left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right| dS_y + \int_S \left|\Delta L_\sigma(x,y)\right| dS_y + \int_S \left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y + \\ + \int_S \left|L_\sigma(x,y)\right| dS_y = A_1 + A_2 + A_3 + A_4.\end{gathered}$$

Estimating these integrals, we get

$$A_1 = \int_S \left|L_\sigma(x,y)\right| dS_y \leqslant \left(\frac{b}{2\sigma} + \frac{ab\sqrt{\sigma\pi}}{\sigma}\right) e^{-\sigma\left(x_2^2 - a^2\right)},$$

$$\begin{gathered}A_2 = \int_S \left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y \leqslant \left[\frac{5b}{2\sigma} + \frac{2ab\sqrt{\sigma\pi}}{\sigma} + 2a^2b + \frac{13ab}{2}(a-x_2) + \frac{2ab(a-x_2)}{\sigma} + \frac{ab\sqrt{\sigma\pi}}{2} + \right. \\ \left. + \frac{3ab}{2\sigma} + a^2b\sqrt{\sigma\pi} + \frac{b\sqrt{\sigma\pi}}{4\sigma}(a-x_2)\right] e^{-\sigma\left(x_2^2 - a^2\right)},\end{gathered}$$

$$\begin{gathered}A_3 = \int_S \left|\Delta L_\sigma(x,y)\right| dS_y \leqslant \left[\frac{17b\sqrt{\sigma\pi}}{4\sigma} + 20ab + a^2b\sqrt{\sigma\pi} + 8ab\sigma\sqrt{\sigma\pi}(a-x_2) + \right. \\ \left. + \frac{5ab\sqrt{\sigma\pi}}{\sigma}(a-x_2)8a^2b\sigma(a-x_2) + 2b(a-x_2)\right] e^{-\sigma\left(x_2^2 - a^2\right)},\end{gathered}$$

$$\begin{gathered}A_4 = \int_S \left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right| dS_y \leqslant \left[16ab\sigma + 42ab(a-x_2) + 28a^2b + \frac{b\sqrt{\sigma\pi}}{\sigma} + \frac{4ab\sqrt{\sigma\pi}}{\sigma(a-x_2)} + \right. \\ + \frac{2b}{\sigma(a-x_2)^2} + 4a^3b\sigma\sqrt{\sigma\pi} + 207ab\sqrt{\sigma\pi} + \frac{16ab\sqrt{\sigma\pi}}{(a-x_2)^2} + 5b\sqrt{\sigma\pi} + 48a^2b\sigma(a-x_2)^2 + 16a^3b\sigma^2 +\end{gathered}$$

$$+2a^3b\sigma+182a^2b\sigma+8a^2b\sigma\sqrt{\sigma\pi}+128ab\sigma(a-x_2)+42b+\frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)}+4b\sigma(a-x_2)^2+$$
$$+16a^2b\sigma^2(a-x_2)^2+16ab\sigma(a-x_2)^3+4ab\sqrt{\sigma\pi}(a-x_2)^2+40ab(a-x_2)^3+\frac{8ab\sqrt{\sigma\pi}}{a-x_2}+$$
$$+16a^2b\sigma^2+2a^2b\sqrt{\sigma\pi}+\frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)+4b\sqrt{\sigma\pi}(a-x_2)+16a^2b\sigma\sqrt{\sigma\pi}(a-x_2)+$$
$$+\frac{16ab\sqrt{\sigma\pi}}{\sigma}+\frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2+16a^3b\sigma^2(a-x_2)+8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2+4b(a-x_2)^2\Big]\,e^{-\sigma\left(x_2^2-a^2\right)}.$$

When evaluating the integrals, polar coordinates were introduced and the inequalities were used

$$\left|\sin 2\sigma y_2\sqrt{u^2+\alpha^2}\right|\leqslant\frac{4\sigma y_2\sqrt{u^2+\alpha^2}}{1+2\sigma y_2\sqrt{u^2+\alpha^2}},$$

