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On a Formula for Calculating the Resultant of Two Entire Functions

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Abstract. Using Newton's recurrent formulae, we find the product of values of an entire function of one variable in zeroes of another entire function. This allows to answer whether they have common zeroes. By that, we propose an approach to construction of the resultant of two entire functions. We also give example illustrating the main result.

Keywords: resultant, entire function, Newton formulae.

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Introduction

An important tool that opened a way to developing algorithmic methods for studying and solving systems of algebraic equations is the concept of the Gröbner basis of the ideal of the ring of polynomials. However, the classical methods for eliminating unknowns from systems of algebraic equations, based on the method of Gröbner bases, cannot be applied for the analysis of essentially nonalgebraic equations (i. e., the equations that cannot be reduced to algebraic equations by changing variables).

However, systems of nonalgebraic equations occur in many areas of knowledge. In particular, in processes described by systems of differential equations with the right-hand sides that can be expanded in Taylor series, it is important to determine the number of stationary states (and localize them) in sets of a certain type. This problem leads to problems of constructing algorithms for determining the number of roots of the given system of equations in various sets, to finding the roots themselves, and to eliminating a part of unknowns. In particular, in [1] one can find numerous examples from chemical kinetics in which algorithms for eliminating unknowns are required. It is important to apply the developed methods for the qualitative and numerical analysis of mathematical models of thermokinetics of the processes of combustion and catalysis with the aim of obtaining conditions of ignition, explosion, and critical phenomena in chemically reacting systems. For applications, in particular, for chemical kinetics equations, an important problem is to explore the dependence of solutions to systems of nonlinear (including nonalgebraic) equations on parameters. This problem is computationally costly. The degree of its complexity strongly depends on the dimension of the space of unknowns. Therefore, the reduction of dimension by eliminating some variables can simplify the original problem.

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A method for eliminating unknowns from systems of nonlinear algebraic equations based on the theory of multidimensional residues was proposed in [2] in 1977. The further modifications of this method were proposed in [1]. These ideas were partially elaborated in [3,4]. An algorithmic method (developed by Aizenberg and Yuzhakov) was proposed in [1]. Its idea is to calculate certain residue integrals related to power sums of the roots (with positive exponents) of the given system of equations without calculating the roots themselves and then apply to them Newton's recurrent formulas. Compared with the classical method, this method reduces the computation time without increasing the root multiplicity.

Another method for the elimination of unknowns is based on constructing the resultant of two entire functions. There is the well-known classical Sylvester's resultant of two polynomials and the elimination method based on it. For nonalgebraic functions, no similar concept was earlier studied. Only recently, an approach to finding the resultant of two entire functions based on Newton's recurrent formulas has been discussed in [5,6].

1. Sylvester's classical resultant and its generalizations

Recall that for the given polynomials f and g, the classical resultant $R(f,g)$ can be defined in various ways using

a) Sylvester's determinant (e. g., see [7,8]);

b) formulas for the product $R(f,g) = \prod\limits_{\{x\colon f(x)=0\}} g(x)$ (e. g., see [7,8]);

c) the Bezout – Caley method (e. g., see [9]).

In this paper, we describe the construction of the resultant of two entire functions on the complex plane. As the main definition of the resultant we use the product formula. This choice is explained by the fact that entire functions are a natural generalization of polynomials in complex analysis.

In a number of works (e. g., see [10–14]), various researchers proposed generalizations of the concept of resultant for analytic functions in the ring of matrix-valued functions, for meromorphic functions on the Riemann surface, and for systems of algebraic equations. In these studies, it was assumed that the number of roots or poles is finite. In the case considered in this paper, the entire functions may have an infinite number of roots. Therefore, for finding the resultant, the passage to the limit is required.

This problem is important because, e. g., in the chemical kinetics equations, there are functions and systems of equations consisting of exponential polynomials [15].

