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Density Problem Some of the Functional Spaces for Studying Dynamic Equations on Time Scales

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Abstract. In this paper we study some topological density properties of certain functional spaces on the time scales and its relationships to Lebesgue spaces in the sense of ∇-integrals on time scales. Our results are provided with applications.

Keywords: time scale, density, measure.

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1. Introduction and some preliminaries

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [9] in 1988, in order to unify and generalize continuous and discrete analysis, see [2, 6]. Bohner and Guseinov have introduced the Lebesgue Δ-integral [6, Chapter 5]. Recent results, by A.Cabada, D.Vivero in [8], are devoted on fundamental relations between Riemann and Lebesgue Δ-integrals. In

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2006, R. Agarwal et *al.* in [1], define and study Sobolev spaces in the sense of Δ-integrals on time scales. After such pioneer work, the study of density properties in the Lebesgue spaces and Sobolev spaces in the sense of the Δ-derivative on time scales was continued by A. Benaissa et al. in [3, 4]. B. Bendouma et al. in [5] presented a relationship between Riemann and Lebesgue ∇-integrals.

The main purpose of this paper is to be investigated some important functional spaces. We deduct some topological density properties of the considered functional spaces.

The paper is organized as follows. In the next section, we give some auxiliary results needed to be proved our main results. Our results are represented in Section 3. We study functional subspaces on the time scales by report Lebesgue spaces in the sense of ∇-integrals on time scales. For example, the space of continuous functions $C(\mathbb{T}, \mathbb{R})$, the space of ld-continuous functions $C_{ld}(\mathbb{T}, \mathbb{R})$, and so on. In the last section, we present the use of density properties and we present diagrams that summarizes our main results.

2. Auxiliary results

A time scale is an arbitrary nonempty closed subset of the real numbers. We will denote it by $\mathbb{T}$. We define the forward and backward jump operators $\sigma, \varrho : \mathbb{T} \to \mathbb{T}$ as follows

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \qquad \text{and} \qquad \varrho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively, where $\sup \mathbb{T} = \inf \emptyset$, $\inf \mathbb{T} = \sup \emptyset$. The point $t \in \mathbb{T}$ is said to be left-dense, if $\varrho(t) = t$ and $t > \inf \mathbb{T}$, left-scattered if $\rho(t) < t$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, and right-scattered if $\sigma(t) > t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ and the backward graininess function $v : \mathbb{T} \to [0, \infty)$ is defined by $v(t) = t - \varrho(t)$. If $\mathbb{T}$ has a right-scattered infimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered supremum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 (Nabla derivative [2, 6]). *Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}_k$. We define*

$$f^{\nabla}(t) = \lim_{s \to t, s \in \mathbb{T}} \frac{f^{\varrho}(t) - f(s)}{\varrho(t) - s},$$

provided the limit exists. We call $f^{\nabla}(t)$ the nabla derivative of f at t. Moreover, we say that f is nabla differentiable on $\mathbb{T}_k$ provided $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$. The function $f^{\nabla} : \mathbb{T}_k \to \mathbb{R}$ is then called the (nabla) derivative of f on $\mathbb{T}_k$.

Definition 2.2 (Delta derivative [2, 6]). *Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$. We define*

$$f^{\Delta}(t) = \lim_{s \to t, s \in \mathbb{T}} \frac{f^{\sigma}(t) - f(s)}{\sigma(t) - s},$$

provided the limit exists. We call $f^{\Delta}(t)$ the delta derivative of f at t. Moreover, we say that f is nabla differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}_k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Definition 2.3 ([2, 6]). *The function $f : \mathbb{T} \to \mathbb{R}$ is called*

1) *ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and right-sided limits exist at right-dense points in $\mathbb{T}$. The space of all ld-continuous functions on $\mathbb{T}$ will be denoted by $\mathcal{C}_{ld}(\mathbb{T}, \mathbb{R})$ or $\mathcal{C}_{ld}(\mathbb{T})$.*

2) *rd-continuous provided it is continuous at each right-dense points and has a left-sided limit at each point. The space of all rd-continuous functions on $\mathbb{T}$ will be denoted by $\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$ or $\mathcal{C}_{rd}(\mathbb{T})$.*

Lemma 2.1 ([5,8])**.** *The set of all right-scattered points and the set of all left-scattered points of $\mathbb{T}$ are at most countable.*

We recall some notions and results related to the theory of ∇-measure (respectively Δ-measure) and Lebesgue ∇-integration (respectively Δ-integration) for an arbitrary bounded time scale $\mathbb{T}$, where $-\infty < a = \inf\mathbb{T} < \sup\mathbb{T} = b < \infty$. For more details we refer the reader to [3,4,8].

