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On a Multidimensional Version of the Principal Theorem of Difference Equations with Constant Coefficients

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Abstract. In the paper we consider systems of linear difference equations with discrete characteristic sets. A multidimensional version of the principal theorem of linear difference equations with constant coefficients is formulated and proved.

Keywords: linear difference equations, characteristic set, multiple roots.

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1. Introduction and formulation of the main result

The main problems of the finite difference theory concern interpolation and summation of functions. The latter is closely related to solving of equations in finite defferences. For linear equations in finite differences a theory has been developed (see, for example, [5]), analogous to theory of ordinary linear differential equations. In case of constant coefficients it is largely completed, and together with the theory of generating functions they are powerful research tools in combinatorial analysis and other branches of mathematics.

Let us recall the principal theorem of difference equations in one variable. First, it states that a generating function of a solution is rational, and second, its Taylor coefficients has the form of an exponential polynomial. Following the book by R. Stanley ([1], Section 4.1.1) we give its formulation. Here $\mathbb{Z}_{\geqslant}$ is the set of nonnegative integers.

Theorem. *Let $c_1, c_2, \cdots, c_d$ be complex numbers such that $d \geqslant 1$ and $c_d \neq 0$. For a function $f\colon \mathbb{Z}_{\geqslant} \to \mathbb{C}$ the following are equivalent*

(i) Generating function of the sequence $f(x)$ is a rational function

$$\sum_{x \geqslant 0} f(x) z^x = \frac{S(z)}{T(z)},$$

where $T(z) = 1 + c_1 z + \cdots + c_d z^d$ and $S(z)$ is a polynomial of degree less than d;

(ii) For any $x \in \mathbb{Z}_{\geqslant}$ there holds

$$f(x+d) + c_1 f(x+d-1) + \cdots + c_d f(x) = 0;$$

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(iii) The function $f(x)$ is an exponential polynomial

$$f(x) = \sum_{j=1}^{m} P_j(x)\gamma_j^x,$$

where γ_j are pairwise distinct reciprocals of the roots of the polynomial

$$T(z) = 1 + c_1 z + \dots + c_d z^d = \prod_{j=1}^{m}(1-\gamma_j z)^{d_j},$$

and P_j are polynomials of degrees deg $P_j(x) < d_j$.

Let us stress out that in (*iii*) the naturals d_j are orders (multiplicities) of roots γ_j of the characteristic polynomial $Q(z) = z^d T\left(\frac{1}{z}\right)$ of the difference equation from (*ii*).

We introduce necessary notation and definitions. Denote by $\mathbb{Z}^n = \mathbb{Z}\times\dots\times\mathbb{Z}$ the n-dimensional integer lattice and by $\mathbb{Z}^n_{\geqslant}$ its subset of points with nonnegative components.

A *difference equation* (with respect to an unknown function $f : \mathbb{Z}^n_{\geqslant} \to \mathbb{C}$) is a relation

$$\sum_{0\leqslant\alpha\leqslant d} c_\alpha f(x + d - \alpha) = 0,\ x \in \mathbb{Z}^n_{\geqslant}, \tag{1}$$

where $c_\alpha \in \mathbb{C}$ are (constant) coefficients of the equation (1), and $\alpha \geqslant \beta$ for multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ denotes that $\alpha_j \geqslant \beta_j$, $j = 1, 2, \dots, n$.

A *characteristic polynomial* for the difference equation (1) is the polynomial

$$\sum_{0\leqslant\alpha\leqslant d} c_\alpha z^{d-\alpha} =: Q(z),$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$. The zeroes of Q are the *characteristic roots*, and the set

$$V = \{z \in \mathbb{C}^n : Q(z) = 0\} \tag{2}$$

of all zeroes Q is called the *characteristic set* of the equation (1).

In the case $n = 1$ it is known (see [5] or the theorem above) that any solution of the equation (1) is a linear combination of functions of the form $f(x) = x^s\gamma_j$, $s = 0, \dots, d_j - 1$, where γ_j^{-1} are roots (of multiplicities d_j) of the characteristic polynomial. Therefore, the dimension of the space of solutions is finite and equal to the degree d of the characteristic polynomial $Q(z)$. For $n > 1$ a similar statement on finite dimension of the space of solutions of an equation of the form (1) is not true, since the characteristic set V is infinite.