since $|\sin x|\leqslant\dfrac{2\,|x|}{1+|x|},\ x>0.$

Adding the estimates obtained, we have

$$|U_\sigma(x)|\leqslant\psi(\sigma,x_2)e^{-\sigma\left(x_2^2-a^2\right)},$$

here

$$\psi\left(\sigma,x_2\right)=\frac{3b}{\sigma}+\frac{19ab\sqrt{\sigma\pi}}{\sigma}+30a^2b+\frac{97ab(a-x_2)}{2}+\frac{ab\sqrt{\sigma\pi}}{2}+\frac{3ab}{2\sigma}+4a^2b\sqrt{\sigma\pi}+$$
$$+\frac{b\sqrt{\sigma\pi}}{4\sigma}(a-x_2)+\frac{21b\sqrt{\sigma\pi}}{4\sigma}+20ab+8ab\sigma\sqrt{\sigma\pi}(a-x_2)+\frac{5ab\sqrt{\sigma\pi}(a-x_2)}{\sigma}+8a^2b\sigma(a-x_2)+$$
$$+2b(a-x_2)+16ab\sigma+\frac{4ab\sqrt{\sigma\pi}}{\sigma(a-x_2)}+\frac{2b}{\sigma(a-x_2)^2}+4a^3b\sigma\sqrt{\sigma\pi}+207ab\sqrt{\sigma\pi}+\frac{16ab\sqrt{\sigma\pi}}{(a-x_2)^2}+$$
$$+5b\sqrt{\sigma\pi}+48a^2b\sigma(a-x_2)^2+16a^3b\sigma^2+2a^3b\sigma+182a^2b\sigma+8a^2b\sigma\sqrt{\sigma\pi}+128ab\sigma(a-x_2)+$$
$$42b+\frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)}+4b\sigma(a-x_2)^2++16a^2b\sigma^2(a-x_2)^2+16ab\sigma(a-x_2)^3+4ab\sqrt{\sigma\pi}(a-x_2)^2+$$
$$40ab(a-x_2)^3+\frac{8ab\sqrt{\sigma\pi}}{a-x_2}+16a^2b\sigma^2++\frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)+4b\sqrt{\sigma\pi}(a-x_2)+$$
$$16a^2b\sigma\sqrt{\sigma\pi}(a-x_2)+\frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2+16a^3b\sigma^2(a-x_2)+8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2+4b(a-x_2)^2.$$

From the integral formula (32) and the condition (9) we obtain

$$|U(x)|\leqslant\delta e^{\sigma\left(a^2-x_2^2\right)}\psi(\sigma,x_2)+M\varphi(\sigma,x_2)e^{-\sigma x_2^2}=\Psi(\sigma,x_2)\left(Me^{-\sigma x_2^2}+\delta e^{\sigma\left(a^2-x_2^2\right)}\right).\qquad(33)$$

The best estimate for the function $|U(x)|$ is obtained in the case, when

$$Me^{-\sigma x_2^2}=\delta e^{\sigma\left(a^2-x_2^2\right)}$$

or

$$\sigma=\frac{1}{a^2}\ln\frac{M}{\delta}.\qquad(34)$$

Substituting the expression for σ from the equality (34) into (33) we obtain the proof of the inequality (30). Theorem 2 is proved. □

Set

$$
\begin{aligned}
U_{\sigma\delta}(x) = \int_S \left[f_{1\delta}(y) \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} - f_{3\delta}(y)\Delta L_\sigma(x,y) \right] dS_y + \\
+ \int_S \left[f_{2\delta}(y) \frac{\partial L_\sigma(x,y)}{\partial n} - f_{4\delta}(y) L_\sigma(x,y) \right] dS_y.
\end{aligned} \tag{35}
$$

Theorem 3. *Let the function $U(y) \in E$ on S satisfy the conditions (2) and instead of the functions $f_i(y)$ their approximations $f_{i\delta}(y)$, $i = 1, 2, 3, 4$ with a given deviation $\delta > 0$, i.e.*

$$
\max_S |f_i(y) - f_{i\delta}(y)| < \delta. \tag{36}
$$

Then, for any $x \in G$ and $\sigma > 0$, the following estimate holds:

$$
|U(x) - U_{\sigma\delta}(x)| \leqslant \Psi(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}. \tag{37}
$$

Proof. From (31) and (35) we get

$$
|U(x) - U_{\sigma\delta}(x)| \leqslant |I_\sigma(x)| + \delta \int_S \left\{ \left| \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| + |\Delta L_\sigma(x,y)| + \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| + |L_\sigma(x,y)| \right\} dS_y.
$$

From Theorems 1 and 2 we obtain

$$
|U(x)| \leqslant \Psi(\sigma, x_2) \left(M e^{-\sigma x_2^2} + \delta e^{\sigma a^2 - \sigma x_2^2} \right),
$$

and choosing $\sigma = \dfrac{1}{a^2} \ln \dfrac{M}{\delta}$, we obtain the proof of Theorem 3. □

Corollary 3. *For each $x \in G$ the equality*

$$
\lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x).
$$

Corollary 4. *If $x \in \bar{G}_\varepsilon$, then the family of functions $\{U_{\sigma\delta}(x)\}$,*

$$
U_{\sigma\delta}(x) \rightrightarrows U(x)
$$

converges uniformly as $\delta \to 0$.

Similarly, one can obtain stability estimates for $\dfrac{\partial U(x)}{\partial x_i}$, $i = 1, 2$, and the following corollaries are true:

Corollary 5. *For each $x \in G$, the equality*

$$
\lim_{\delta \to 0} \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \quad i = 1, 2.
$$

Corollary 6. *If $x \in \overline{G}_\varepsilon$, then the family of functions*

$$
U_{\sigma\delta}(x) \rightrightarrows U(x), \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \quad i = 1, 2
$$

converge uniformly at $\delta \to 0$.