Lemma 2.2 ([5,8])**.** *Let $A \subset \mathbb{T}$. Then the following properties are equivalent*

1) *A is a ∇-measurable,*
2) *A is a Δ-measurable,*
3) *A is Lebesgue measurable.*

Notation 2.1. *For simplification, we note*

$$\mathcal{R} = \{t \in \mathbb{T}, \sigma(t) > t\} \quad \textit{and} \quad \mathcal{L} = \{t \in \mathbb{T}, \varrho(t) < t\}.$$

Proposition 2.1 ([5,8])**.** *Let $A \subset \mathbb{T}$ be a Lebesgue measurable set. Then the following properties hold.*

1) *If $a \notin A$, then $\mu_{\nabla}(A) = \mu_L(A) + \sum\limits_{s\in\mathcal{L}\cap A} \nu(s)$,*
2) *If $b \notin A$, then $\mu_{\Delta}(A) = \mu_L(A) + \sum\limits_{s\in\mathcal{R}\cap A} \mu(s)$,*
3) *$\mu_{\nabla}(A) = \mu_L(A)$ if and only if $a \notin A$ and A has no left-scattered points,*
4) *$\mu_{\Delta}(A) = \mu_L(A)$ if and only if $b \notin A$ and A has no right-scattered points.*

Theorem 2.1 ([8])**.** *Let $A \subset \mathbb{T}$ be a ∇-measurable set such that $a \notin A$. Let also, $f : \mathbb{T} \to \mathbb{R}$ be a ∇-measurable function. Then*

$$\int_A f(t)\,\nabla t = \int_A f(t)\,dt + \sum_{s\in\mathcal{L}\cap A} \nu(s)\,f(s). \tag{1}$$

Theorem 2.2 ([5])**.** *Let $A \subset \mathbb{T}$ be a Δ-measurable set such that $b \notin A$. Let also, $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then*

$$\int_A f(t)\,\Delta t = \int_A f(t)\,dt + \sum_{s\in\mathcal{R}\cap A} \mu(s)\,f(s)\,. \tag{2}$$

We state some of their properties.

Definition 2.4 ([3])**.** *Let $p \in [1,+\infty)$. Then, the set $L^p_{\Delta}(\mathbb{T},\mathbb{R})$ is a Banach spaces together with the norm defined for every $f \in L^p_{\Delta}(\mathbb{T},\mathbb{R})$ as follows*

$$\|f\|^p_{L^p_{\Delta}} = \int_{\mathbb{T}} |f(s)|^p \Delta s < \infty.$$

Theorem 2.3 ([4])**.** *Let $p \in [1,\infty)$. Then $C(\mathbb{T},\mathbb{R})$ is dense in $L^p_{\Delta}(\mathbb{T},\mathbb{R})$.*

3. Main results

In this section, assume that $\mathbb{T}$ is a bounded time scale with $a = \sup \mathbb{T} < \infty$ and $b = \inf \mathbb{T} > -\infty$. For simplification, we note $\mathbb{T}_a = \mathbb{T} \cap (a, b]$ and $\mathbb{T}_b = \mathbb{T} \cap [a, b)$.

Definition 3.1. *Let $p \in [1, +\infty)$ and $f : \mathbb{T} \to \mathbb{R}$ be a ∇-measurable function. We say that $f \in L^p_\nabla(\mathbb{T}, \mathbb{R})$ provided*

$$\|f\|^p_{L^p_\nabla} = \int_{\mathbb{T}_a} |f(s)|^p \nabla s < +\infty. \tag{3}$$