However, as the main result of this paper shows, a multidimensional analog of this theorem exists, for example, when instead of one equation (1) for a function f in n variables one considers a system of n equations. To each such system we can put into one-to-one correspondence a system of algebraic equations. Consider polynomials $Q = (Q_1, \dots, Q_n)$ of the form

$$Q_i(z) = \sum_{0\leqslant\alpha\leqslant d^i} c^i_\alpha z^\alpha,\ i = 1, 2, \dots, n,$$

where d^i are vectors from $\mathbb{Z}^n_{\geqslant}$. We shall assume that $c^i_0 = 1, c^i_{d^i} \neq 0$.

Denote by V_Q the zero set of the system

$$Q_1(z) = Q_2(z) = \dots = Q_n(z) = 0, \tag{3}$$

which we shall call *characteristic*.

Let $Q : \mathcal{U}_a \to \mathbb{C}^n$ be a holomorphic mapping of a neighborhood $\mathcal{U}_a \subset \mathbb{C}^n$ of a such that $Q(a) = 0$. The point a is called a *simple zero* of Q if the Jacobian $J_Q = \dfrac{\partial Q}{\partial z}$ does not vanish at

a. From the implicit function theorem it follows that a simple zero is always an isolated one. If $J_Q(a) = 0$, than a is called a *multiple zero* of the mapping Q.

Consider the case when discrete zeroes of the characteristic system (2) may be multiple. In one-dimensional case there are two equivalent definitions for multiplicity (order) of a root:

(i) $z = a$ is a zero of order (multiplicity) d of the polynomial $Q(z)$ if $Q(z)$ can be represented as

$$Q(z) = (z - a)^d \varphi(z), \qquad \varphi(a) \neq 0;$$

(ii) $z = a$ is a zero of order (multiplicity) d of the polynomial $Q(z)$ if for its derivatives we have

$$Q(a) = Q^{'}(a) = \ldots = Q^{(d-1)}(a) = 0, \qquad Q^{(d)}(a) \neq 0.$$

Note that in multidimensional case the notions of order and multiplicity of a zero do not coincide (see Section 2 below). In this case as an analog of (ii) we require that the roots $a \in V_Q$ of the characteristic system (3) satisfy the following: *for some* $d_a = (d_{1,a}, \ldots, d_{n,a}) \in \mathbb{Z}^n_>$ *we have*

$$\frac{\partial^\alpha Q_i}{\partial z^\alpha}(a) = 0 \;\; for \;\; 0 \leqslant \alpha \leqslant d_a - I, \;\; i = 1, \ldots, n, \tag{4}$$

$$\Delta_{d_a}(z) = det \left\| \frac{\partial^{d_{l,a}} Q_i(z)}{\partial z_l^{d_{l,a}}} \right\|_{z=a} \neq 0. \tag{5}$$

Here in (4) $I = (1, \ldots, 1) \in \mathbb{Z}^n$ and in (5) the indices l and i independently run over $1, 2, \ldots, n$. In the case $d_a = I$ the determinant (5) coincides with the Jacobian $\dfrac{\partial Q}{\partial z}$ at a, therefore the point $z = a$ is a simple zero of the characteristic system (2).

Besides that we shall assume that the roots $\gamma_{(j)}$ of the system of algebraic equations (3) do not lie on coordinate planes, i.e. that all coordinates of $\gamma_{(j)}$ are nonzero.

Let us now formulate the main result of the paper that generalizes the (one-dimensional) principal theorem to the case of several variables.

Theorem 1. For a function $f(x) = f(x_1, \ldots, x_n)\colon \mathbb{Z}^n_\geqslant \to \mathbb{C}$ the following are equivalent:

(i) Generating function for the sequence $f(x)$ is a rational function of the form

$$F(z) = \sum_{x \geqslant 0} f(x) z^x = \sum_{j=1}^{m} \frac{b_j(z)}{(I - \gamma_{(j)} z)^{d_{(j)}}},$$

where

$$(I - \gamma_{(j)} z)^{d_{(j)}} = (1 - \gamma_{(j),1} z_1)^{d_{(j),1}} (1 - \gamma_{(j),2} z_2)^{d_{(j),2}} \cdots (1 - \gamma_{(j),n} z_n)^{d_{(j),n}},$$

$b_j(z)$ are polynomials of the form $\sum\limits_{0 \leqslant \alpha < d_{(j)}} b^j_\alpha z^\alpha$;

(ii) For any $x \in \mathbb{Z}^n_\geqslant$ the function $f(x)$ satisfies the system of difference equations

$$\sum_{0 \leqslant \alpha \leqslant d^i} c^i_\alpha f(x + d^i - \alpha) = 0, \;\; i = 1, 2, \ldots, n, \tag{6}$$

with characteristic roots satisfying the conditions (4), (5).

