УДК 517.55

An Analogue of the Hartogs Lemma for $R$-Analytic Functions

Alimardon A. Atamuratov
V. I. Romanovsky Institute of Mathematics
Tashkent, Uzbekistan

Urgench State University
Urgench, Uzbekistan

Djurabay K. Tishabaev
National University of Uzbekistan
Tashkent, Uzbekistan

Received 02.09.2021, received in revised form 13.11.2021, accepted 20.12.2021

Abstract. The paper is devoted to the problem of $R$-analytic continuation of functions of several real variables which admit $R$-analytic continuation along parallel sections. We prove an analogue of the well-known Hartogs lemma for $R$-analytic functions.

Keywords: $R$-analytic functions, holomorphic functions, plurisubharmonic functions, pluripolar sets, Hartogs series.


1. Introduction and preliminaries

In this paper we consider the $R$-analytic continuation of functions of several real variables that admit $R$-analytic continuation along parallel sections. Regarding to holomorphic functions, the first result in this direction is due to Hartogs [1]: if a holomorphic function $f(tz, z_n)$ in the domain $U \times \{|z_n| < r\} \subset \mathbb{C}^n \times \mathbb{C}$, where $t = (z_1, z_2, \ldots, z_{n-1})$, for each fixed $t \in (U)$ by $z_n$ extends holomorphically to the disk $|z_n| < R$, $R > r > 0$, then it is holomorphic with respect to all variables in the domain $U \times \{|z_n| < R\}$.

The following Forelli’s theorem [2] is also directly related to Hartogs theorem: if $f$ is infinitely smooth at a point $0 \in \mathbb{C}^n, f \in C^\infty \{0\}$, and the restrictions $f|_l$ are holomorphic in the disc $U(0,1) = l \cap B(0,1)$ for all complex lines $l \ni 0$, then $f$ can be holomorphically extended to the ball $B(0,1) \subset \mathbb{C}^n$.

In a recent paper [3] A. Sadullaev proved the following analogue of Forelli’s theorem for $R$-analytic functions.

**Theorem 1.** Let a function $f(x)$, $x = (x_1, x_2, \ldots, x_n)$ be smooth in some neighborhood of the origin $0 \in \mathbb{R}^n, f(x) \in C^\infty \{0\}$ and let for any real line $l : x = \lambda t$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S(0,1) \subset \mathbb{R}^n, f|_l \in C^\infty \{0\}$, then $f$ can be holomorphically extended to the ball $B(0,1) \subset \mathbb{C}^n$.
\( \mathbb{R}^n, t \in \mathbb{R} \) is a parameter, the restriction \( f|_t = f(x) \) is real-analytic (R-analytic) in the interval \( t \in (-1, 1) \). Then there is a closed pluripolar set \( S \subset B(0, 1) \) such that \( f(x) \) is R-analytic in \( B(0, 1) \setminus S \), where \( B(0, 1) \subset \mathbb{R}^n \) is the unit ball and \( S(0, 1) = \partial B(0, 1) \) is the unit sphere.

Note that the well-known terminology is used here, a set \( S \subset \mathbb{R}^n \) is called pluripolar if it is pluripolar in the ambient complex space \( \mathbb{C}^n \). A. Sadullaev, namely, the embedding of a real space \( \mathbb{R}^n \subset \mathbb{C}^n, z = x + iy \). An example of a function \( f(x_1, x_2) = \frac{x_1^{k+1}}{(x_2 - 1)^2 + x_1^2} \) shows that exact analogues of Forelli’s Theorem and Hartogs’ Theorem for R-analytic functions are not true. The function \( f(x_1, x_2) \) is real-analytic in the domain \( \mathbb{R} \times \{|x_2| < \frac{1}{2}\} \), the restriction \( f(x_1^0, x_2) \) is real-analytic on the whole line \( \mathbb{R} \). However, \( f \) is not real-analytic at the point \((0, 1)\).

The main result of this work is

**Theorem 2.** Let a function \( f(x) = f'(x, x_n) \) satisfy the following conditions:

1) The function \( f(x, x_n) \) is R-analytic in a polyhedron \( U = \langle U \rangle \times \{\{x_n| < r_n\}, r_n > 0\} \), where \( x = (x_1, x_2, \ldots, x_n-1) \) and

\[
\begin{align*}
\langle U \rangle &= \{ x \in \mathbb{R}^{n-1} : |x_1| < r_1, |x_2| < r_2, \ldots, |x_{n-1}| < r_{n-1} \} \\
&= \{ x \in \mathbb{R}^{n-1} : r_1 < x_1 < r_2, r_2 < x_2, \ldots, r_{n-1} < x_{n-1} < r_{n-1} \}
\end{align*}
\]

2) For each fixed \( (x_0) \in \langle U \rangle \) the function \( f'(x_0, x_n) \) that is R-analytic in the interval \( |x_n| < r_n \). R-analytically continues into a larger interval \( |x_n| < R_n, R_n > r_n \).