Remark 3.1. *Let $p \in [1, +\infty)$. Then the spaces $L^p_\nabla(\mathbb{T}, \mathbb{R})$ and $L^p_\Delta(\mathbb{T}, \mathbb{R})$ are not even spaces. Really, let $\mathbb{T} = \{a_1, a_2, a_3, a_4\}$ be such that $a_1 < \ldots < a_4$ and let $f_1, f_2 : \mathbb{T} \to \mathbb{R}$ be two functions such that $f_1(t) = (t - a_1)^{-1}$, $f_2(t) = (t - a_4)^{-1}$. Then $f_1 \in L^p_\nabla(\mathbb{T}, \mathbb{R})$, $f_1 \notin L^p_\Delta(\mathbb{T}, \mathbb{R})$, $f_2 \notin L^p_\nabla(\mathbb{T}, \mathbb{R})$ and $f_2 \in L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

Lemma 3.1. *Let $p \in [1, +\infty)$, and $f : \mathbb{T} \to \mathbb{R}$ be a ∇-measurable function. Then $f \in L^p_\nabla(\mathbb{T}, \mathbb{R})$ if and only if $f^\sigma \in L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. Let s is left-scattered. Then $\rho(s)$ is right-scattered and $\sigma(\rho(s)) = s$. Hence,

$$v(s) = s - \rho(s) = \sigma(\rho(s)) - \rho(s) = \mu(\rho(s))$$

and $\rho(\mathcal{L}) \subseteq \mathcal{R}$. Take $f \in L^p_\nabla(\mathbb{T}, \mathbb{R})$ arbitrarily. By Definition 3.1, Theorem 2.1 and Theorem 2.2, we have

$$\begin{aligned} \|f\|^p_{L^p_\nabla} &= \int_{\mathbb{T}_a} |f(s)|^p \, dt + \sum_{s \in \mathcal{L}} v(s) |f(s)|^p = \\ &= \int_{\mathbb{T}_b} |f^\sigma(s)|^p \, dt + \sum_{s \in \mathcal{L}} \mu(\varrho(s)) |f^\sigma(\varrho(s))|^p . \end{aligned}$$

If $s \in \mathcal{R}$, then $\sigma(s) \in \mathcal{L}$ and $\varrho(\sigma(s)) = s$. Therefore $\mathcal{R} \subseteq \varrho(\mathcal{L})$ and $\varrho(\mathcal{L}) = \mathcal{R}$. Consequently

$$\|f\|^p_{L^p_\nabla} = \int_{\mathbb{T}_b} |f^\sigma(t)|^p \, dt + \sum_{\tau \in \mathcal{R}} \mu(\tau) |f^\sigma(\tau)|^p = \|f^\sigma\|^p_{L^p_\Delta} .$$

This completes the proof. □

Corollary 3.1. *Let $p \in [1, \infty)$. Then $L^p_\nabla(\mathbb{T}, \mathbb{R})$ is a Banach spaces equipped with the norm* (3).

Proof. It is clear that $L^p_\nabla(\mathbb{T}, \mathbb{R})$ is a normed space. Let $(f_\varepsilon)_\varepsilon$ be a Cauchy sequence in $L^p_\nabla(\mathbb{T}, \mathbb{R})$. By Lemma 3.1, we deduce that $(f^\sigma_\varepsilon)_\varepsilon$ is a Cauchy sequence in $L^p_\Delta(\mathbb{T}, \mathbb{R})$, which is a Banach space. Then there exists $g^\sigma \in L^p_\Delta(\mathbb{T}, \mathbb{R})$ such that $\lim\limits_{\varepsilon \to 0} f^\sigma_\varepsilon = g^\sigma$ in $L^p_\Delta(\mathbb{T}, \mathbb{R})$. Hence, $\lim\limits_{\varepsilon \to 0} f_\varepsilon = g$ in $L^p_\nabla(\mathbb{T}, \mathbb{R})$. This completes the proof. □

Lemma 3.2. *Let $p \in [1, +\infty)$ and $f \in \mathcal{C}_{ld}(\mathbb{T})$. For $\varepsilon > 0$, there is a continuous function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ such that $\|f - f_\varepsilon\|_{L^p_\nabla} \to 0$, as $\epsilon \to 0$.*