(iii) The function $f(x)$ is an exponential polynomial

$$f(x) = \sum_{j=1}^{m} P_j(x)\gamma_{(j)}^x, \tag{7}$$

where $\gamma_{(j)}^x = \gamma_{(j),1}^{x_1} \cdot \ldots \cdot \gamma_{(j),n}^{x_n}$, and $P_j(x)$ are polynomials of the form $\sum\limits_{0\leqslant k<d_{(j)}} P_k^{(j)} x^k$.

Remark. In the proof of this theorem we will establish that the general form (7) of a solution to a system of difference equations (6) is defined by the characteristic roots and by polynomials $P_j(z)$ that constitute the space of remainders after division by the ideal generated by the system Q in local rings O_a of germs of holomorphic functions. This fact is closely related to the problem of multidimensional interpolation studied in the recent papers [12, 13].

2. Relations between notions of order and multiplicity of roots of the characteristic system (3)

For the proof of Theorem 1 we need to relate conditions (4), (5) with dimension of spaces of polynomials $c_j(x)$ in *(iii)* of the theorem. In fact, each such space coincides with the set of remainders $R_{Q,a}$ after division of the local ring $\mathcal{O}_a$ of germs of holomorphic function by the ideal $I_a(Q) \subset \mathcal{O}_a$ generated by the characteristic system $Q = (Q_1, \ldots, Q_n)$, where $a = \gamma_{(j)}$ is a characteristic root.

Recall (see [10] or [7]) that the *multiplicity* of an isolated zero $a \in \mathbb{C}^n$ of a germ of a holomorphic mapping $Q : (\mathbb{C}^n, a) \to (\mathbb{C}^n, 0)$ is the limit

$$\mu_a(Q) = \overline{\lim_{\xi\to a}}\ \#\{\mathcal{U}_a \cap Q^{-1}(\xi)\},$$

where $\#$ denotes the cardinality of a set, $\mathcal{U}_a$ is a neighborhood of a whose closure does not contain any zero of Q except for a. The geometric multiplicity $\mu_a(Q)$ is the number of simple roots we get from a multiple root a under small perturbations of the mappings Q, i.e. after passing from Q to $Q - \xi$ (the roots of the latter is the preimage $Q^{-1}(\xi)$)

Lemma 1. *Provided the conditions (4), (5) are fulfilled the multiplicity of the zero a of Q is equal to product of orders: $\mu_a(Q) = d_{1,a} \cdot \ldots \cdot d_{n,a}$.*

Proof. By the *order of a holomorphic function* g at a we call the least order of derivatives of g that do not vanish at a; we denote this order by $d_a(g)$. Thus, the Taylor expansion of g centered at a starts with terms of degree $d_a(g)$. The sum of terms of this degree is naturally called the *initial polynomial* for g at a; we denote it by $(g)_*$. Obviously, $(g)_*$ is a homogeneous polynomial, the number $d_a(g)$ is usually called the degree of the polynomial $(g)_*$ in all variables.

Consider the system of initial polynomials of the system $Q = (Q_1, \ldots, Q_n)$ at a

$$(Q)_* = ((Q_1)_*, \ldots, (Q_n)_*).$$

It is known that (see [9] or [11], p. 10.3) that the *multiplicity $\mu_a(Q)$ of the system Q is equal to product of orders*:

$$\mu_a(Q) = d_a(Q_1) \cdot \ldots \cdot d_a(Q_n), \tag{8}$$

if and only if the a is an isolated zero of the system $(Q)_*$ of initial polynomials.

According to (4), (5) the Taylor expansions (at a) of $Q_i's$ have the following form:

$$Q_i(z) = (Q_i)_* + \Theta_i(z), \quad i = 1, \ldots, n, \tag{9}$$

where

$$(Q_i)_* = b_{i1}(z_1 - a_1)^{d_{1,a}} + \ldots + b_{in}(z_n - a_n)^{d_{n,a}}, \quad i = 1, \ldots, n, \tag{10}$$

moreover, by Taylor's formula $b_{il} = \dfrac{1}{d_{l,a}!}\dfrac{\partial^{d_{l,a}} Q_i}{\partial z_l^{d_{l,a}}}(a)$. Taking into account (5), we have $det(b_{il}) \neq 0$.