Then there exists a closed pluripolar set \( S \subset \langle U \rangle \) such that the function \( f'(x, x_n) \) R-analytically with respect to all variables \( (x, x_n) \) continues into the domain \( \langle U \rangle \times \{ |x_n| < R_n \} \setminus \{ S \times \{ x_n \geq r_n \} \} \).

The proof of Theorem 2 essentially uses the method of proving Theorem 1 proposed by A. Sadullaev, namely, the embedding of a real space \( \mathbb{R}^n \subset \mathbb{C}^n, z = x + iy \), and the natural holomorphic continuation of R-analytic functions into \( \mathbb{C}^n \), the holomorphic continuation of the Hartogs series and methods of pluripotential theory (see [4-5]).

Note that using the local transformation of the pencil of lines \( l \ni 0 \), into parallel ones, from Theorem 2 one can obtain a proof of Theorem 1.

Real analytic functions were also studied in the work of J. Sichak [6], where he proved that if the function \( f(x) \) is smooth in a domain \( D \subset \mathbb{R}^n f \in C^\infty(D) \) and for each real line \( l : x = x_0 + \lambda t, x_0 \in D, \lambda \in \mathbb{R}^n, |\lambda| = 1, t \in \mathbb{R} \), the restriction \( f|_l \) is R-analytic by \( t \) in some neighborhood of zero, then \( f(x) \) is R-analytic in \( D \).

### 2. Domain of holomorphy of Hartogs series

Let \( U = \langle U \rangle \times U_n \) be a domain in \( \mathbb{C}_{x_n-1}^n \times \mathbb{C}_{z_n}^n \), where \( U_n \) is a disc centered at the point \( z_n = 0 \) and with a radius \( \delta > 0 \). If the function \( f'(z, z_n) \) is holomorphic in \( U \), then it can be expanded in a Hartogs series:

\[
f'(z, z_n) = \sum_{k=0}^\infty c_k(z) z_n^k,
\]

where, the coefficients \( c_k(z) \) are holomorphic in \( \langle U \rangle \) and determined by the formula

\[
c_k(z) = \frac{1}{2\pi i} \int_{|\xi| = \delta'} \frac{f'(z, \xi)}{\xi^{k+1}} d\xi, \quad 0 < \delta' < \delta, \quad k = 0, 1, 2, \ldots
\]
Then, it is known that if \( R(z) \) is the radius of convergence of series (1), then the function 
\[ u(z) = -\ln R(z) \]
is plurisubharmonic in \( \mathcal{U} \), and the set \( \{ z \in \mathcal{U} : R(z) < R(z) \} \) is pluripolar. Here \( R(z) = \lim_{w \to z} R(w) \) is the lower regularization. Moreover, the series (1) converges uniformly on any compact subset \( K \subset \mathcal{U} \times \{ |z_n| < R_\ast(z) \} \). The proof of this fact can be found, for example, in [7, 8].

The following lemma, which plays the key role in the proof of Theorem 2, is widely used in the theory of analytic continuation.

**Lemma 1.** Let a function \( f(z, z_n) \) be holomorphic in the domain \( \mathcal{U} \times \{ z_n \in \mathbb{C} : |z_n| < \delta \} \), \( \mathcal{U} \subset \mathbb{C}^{n-1} \). If for each fixed \( t \) \( z_0 \in (\mathcal{U}_0) \) from some non-pluripolar set \( \mathcal{U}_0 \subset (\mathcal{U}) \) the function \( f(t \ x_0, z_n) \) of variable \( z_n \), extends holomorphically to the larger disc \( \{ z_n \in \mathbb{C} : |z_n| < \Delta \} \), \( \Delta > \delta > 0 \), then the function \( f(z, z_n) \) holomorphically extends to the domain \( \{ t \in \mathcal{U}, |z_n| < \delta - \omega(z, U_0, U) \cdot \Delta^{1-\omega(z, U_0, U)} \} \), where \( \omega(z, U_0, U) \) is the well-known plurisubharmonic measure of the set \( \mathcal{U}_0 \) with respect to the domain \( \mathcal{U} \), that is defined by the following formula:

\[ \omega(z, U_0, U) = (\sup \{ u(z) \in \text{psh}(\mathcal{U}) : u(z)_{|U} < 1, u(z)_{|U_0} \leq 0 \})^\ast. \]