Proof. Let s be right-dense. Then there exist an $\epsilon > 0$ and an element $s_\varepsilon \in \mathbb{T}$ such that $0 < s_\varepsilon - s < \varepsilon$. Consider the function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ defined by

$$f_\varepsilon(t) = \begin{cases} f(s) + \dfrac{f(s) - f(s_\varepsilon)}{s - s_\varepsilon}(t - s), & t \in [s, s_\varepsilon]_{\mathbb{T}}, \, s \in \mathbb{T} \backslash \mathcal{R}, \\ f(t), & \text{otherwise}. \end{cases}$$

Note that $f_\varepsilon \in \mathcal{C}_{ld}(\mathbb{T})$. Let $t \in [s, s_\varepsilon]_\mathbb{T}$ and $s \in \mathbb{T}\backslash\mathcal{R}$. Then

$$\begin{aligned} |f_\varepsilon(t) - f(t)| &\leqslant |f(s)| + |f(t)| + |f(s) - f(s_\epsilon)| \left|\frac{t-s}{s-s_\epsilon}\right| \leqslant \\ &\leqslant |f(s)| + |f(t)| + |f(s) - f(s_\epsilon)| \leqslant \\ &\leqslant |f(s)| + |f(t)| + |f(s)| + |f(s_\epsilon)| \leqslant \\ &\leqslant 4\|f\|_\infty. \end{aligned}$$

Set $B_\varepsilon = \bigcup_{s\in\mathcal{L}} [s, s_\varepsilon]_\mathbb{T}$. By Proposition 2.1, we get

$$\begin{aligned} \mu_\nabla(B_\varepsilon) &= \sum_{s\in\mathbb{T}\backslash\mathcal{R}} \mu_\nabla([s, s_\varepsilon]_\mathbb{T}) = \sum_{s\in\mathcal{L}} \lambda([s, s_\varepsilon]) + \sum_{s\in\mathcal{L}} \sum_{t\in\mathcal{L}\cap[s,s_\epsilon]} v(t) \leqslant \\ &\leqslant \sum_{s\in\mathbb{T}\backslash\mathcal{R}} \epsilon + \sum_{s\in\mathbb{T}\backslash\mathcal{R}} \epsilon^2 = \\ &= \sum_{s\in\mathbb{T}\backslash\mathcal{R}} (\epsilon + \epsilon^2) \leqslant \\ &\leqslant (\epsilon + \epsilon^2)(b-a). \end{aligned}$$

As a result, we find

$$\|f - f_\varepsilon\|_{L^p_\nabla} \leqslant 4\|f\|_\infty [\mu_\Delta(B_\varepsilon)]^{\frac{1}{p}} \leqslant 4\|f\|_\infty \left(\epsilon + \epsilon^2\right)^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \to 0, \quad \text{as} \quad \epsilon \to 0.$$

This completes the proof. □

For $t_0 \in \mathbb{T}$, define the generalized polynomials as follows

$$h_0(t, t_0) = 1 \quad \text{and} \quad h_n(t, t_0) = \int_{t_0}^{t} h_{n-1}(\tau, t_0) \nabla\tau \text{ for } n \in \mathbb{N},\ t \in \mathbb{T}. \tag{4}$$

Lemma 3.3. *Let $t_0, t \in \mathbb{T}$ be such that $t \geqslant t_0$. Then, for all $n \in \mathbb{N}$, there are positive constants α_n and β_n such that*

$$\beta_n (t - t_0)^n \leqslant h_n(t, t_0) \leqslant \alpha_n (t - t_0)^n \quad \textit{for all } t \geqslant t_0. \tag{5}$$

Proof. For $n = 1$, we have $h_1(t, t_0) = t - t_0$ and for $\alpha_1 = \beta_1 = 1$, we get (5). Now, assume that (5) holds for $n-1$ for some $n \in \mathbb{N}$, $n \geqslant 2$. We will prove (5) for n. From Theorem 2.1, we have

$$\begin{aligned} h_n(t, t_0) &\leqslant \alpha_{n-1} \int_{t_0}^{t} (\tau - t_0)^{n-1} \nabla\tau = \qquad (6) \\ &= \alpha_{n-1} \int_{t_0}^{t} (\tau - t_0)^{n-1} d\tau + \alpha_{n-1} \sum_{\tau\in[t_0,t]\cap\mathcal{L}} v(\tau) (\tau - t_0)^{n-1} \leqslant \\ &\leqslant \alpha_{n-1} \frac{n+1}{n} (t - t_0)^n \text{ for } n \in \mathbb{N},\ t \geqslant t_0. \qquad (7) \end{aligned}$$