The polynomials (10) are not homogeneous (in all variables), therefore formula (8) for multiplicity of Q at a can not be applied immediately. However, thay are weighted homogeneous with respect to the weight $w = (w_1, \ldots, w_n)$ with

$$w_i = d_{1,a} \ldots [i] \ldots d_{n,a}, \quad i = 1, \ldots, n.$$

Thanks to (4), the remainders Θ_i do not contain monomials with exponents from the parallelepiped $\{0 \leqslant \alpha \leqslant d\}$, therefore their w-weighted degrees a large than those of $(Q_i)_*$.

Consider a superposition $Q(a + \xi^w)$ of the mapping $Q(z)$ with the mapping

$$z_i = a_i + \xi_i^{w_i}, \quad i = 1, \ldots, n.$$

After substitution $z = a + \xi^w$ the representation (9) becomes

$$Q_i(a + \xi^w) = \widetilde{(Q_i)}_*(\xi) + \widetilde{\Theta}_i(\xi), \quad i = 1, \ldots, n,$$

where

$$\widetilde{(Q_i)}_*(\xi) = b_{i1}\xi_1^u + \ldots + b_{in}\xi_n^u, \quad i = 1, \ldots, n, \tag{11}$$

here $u = d_{1,a} \cdot \ldots \cdot d_{n,a}$ and $d_0(\widetilde{\Theta}_i) > u$. The system (11) consists of usual (with respect to the weight $(1, \ldots, 1)$) homogeneous polynomials. Since $det(b_{il}) \neq 0$, this system does not have common roots except for $\xi = 0$. Therefore, the multiplicity of the zero $\xi = 0$ of the mapping $Q(a + \xi^w)$ is given by formula (8), i.e. it is u^n. Under superpositions the multiplicities are multiplied, and since the multiplicity of $\xi = 0$ of the system $(\xi_1^{w_1}, \ldots, \xi_n^{w_n})$ equals $w_1 \cdot \ldots \cdot w_n = u^{n-1}$, we get

$$\mu_a(Q) = \frac{u^n}{u^{n-1}} = d_{1,a} \cdot \ldots \cdot d_{n,a}.$$

The lemma is proved. □

3. The proof of the main result and an example

Now we are ready to prove Theorem 1. Fix a mapping Q with properties (4), (5) and introduce three vector spaces:

$$V_1 = \{f : \mathbb{Z}^n_{\geqslant} \Rightarrow \mathbb{C}^n, \ f \textit{ satisfies (i)}\},$$

$$V_2 = \{f : \mathbb{Z}^n_{\geqslant} \Rightarrow \mathbb{C}^n, \ f \textit{ satisfies (ii)}\},$$

$$V_3 = \{f : \mathbb{Z}^n_{\geqslant} \Rightarrow \mathbb{C}^n, \ f \textit{ satisfies (iii)}\}.$$

Let us first show that they all have the same dimension:

$$\dim V_1 = \dim V_2 = \dim V_3 = \sum_{j=1}^{m} d_{1,(j)} \cdot \ldots \cdot d_{n,(j)}. \tag{12}$$

Indeed, the elements of V_1 are defined by polynomials $b_1(z), \ldots, b_m(z)$, each running over a space of polynomials with $d_{1,(j)} \cdot \ldots \cdot d_{n,(j)}$ independent coefficients $b_\alpha^{(j)}$.

To compute the dimension of V_2 we denote by $I(Q) = (Q_1, \ldots, Q_n)$ the ideal in the ring $\mathbb{C}[z_1, \ldots, z_n]$ generated by polynomials $Q_1, \ldots, Q_n$. Use the following fact ([3], Proposition 2):

If the quotient ring

$$R = \mathbb{C}[z_1, \ldots, z_n]/I(Q) \tag{13}$$

has finite dimension and $\{z^\beta;\ \beta \in B \subset \mathbb{Z}^n\}$ is its monomial basis, then any solution $f(x)$ of a system of difference equations

$$Q_i(\delta)f(x) = 0, \ \ i = 1, \ldots, n, \tag{14}$$

is uniquely determined by its values on the set B.