The first step in finding the resultant of two entire functions was the work [5], in which the case when one of the functions is entire and another one is a polynomial (or an entire function with a finite number of zeros) was considered. The condition of finiteness of the number of zeros of an entire function was studied in [16]. In [17], the results of [5] are generalized for the case when one of the entire functions satisfies certain severe constraints but may have an infinite number of zeros.

The advantage of the approach proposed in this paper is that it makes possible to answer the question whether or not entire functions have common zeros without calculating the zeros themselves. The final formula for the resultant includes power sums of the roots that can be calculated using Newton's formulas without finding the zeros themselves.

2. Resultant of two entire functions

Consider the system of equations consisting of a fourth degree polynomial $f(z)$ and a polynomial $g(z)$ of degree n:

$$\begin{cases} f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + z^4, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases} \tag{1}$$

Denote the roots of $f(z)$ by z_1, z_2, z_3, z_4.

Recall the classical Newton's recurrent formulas for polynomials. They relate the coefficients of the polynomial to the power sums of its roots.

Let

$$P(z) = z^m + c_1 z^{m-1} + \ldots + c_{m-1} z + c_m.$$

Denote its roots by $z_1, z_2, \ldots, z_m$ (there may be multiple roots among them). Define the power sum of the roots

$$S_k = z_1^k + \ldots + z_m^k, \quad k \in \mathbb{N}, \quad S_0 = m.$$

The power sums S_k and the coefficients c_j are connected by Newton's recurrent formulas:

$$\begin{cases} S_j + \sum\limits_{i=1}^{j-1} S_{j-i} c_i + j c_j = 0, \quad 1 \leqslant j \leqslant m, \\ S_j + \sum\limits_{i=1}^{m} S_{j-i} c_i = 0, \quad j > m. \end{cases} \tag{2}$$

Using the definition of the resultant as a product, we obtain the following result.

Theorem 2.1. *The resultant $R(f,g)$ of the system of polynomials* (1) *is found by the formula*

$$\begin{aligned} R(f,g) = \prod_{i=1}^{4} g(z_i) = \sum_{s=0}^{n} b_s^4 a_0^s + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} b_s^3 b_t a_0^s S_{t-s} + \sum_{s=0}^{n} \sum_{t=s+1}^{n} b_s b_t^3 a_0^s \frac{1}{4} \left(S_{t-s}^3 - 3S_{2t-2s} \cdot S_{t-s} + 2S_{3t-3s}\right) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} b_s^2 b_t^2 a_0^s \frac{1}{2} \left(S_{t-s}^2 - S_{2t-2s}\right) + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} b_s^2 b_t b_p a_0^s \left(S_{p-s} \cdot S_{t-s} - S_{p+t-2s}\right) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} b_s b_t^2 b_p a_0^s \frac{1}{2} \left(S_{t-s}^2 \cdot S_{p-s} - S_{2t-2s} \cdot S_{p-s} - 2S_{t+p-2s} \cdot S_{t-s} + 2S_{2t+p-3s}\right) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} b_s b_t b_p^2 a_0^s \frac{1}{2} \left(S_{p-s}^2 \cdot S_{t-s} - S_{2p-2s} \cdot S_{t-s} - 2S_{t+p-2s} \cdot S_{p-s} + 2S_{2p+t-3s}\right) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} \sum_{r=p+1}^{n} b_s b_t b_p b_r a_0^s \left(S_{t-s} \cdot S_{p-s} \cdot S_{r-s} - S_{t+p-2s} \cdot S_{r-s} - \right. \\ \left. - S_{t+r-2s} \cdot S_{p-s} - S_{p+r-2s} \cdot S_{t-s} + 2S_{t+p+r-3s}\right), \end{aligned} \tag{3}$$

where the power sums S_j of the roots of the polynomial $f(z)$ are determined by Newton's recurrent formulas.

This statement can be obtained as a special case of a more general theorem considered in [18]. By passing to the limit in (3) over n, we obtain the following theorem concerning the resultant of an entire function and a polynomial (or an entire function with a finite number of zeros).