Moreover, we have

$$\begin{aligned} h_n(t, t_0) &\geqslant \beta_{n-1} \int_{t_0}^{t} (\tau - t_0)^{n-1} \nabla\tau = \\ &= \beta_{n-1} \int_{t_0}^{t} (\tau - t_0)^{n-1} d\tau + \beta_{n-1} \sum_{\tau\in[t_0,t]\cap\mathcal{L}} v(\tau) (\tau - t_0)^{n-1} \geqslant \\ &\geqslant \frac{\beta_{n-1}}{n} (t - t_0)^n \text{ for } n \in \mathbb{N},\ t \geqslant t_0. \end{aligned}$$

By the last inequality and by (7), we get that (5) holds for n. Hence, we conclude that (5) holds for any $n \in \mathbb{N}$. This completes the proof. $\square$

Lemma 3.4. *Let $p \in [1, +\infty)$ and $f : \mathbb{T} \to \mathbb{R}$ be ∇-differentiable such that $f^{\nabla} \in \mathcal{C}_{ld}(\mathbb{T})$. For any $\varepsilon > 0$, there is a function $f_{\varepsilon} : \mathbb{T} \to \mathbb{R}$ that is ∇-differentiable and $f_{\varepsilon}^{\nabla} \in \mathcal{C}(\mathbb{T})$, $\|f - f_{\varepsilon}\|_{L^p_{\nabla}} \to 0$ and $\|f^{\nabla} - f_{\varepsilon}^{\nabla}\|_{L^p_{\nabla}} \to 0$, as $\varepsilon \to 0$.*

Proof. Let s be right-dense. Then for any $\epsilon > 0$ there exists $s_{\varepsilon} \in \mathbb{T}$ such that $0 < s_{\varepsilon} - s < \varepsilon$. Consider the function $Q_{\varepsilon}(s, .) : [s, s_{\varepsilon}] \to \mathbb{R}$ defined by

$$Q_{\varepsilon}(s,t) = f(s) + f^{\nabla}(s)(t-s) + \lambda_{\varepsilon} h_2(t,s) + \gamma_{\varepsilon} h_3(t,s)$$

and the function $f_{\varepsilon} : \mathbb{T} \to \mathbb{R}$ given by

$$f_{\varepsilon}(t) = \begin{cases} Q_{\varepsilon}(s,t), & t \in [s, s_{\varepsilon}]_{\mathbb{T}},\ s \in \mathbb{T} \backslash \mathcal{R}, \\ f(t), & \text{otherwise.} \end{cases} \tag{8}$$

It is clear that the function f_{ε} is continuous at all points $t \in \mathbb{R} \backslash \{s_{\varepsilon} : s \in \mathbb{T} \backslash \mathcal{R}\}$.We choose the constants $\lambda_{\varepsilon}, \gamma_{\varepsilon}$ such that $\lim\limits_{t \to s_{\varepsilon}^{+}} f(t) = \lim\limits_{t \to s_{\varepsilon}^{-}} Q_{\varepsilon}(s,t)$. We find

$$\lambda_{\varepsilon} h_2(s_{\varepsilon}, s) + \gamma_{\varepsilon} h_3(s_{\varepsilon}, s) = f(s_{\varepsilon}) - f(s) - f^{\nabla}(s) h_1(s_{\varepsilon}, s). \tag{9}$$

Moreover, f_{ε} is ∇-differentiable at all points $t \in \mathbb{R} \backslash \{s_{\varepsilon} : s \in \mathbb{T} \backslash \mathcal{R}\}$ and f_{ε}^{∇} is given by

$$f_{\varepsilon}^{\nabla}(t) = \begin{cases} f^{\nabla}(s) + \lambda_{\varepsilon} h_1(t,s) + \gamma_{\varepsilon} h_2(t,s), & t \in [s, s_{\varepsilon}]_{\mathbb{T}},\ s \in \mathbb{T} \backslash \mathcal{R}, \\ f^{\nabla}(t), & \text{otherwise.} \end{cases}$$