Here the symbol $\delta = (\delta_1, \ldots, \delta_n)$ denotes a vector of atomic difference operators (shifts)

$$\delta_j f(x) = f(x + e_j) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n), \ \ j = 1, \ldots, n.$$

It is known (see [8], Sec. 5, and [7], Sec. 19) that dimension of R equals to the number of zeroes of the system Q (with multiplicities). Therefore from Lemma 1 we get $\dim V_2 = \dim V_1$. Finally, in the representation (iii) we may choose coefficients of $P_j(x)$ arbitrarily, and their number is equal to $d_{1,(j)} \cdot \ldots \cdot d_{n,(j)}$, therefore $\dim V_3 = \dim V_2 = \dim V_1$.

The proof of equivalence $(iii) \approx (i)$. Let $f(x)$ has the form of an exponential polynomial as in (iii):

$$f(x) = \sum_{j=1}^{m} c_j(x)\gamma_{(j)}^x = \sum_{j=1}^{m} \sum_{0 \leqslant k < d_{(j)}} l_k^{(j)} x^k \gamma_{(j)}^x.$$

We multiply both sides of the equality by z^x and sum over $x = (x_1, \ldots, x_n)$. By this, we get for the generating function $F(z)$ the following representation:

$$\sum_{j=1}^{m} \sum_{0 \leqslant k < d_{(j)}} l_k^{(j)} \sum_{x \geqslant 0} (x^k \gamma_{(j)}^x z^x) = \sum_{j=1}^{m} \sum_{0 \leqslant k < d_{(j)}} l_k^{(j)} \sum_{x_1 \geqslant 0} (x_1^{k_1} \gamma_{(j),1}^{x_1} z_1^{x_1}) \cdot \ldots \cdot \sum_{x_n \geqslant 0} (x_n^{k_n} \gamma_{(j),n}^{x_n} z_n^{x_n}). \tag{15}$$

For a fixed k we consider one of series in x_j separately, for brevity we omit the index j:

$$\sum_{x \geqslant 0} x^k \gamma^x z^x = \sum_{x \geqslant 0} x^k w^x, \tag{16}$$

where $w = \gamma z$, and k is a scalar, since it is the coordinate k_j. To compute the last series, we write down the geometric series in w, take its derivative and multiply both sides of it by w:

$$\frac{1}{1-w} = \sum_{x=0}^{\infty} w^x,$$

$$\left(\frac{1}{1-w}\right)' = \frac{1}{(1-w)^2} = \sum_{x=0}^{\infty} x w^{x-1},$$

$$w \cdot \frac{1}{(1-w)^2} = \sum_{x=0}^{\infty} x w^x.$$

Thus, we get an expression for the series (16) for $k = 1$. Repeating this procedure, we get an expression for $k = 2$:

$$\left(\frac{w}{(1-w)^2}\right)' = \frac{1+2w}{(1-w)^3} = \sum_{x=0}^{\infty} x^2 w^{x-1}, \quad \sum_{x=0}^{\infty} x^2 w^x = w \cdot \frac{1+2w}{(1-w)^3}.$$

Iterating this procedure, for the ith series in (15) we get a representation as a rational function:

$$\sum_{x_j \geqslant 0} x_i^{k_i} \gamma_{(j),i}^{x_i} z_i^{x_i} = \frac{M_i(\gamma_{(j),i} z_i)}{(1 - \gamma_{(j),i} z_i)^{k_j + 1}},$$

where $M_i(w)$ is a polynomial of degree k.

Now we turn back to the equality (15). After summation over k (as a multi-index) for the generating function $F(z)$ we get an expression

$$F(z) = \sum_{j=1}^{m} \sum_{0 \leqslant k < d_{(j)}} l_k \frac{\prod\limits_{i=1}^{n} M_i(\gamma_{(j),i} z_i)}{(I - \gamma_{(j)} z)^{k+I}} = \sum_{j=1}^{m} \frac{b_j(z)}{(I - \gamma_{(j)} z)^{\mu_{(j)}}}$$

with polynomials $b_j(z)$ of the form $\sum\limits_{0 \leqslant \alpha < d_{(j)}} b_\alpha^j z^\alpha$ as in (i).