Indeed, if we expand the function \( f(z, z_n) \) in a Hartogs series of the form (1) in the domain \( \mathcal{U} \times \{ z_n \in \mathbb{C} : |z_n| < \delta \} \), then the function \( u(z) = -\ln R_\ast(z) \) is plurisubharmonic in the domain \( \mathcal{U} \) and by the conditions of the lemma \( u(z)_{|U} \leq -\ln \delta, u(z)_{|U_0} \leq -\ln \Delta \). According to the theorem on two constants (see [9], p. 103), we obtain the inequality

\[ u(z) \leq (1 - \omega(z, U_0, U)) \cdot (-\ln \Delta) + \omega(z, U_0, U) \cdot (-\ln \delta). \]

Hence it follows that

\[ \ln R_\ast(z) \geq (1 - \omega(z, U_0, U)) \cdot (-\ln \Delta) + \omega(z, U_0, U) \cdot (-\ln \delta), \]
or \( R_\ast(z) \geq \delta - \omega(z, U_0, U) \cdot \Delta^{1-\omega(z, U_0, U)} \). Thus in accordance with above mentioned, the function \( f(z, z_n) \) extends holomorphically to the domain

\[ \mathcal{U} \times \{ |z_n| < R_\ast(z) \} \supset (\mathcal{U}) \times \{ |z_n| < \delta - \omega(z, U_0, U) \cdot \Delta^{1-\omega(z, U_0, U)} \}. \]

3. **Proof of the main result**

Without loss of generality we assume that for each fixed \( t x \in (\mathcal{U}) \) the function \( f(t x, x_n) \) is \( R \)-analytic in the interval \(( -R_n - \varepsilon, R_n + \varepsilon ) \), \( \varepsilon > 0 \). The proof of the theorem will be implemented in several steps.

**Step 1.** We embed the real space \( \mathbb{R}_x^n \) into the complex space \( \mathbb{C}_z^n \), \( \mathbb{R}_x^n \subset \mathbb{C}_z^n, z = x + iy \). Then, by definition of \( R \)-analyticity of a function \( f(t x, x_n) \), there exists a domain \( \mathcal{U} \subset \mathbb{C}^n \), \( \mathcal{U} \supset \mathcal{U} \) and a holomorphic function \( F(z) = F(t x, z_n) \in O(\mathcal{U}) \) such that \( F(t x, z_n)_{|U} = f(t x, x_n) \).

It follows that from the conditions of the theorem the function \( F(z) = F(t z, z_n) \) satisfies the following conditions:

1. \( F(z) \in O(\mathcal{U}) \).
2. For each fixed \( t z = (t x) \in (\mathcal{U}) \) the function \( F(t x, z_n) \) of the variable \( z_n \), can be extended holomorphically into the ellipse of type \( E_j : \frac{(Re z_n)^2}{R_j^2} + j^2(Im z_n)^2 < 1, j \in \mathbb{N}, \) such that \( E_j \supset \{ |x_n| \leq R_n \} \forall j \in \mathbb{N}. \)
We put $\hat{U} = \hat{U} \cap \mathbb{C}_z^{n-1}$ and fix a subdomain $\hat{V} \subseteq \hat{U}$ such that $V = (\hat{V}) \cap (\hat{U}) \neq \emptyset$. Then there is a circle $\{ |z_n| < \sigma \}, \sigma > 0$, such that $\hat{V} \times \{ |z_n| < \sigma \} \subset \hat{U}$, i.e. the function $F(z) = F'(z, z_n)$ is holomorphic with respect to the $(z, z_n)$ in $\hat{V} \times \{ |z_n| < \sigma \}$. We fix the number $j \in \mathbb{N}$ and denote by $V_j$ the set of points $x$ from $V = (\hat{V}) \cap (\hat{U})$ for which the function $F'(x, z_n)$ of variable $z_n$ extends holomorphically into the ellipse $E_j$, i.e.

$$V_j = \{ x \in (V) : F'(x, z_n) \in O(E_j) \}$$

It is obvious that

$$V_j \subset V_{j+1} \forall j \in \mathbb{N}$$

and

$$\bigcup_{j=1}^{\infty} (V_j) = V.$$

**Step 2.** Since an open non-empty subset $V \subseteq \mathbb{R}^{n-1}$ is not pluripolar in $\mathbb{C}^{n-1}$, then there exists a number $j_0 \in \mathbb{N}$ such that for all $j > j_0$ the sets $V_j \subset (V)$ will be non-pluripolar in $\mathbb{C}^n$.