Theorem 2.2. *Let $g(z)$ be an entire function on the complex plane $\mathbb{C}$*

$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots,$$

and let $f(z)$ be a polynomial of form (1). Then the resultant $R(f,g)$ is given by formula (3), where we pass to the limit as n tends to infinity.

3. Example

Consider the system of equations ($n = 5$)

$$\begin{cases} f(z) = z^2(z-1)(z+1) = z^2\left(z^2-1\right) = z^4 - z^2, \\ g(z) = z^5 + 2. \end{cases}$$

In this case,

$$a_0 = a_1 = a_3 = 0, \quad a_2 = -1, \quad b_0 = 2, \quad b_1 = b_2 = b_3 = b_4 = 0, \quad b_5 = 1.$$

Since $a_0 = 0$, then expression (3) will contain only the terms corresponding to $s = 0$. We also note that since only b_0 and b_5 are nonzero, only the terms with $t = 5$ will remain in the corresponding sums. So we have

$$\sum_{s=0}^{5} b_s^4 a_0^s = b_0^4 = 16,$$

$$\sum_{s=0}^{5}\sum_{t=s+1}^{5} b_s^3 b_t a_0^s S_{t-s} = \sum_{t=1}^{5} b_0^3 b_t S_t = b_0^3 b_5 S_5 = 8S_5,$$

$$\sum_{s=0}^{5}\sum_{t=s+1}^{5} b_s b_t^3 a_0^s \frac{1}{4}\left(S_{t-s}^3 - 3S_{2t-2s}\cdot S_{t-s} + 2S_{3t-3s}\right) = \sum_{t=1}^{5} b_0 b_t^3 \frac{1}{4}\left(S_t^3 - 3S_{2t}\cdot S_t + 2S_{3t}\right) =$$

$$= \frac{1}{4} b_0 b_5^3\left(S_5^3 - 3S_{10}\cdot S_5 + 2S_{15}\right) = \frac{1}{2}\left(S_5^3 - 3S_{10}\cdot S_5 + 2S_{15}\right),$$

$$\sum_{s=0}^{5}\sum_{t=s+1}^{5} b_s^2 b_t^2 a_0^s \frac{1}{2}\left(S_{t-s}^2 - S_{2t-2s}\right) = \sum_{t=1}^{5} \frac{1}{2} b_0^2 b_t^2\left(S_t^2 - S_{2t}\right) = \frac{1}{2} b_0^2 b_5^2\left(S_5^2 - S_{10}\right) = 2\left(S_5^2 - S_{10}\right).$$

All other triple and quadruple sums in expression (3) vanish due to the fact that the previous sums are nonzero only if the values of the indices are $s = 0$, $t = 5$.

Thus,

$$R(f,g) = 16 + 8S_5 + \frac{1}{2}S_5^3 - \frac{3}{2}S_{10}\cdot S_5 + S_{15} + 2S_5^2 - 2S_{10}.$$

Find the power sums of the roots S_5, S_{10}, S_{15} without using the roots themselves. But using Newton's recurrent formulas (2), in which it is necessary to assign

$$m = 4, \quad c_1 = a_3 = 0, \quad c_2 = a_2 = -1, \quad c_3 = a_1 = 0, \quad c_4 = a_0 = 0.$$

For the case under consideration, we have

$$S_5 = 0, \quad S_{10} = 2, \quad S_{15} = 0.$$

Thus, $R(f, g) = 12$.

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Об одной формуле вычисления результанта двух целых функций

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Аннотация. На основе рекуррентных формул Ньютона на комплексной плоскости найдены суммы значений одной целой функции в нулях другой целой функции. Это позволяет ответить на вопрос, имеют ли эти функции общие нули или нет. Тем самым разработан подход к построению результанта двух целых функций. Приведен пример, демонстрирующий данный подход.

Ключевые слова: результант, целая функция, формулы Ньютона.