Note that f_{ε}^{∇} is continuous at all points $t \in \mathbb{R} \backslash \{s_{\varepsilon} : s \in \mathbb{T} \backslash \mathcal{R}\}$. If $\lim\limits_{t \to s_{\varepsilon}^{+}} f^{\nabla}(t) = \lim\limits_{t \to s_{\varepsilon}^{-}} Q_{\varepsilon}^{\nabla}(s,t)$, we find

$$\lambda_{\varepsilon} h_1(s_{\varepsilon}, s) + \gamma_{\varepsilon} h_2(s_{\varepsilon}, s) = f^{\nabla}(s_{\varepsilon}) - f^{\nabla}(s). \tag{10}$$

By (9) and (10), we have

$$\begin{bmatrix} \lambda_{\varepsilon} \\ \gamma_{\varepsilon} \end{bmatrix} = \begin{bmatrix} h_2(s_{\varepsilon}, s) & h_3(s_{\varepsilon}, s) \\ h_1(s_{\varepsilon}, s) & h_2(s_{\varepsilon}, s) \end{bmatrix}^{-1} \begin{bmatrix} f(s_{\varepsilon}) - f(s) - f^{\nabla}(s) h_1(s_{\varepsilon}, s) \\ f^{\nabla}(s_{\varepsilon}) - f^{\nabla}(s) \end{bmatrix}.$$

By Lemma 3.3, we obtain

$$|\lambda_{\varepsilon}| \leqslant \frac{r}{\varepsilon} \left[\|f\|_{\infty} + \|f^{\nabla}\|_{\infty}\right] \quad \text{and} \quad |\gamma_{\varepsilon}| \leqslant \frac{r}{\varepsilon} \left[\|f\|_{\infty} + \|f^{\nabla}\|_{\infty}\right] \tag{11}$$

for some positive constant r. Thus,

$$|f_{\varepsilon}(t) - f(t)| \leqslant \xi = C\left[\|f\|_{\infty} + \|f^{\nabla}\|_{\infty}\right] \quad \text{and} \quad \left|f_{\varepsilon}^{\nabla}(t) - f^{\nabla}(t)\right| \leqslant \xi$$

for some positive constant C. Then

$$\|f - f_{\varepsilon}\|_{L^p_{\nabla}} \leqslant \xi \left[\mu_{\Delta}(B_{\varepsilon})\right]^{\frac{1}{p}} \to 0, \quad \text{as} \quad \epsilon \to 0,$$

and

$$\|f^{\nabla} - f_{\varepsilon}^{\nabla}\|_{L^p_{\nabla}} \leqslant \xi \left[\mu_{\Delta}(B_{\varepsilon})\right]^{\frac{1}{p}} \to 0, \quad \text{as} \quad \epsilon \to 0,$$

where B_{ε} is defined as in the proof of Lemma 3.2. This completes the proof. $\square$

Lemma 3.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be continuous. For any $\varepsilon > 0$, there is a function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ that is ∇-differentiable and $f_\varepsilon^\nabla \mathcal{C}_{ld}(\mathbb{T})$, $\|f - f_\varepsilon\|_\infty \to 0$, as $\varepsilon \to 0$.*

Proof. Define the function $g : \mathbb{T} \to \mathbb{R}$ as follows

$$g(t) = \begin{cases} f(s) + f^\nabla(s)(t-s), & t \in (\varrho(s), s], s \in \mathcal{L}, \\ f(t), & \text{otherwise} . \end{cases}$$

Then $g \in \mathcal{C}(\mathbb{T})$. Since $C^1(\mathbb{T})$ is dense in $C(\mathbb{T})$, for $\varepsilon > 0$, we have $B_\varepsilon(g) \cap C^1(\mathbb{T}) \neq 0$, where $B_\varepsilon(g) = \{h \in C(\mathbb{T}) : \|g - h\|_\infty < \varepsilon\}$. Then there is $g_\varepsilon \in C^1(\mathbb{T})$ such that $\|g_\varepsilon - g\|_\infty < \varepsilon$. Now, consider the function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$, defined by $f_\varepsilon = g_\varepsilon$. Then f_ε is ∇-differentiable on $\mathbb{T}_k$ and f_ε^∇ is given by

$$f_\varepsilon^\nabla(t) = \begin{cases} g_\varepsilon^{'}(t), & t \in \mathbb{T} \backslash \mathcal{L}, \\ \dfrac{g_\varepsilon(\varrho(s)) - g_\varepsilon(s)}{v(s)}, & t \in \mathcal{L}. \end{cases}$$