Let us fix $j \in \mathbb{N}$, $j > j_0$ and let the function $w = g_j(z_n)$ conformally maps the ellipse $E_j$ into the unit circle $\{ |w| < 1 \}$, $g_j(0) = 0$. Since the function $F'(z, z_n)$ is holomorphic in the neighborhood $\hat{V} \times \{ |z_n| < \sigma \}$, the function $\Phi'(z, w) = F'(z, g_j^{-1}(w))$ is holomorphic in the domain $\hat{V} \times g_j^{-1}(\{ |z_n| < \sigma \})$. Since $g_j(0) = 0$, there is a number $\delta_j > 0$ such that $\{ |w| < \delta_j \} \subset (\hat{V}) \times g_j^{-1}(\{ |z_n| < \sigma \})$, i.e. the function $\Phi'(z, w)$ is holomorphic in the domain $\hat{V} \times \{ |w| < \delta_j \}$. In addition, for each fixed variable $z = (x) \in (V_j)$, the function $\Phi'(x, w)$ of the variable $w$ extends holomorphically to the circle $\{ |w| < 1 \}$.

By Lemma 1, where $\delta = \delta_j$, $\Delta = 1$, the function $\Phi'(z, w)$ is holomorphic in the domain

$$\{ z' \in \hat{V}, |z_n| < \delta_j^{\omega^*} (\hat{V}, V_j) \}.$$ 

Thus, if we substitute into $\Phi'(z, w)$ the value $w = g_j(z_n)$, then we obtain that the function $F'(z, z_n)$ extends holomorphically to the domain

$$G_j = \{ (z, z_n) \in \mathbb{C}^n : (z) \in (\hat{V}), |g_j(z_n)| < \delta_j^{\omega^*} (\hat{V}, V_j) \}.$$ 

Note that if the point $'x \in (V_j)$ is pluriregular, i.e. $\omega^* (x', V_j, \hat{V}) = 0$, then, according to (3), the ellipse $\{ x' \times \{ |g_j(z_n)| < 1 \} \subset G_j$. Consequently, the domain $G_j$ contains some neighborhood of the segment $\{ x' \} \times [-R_n, R_n]$.

**Step 3.** By the construction of the domain $G_j$, $F$ can be extended holomorphically to the domain $G \cap V = \bigcup_{j=j_0}^{\infty} G_j$ as well. Let us denote by $P_j$ the set of irregular points $x' \in (V_j)$ and by $P_V = \bigcup_{j=j_0}^{\infty} P_j$ the union of these sets $P_V \subset (V)$. It is a pluripolar set in $\mathbb{C}_z^{n-1}$. For each fixed point $z = (x') \in (\hat{V}) \setminus P_V$, the union $G \cap V = \bigcup_{j=j_0}^{\infty} G_j$ contains a neighborhood of the segment $\{ x' \} \times [-R_n, R_n]$.

**Step 4.** We take a sequence of domains $\hat{V}_k \subseteq \hat{V}$, $\hat{V}_{k+1} \subseteq \hat{U}$ such that $\bigcup_{k=1}^{\infty} (\hat{V}_k) = \hat{U}$ and put $P_\hat{V} = \bigcup_{k=1}^{\infty} P_{\hat{V}_k}$. Then $P \subseteq (\hat{U})$ is pluripolar set in $\mathbb{C}_z^{n-1}$. According to Step 3, the function $F$ extends holomorphically to the domain $G = \bigcup_{k=1}^{\infty} G \setminus P_{\hat{V}_k}$, and for each fixed point $z = (x') \in (\hat{U}) \setminus P$ the union $G = \bigcup_{k=1}^{\infty} G \setminus P_{\hat{V}_k}$ contains a neighborhood of the segment $\{ x' \} \times [-R_n, R_n]$. Therefore, for such points the given function $f (x, x_n)$ is $R$-analytic in the set of variables in the neighborhood of the segment $\{ x' \} \times [-R_n, R_n]$. 

– 197 –
We note that the complement \( S = [U \times \{ |x_n| < R_n \}] \setminus [G \cap \mathbb{R}^n] \) is a closed pluripolar set, \( S \subset P \times \{ |x_n| \geq r_n \} \), and the function \( f(x, x_n) \) is \( R \)-analytically extended to \([U \times \{ |x_n| < R_n \}] \setminus S \). The theorem is proved.

The authors are grateful to Professor A. Sadullaev for useful advices and comments to the article.

References


Об аналоге леммы Гартогса для \( R \)-аналитических функций

Алимардон А. Атамуратов
Институт математики им. В. И. Романовского АН РУз
Ташкент, Узбекистан
Ургенчский Государственный Университет
Ургенч, Узбекистан
Джурабай К. Тишабаев
Тахир Т. Туйчиев
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. Работа посвящена задачам \( R \)-аналитического продолжения функций многих действительных переменных, допускающих \( R \)-аналитическое продолжение на параллельные сечения. В ней доказывается аналог известной теоремы Гартогса для \( R \)-аналитических функций.

Ключевые слова: \( R \)-аналитические функции, голоморфные функции, plurisubharmonic functions, pluripolar sets, Гартогса.