References

[1] L.A.Aizenberg, Carleman's formulas in complex analysis, Moscow, Nauka, 1990 (in Russian).

[2] T.Carleman, Les Functions quasi analytiques, Paris, 1926.

[3] G.M.Goluzin, V.I.Krylov, Generalized Carleman's formula and its application to analytic continuation of functions, *Mat. sb.*, **40**(1933), 144–149 (in Russian).

[4] A.N.Tikhonov, On the stability of inverse problems, *DAN SSSR*, **39**(1943), no. 5, 147–160 (in Russian).

[5] A.N.Tikhonov, V.Ya.Arsenin, Methods for solving ill-posed problems, Moscow, Nauka, 1995 (in Russian).

[6] A.N.Tikhonov, A.A.Samarskiy, Equations of mathematical physics, Moscow, Nauka, 1974 (in Russian).

[7] M.M.Lavrent'ev, On the Cauchy problem for the Laplace equation, *Izv. AN SSSR*, **20**(1956), no. 6, 819–842 (in Russian).

[8] M.M.Lavrent'ev, On some ill-posed problems of mathematical physics, Ed. SO AN SSSR, Novosibirsk, 1962 (in Russian).

[9] Sh.Yarmukhamedov, On the harmonic continuation of differentiable functions given on a piece of the boundary, *Siberian Math. J*, **43**(2002), no. 1, 183–193.
DOI: 10.1023/A:1013849310522

[10] Sh.Yarmukhamedov, Representation of a harmonic function in the form of potentials and the Cauchy problem, *Mathematical Notes*, **83**(2008), no. 5, 693–706.
DOI: 10.1134/S0001434608050131

[11] M.Ikehata, Inverse conductivity problem in the infinite slab, *Inverse Problems*, **17**(2001), no. 3, 437–454. DOI: 10.1088/0266-5611/17/3/305

[12] S.I.Kabanikhin, Inverse and ill-posed problems. Novosibirsk: Siberian Scientific Publishing House, 2009 (in Russian).

[13] L.A.Aizenberg, N.N.Tarkhanov, Abstract Carleman formula, *Dokl. Math.*, **37**(1988), no. 1, 235–238.

[14] A.M.Kytmanov, T.N.Nikitina, Analogs of the Carleman formula for classical domains, *Mat. Notes*, **45**(1989), no. 3, 243–248. DOI: 10.1007/BF01158560

[15] A.N.Polkovnikov, A.A.Shlapunov, Construction of Carleman formulas by using mixed problems with parameter-dependent boundary conditions, *Siberian Mathematical Journal*, **58**(2017), no. 4, 676–686. DOI: 10.17377/smzh.2017.58.414

[16] V.V.Tikhomirov, N.N.Ochilov, Regularization of the Cauchy problem for the Laplace equation, *Differential Equations*, **50**(2014), no. 8, 1128–1132.

[17] I.N.Vekua, New methods for solving elliptic equations. OGIZ State publishing house of technology-theoretical literature, Moscow, 1948 (in Russian).

[18] L.Marin, Relaxation procedures for an iterative MFS algorithm for two-dimensional steady-state isotropic heat conduction Cauchy problems, *Eng. Anal. Bound. Elem.*, **35**(2011), 415–429. DOI: 10.1134/S0012266114080114

[19] K.S.Fayazov, I.O.Khazhiev, Estimation of conditional stability and approximate solution of a boundary value problem for a fourth-order partial differential equation, *Mathematical notes of NEFU,* **22(85)**(2015), no. 1, 78–88 (in Russian).

[20] A.B Khasanov, F.R.Tursunov, On the Cauchy problem for the Laplace equation, *Ufa Math. J.*, **11**(2019), no. 4, 92–107. DOI: 10.13108/2019-11-4-91

О задаче Коши для бигармонического уравнения

Дильшод С. Шодиев
Самаркандский государственный университет
Самарканд, Узбекистан

Аннотация. Работа посвящена исследованию продолжения и оценки устойчивости решения задачи Коши для бигармонического уравнения в области G по его известным значениям на гладкой части границы ∂G. Рассматриваемая задача относится к задачам математической физики, в которых отсутствует непрерывная зависимость решений от начальных данных. В данной работе с помощью функции Карлемана восстанавливается не только сама бигармоническая функция, но и ее производные по данным Коши на части границы области. Получены оценки устойчивости решения задачи Коши в классическом смысле.

Ключевые слова: бигармонические уравнения, задача Коши, некорректные задачи, функция Карлемана, регуляризованные решения, регуляризация, формулы продолжения.