The proof of equivalence $(iii) \approx (ii)$. Let $\alpha, k \in \mathbb{Z}^n_{\geqslant}$, the inequality $\alpha \leqslant k$ means that $\alpha_j \leqslant k_j$ for $j = 1, \ldots, n$, and we write $\alpha \nleqslant k$ if for some $j_0, 1 \leqslant j_0 \leqslant n$ we have $\alpha_{j_0} > k_{j_0}$.

For a difference operator $\delta - a = (\delta_1 - a_1, \ldots, \delta_n - a_n)$, $a \in \mathbb{C}^n$ there holds the following property: for any $\alpha \nleqslant k$ one has $(\delta - a)^\alpha x^k a^x = 0$, $x \in \mathbb{Z}^n_{\geqslant}$.

For $n = 1$ this is checked by direct computation, and for $n > 1$ it follows from $(\delta - a)^\alpha x^k a^x = \prod\limits_{j=1}^{n} (\delta_j - a_j)^{\alpha_j} x^{k_j} a_j^{x_j}$.

If a polynomial $Q_i(z)$ satisfies (4), (5) of the principal theorem at the point $z = a$, then it can be represented as $Q_i(z) = \sum\limits_{\substack{\alpha \geqslant 0 \\ \alpha \nleqslant d_{i,a}}} c_\alpha^{(i)} (z - a)^\alpha$ for some $d_a = (d_{a,1}, \ldots, d_{a,n})$. Thus, for any $k < d_a$ and any $\alpha \geqslant 0, \alpha \nleqslant d_{i,a}$ there exists $\alpha_{j_0} > k_{j_0}$, therefore $Q(\delta)[x^k a^x] = 0$ for $x \in \mathbb{Z}^n_{\geqslant}$.

If $a \in V_Q$ is a point of the characteristic set of the system (ii) and $f(x) \in V_3$, then from what we have shown above follows that $Q_i(\delta) f(x) = 0$, $x \in \mathbb{Z}^n_{\geqslant}$, $i = 1, 2, \ldots, n$. Therefore, $V_3 \subset V_2$ and due to $dim\, V_3 = dim\, V_2$ we get $V_3 = V_2$.

As an example we consider the following system of equations satisfying (4) and (5):

$$\begin{cases} Q_1 = (z_1 - 1)^2 + (z_2 - 1)^2 + \dfrac{1}{3}(z_1 - 1)^2 (z_2 - 1)^2; \\ Q_2 = (z_1 - 1)^2 - (z_2 - 1)^2 + \dfrac{2}{3}(z_1 - 1)^2 (z_2 - 1)^2. \end{cases} \tag{17}$$

This system is a superposition of the mappings

$$\begin{cases} x_1 = (z_1 - 1)^2; \\ x_2 = (z_2 - 1)^2 \end{cases} \quad \text{and} \quad \begin{cases} y_1 = x_1 + x_2 + \dfrac{1}{3} x_1 x_2; \\ y_2 = x_1 - x_2 + \dfrac{2}{3} x_1 x_2 \end{cases}.$$

The second mapping has in $\mathbb{C}^2$ two zeroes: $(0, 0)$ and $(6, -2)$ (and two zeroes at infinity in accordance with Bezout's theorem). Therefore the system (17) has in $\mathbb{C}^2$ eight zeroes: $(1, 1)$ of multiplicity 4 (in accordance with formula (8)) and 4 simple zeroes $(\gamma_1^\pm, \gamma_2^\pm) = (1 \pm \sqrt{6}, 1 \pm i\sqrt{2})$.

The series representation of the generating function from (i) in Theorem 1 is

$$F(z) = \sum_{\varepsilon, \delta} \frac{b_{\varepsilon\delta}}{(1 - \gamma_1^\varepsilon z_1)(1 - \gamma_2^\delta z_2)} + \frac{b_5(z)}{(1 - z_1)^2 (1 - z_2)^2},$$

where $b_{\varepsilon\delta}$ are constants (polynomials of zero degree), and $b_5(z)$ is a polynomial of the form $b_{00} + b_{10}(z_1 - 1) + b_{01}(z_2 - 1) + b_{11}(z_1 - 1)(z_2 - 1)$.

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О многомерном варианте основной теоремы разностных уравнений с постоянными коэффициентами

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Аннотация. В работе рассматриваются системы линейных разностных уравнений с дискретным характеристическим множеством. Сформулирован и доказан многомерный вариант основной теоремы линейных разностных уравнений с постоянными коэффициентами.

Ключевые слова: линейные разностные уравнения, характеристическое множество, кратность корней.