We have that $f_\varepsilon \in C_{ld}^1(\mathbb{T}, \mathbb{R})$ and $\|f_\varepsilon - f\|_\infty \leqslant \|g_\varepsilon - g\|_\infty < \varepsilon$. This completes the proof. □

Lemma 3.6. *Let $f : \mathbb{T} \to \mathbb{R}$ be continuous. For any $\varepsilon > 0$, there is a function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ that is ∇-differentiable and $f_\varepsilon^\nabla \in \mathcal{C}_{ld}(\mathbb{T})$ and $\|f - f_\varepsilon\|_{L_\nabla^p} \to 0$, as $\varepsilon \to 0$.*

Proof. Let $f : \mathbb{T} \to \mathbb{R}$ be continuous and $\varepsilon > 0$. From Lemma 3.5, it follows that there is $f_\varepsilon \in C_{ld}(\mathbb{T}, \mathbb{R})$ such that $\|f - f_\varepsilon\|_\infty \to 0$, as $\varepsilon \to 0$. Hence, $\|f - f_\varepsilon\|_{L_\nabla^p} \leqslant (b-a)\|f - f_\varepsilon\|_\infty \to 0$, as $\varepsilon \to 0$. This completes the proof. □

Remark 3.2. *Note that $C(\mathbb{T}, \mathbb{R}), C_{rd}(\mathbb{T}, \mathbb{R})$ and $C_{ld}(\mathbb{T}, \mathbb{R})$ are Banach spaces together with the norm defined by*

$$\|f\|_\infty = \sup\{|f(t)| : t \in \mathbb{T}\}.$$

Lemma 3.7. *Let $p \in [1, +\infty)$, $f : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable and $f^\nabla \in \mathcal{C}_{ld}(\mathbb{T})$. For any $\varepsilon > 0$ there is a function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ that is ∇-differentiable and f_ε^∇ is continuous such that $\|f - f_\varepsilon\|_\infty \to 0$, as $\varepsilon \to 0$.*

Proof. Consider the function $f_\varepsilon : \mathbb{T} \to \mathbb{R}$ given by (8). By the proof of Lemma 7, we find

$$\|f_\varepsilon - f\|_\infty \leqslant \lambda\varepsilon \left[\|f\|_\infty + \left\|f^\nabla\right\|_\infty\right]$$

for some positive constant λ. This completes the proof. □

The next result is a generalization of the theoretical density(see [7, Theorem 4.3]).

Theorem 3.1. *Let $p \in [1, \infty)$. Then $C_{ld}(\mathbb{T}, \mathbb{R})$ is dense in $L_\nabla^p(\mathbb{T}, \mathbb{R})$.*

Proof. Let $f \in L_\nabla^p(\mathbb{T}, \mathbb{R})$. By Lemma 3.1, we have $f^\sigma \in L_\Delta^p(\mathbb{T}, \mathbb{R})$. Then there exists a sequence $(g_n^\sigma)_n \in C(\mathbb{T}, \mathbb{R})$ that converges to f^σ in $L_\Delta^p(\mathbb{T}, \mathbb{R})$. Set $(f_n)_n = (g_n)_n$. We have $(f_n)_n \subseteq C_{ld}(\mathbb{T}, \mathbb{R})$. Therefore

$$\|f - f_n\|_{L_\nabla^p}^p = \|f^\sigma - g_n^\sigma\|_{L_\Delta^p}^p.$$

Hence, $(f_n)_n$ converges to f in $L_\nabla^p(\mathbb{T}, \mathbb{R})$. This completes the proof. □

We denote by

$$C^1_\nabla(\mathbb{T},\mathbb{R}) = \left\{ f : \mathbb{T} \to \mathbb{R} : \text{such as } f^\nabla \in C(\mathbb{T}_k,\mathbb{R}) \right\},$$
$$C^1_{rd}(\mathbb{T},\mathbb{R}) = \left\{ f : \mathbb{T} \to \mathbb{R} : \text{such as } f^\nabla \in C_{rd}(\mathbb{T}_k,\mathbb{R}) \right\}.$$

The following result is a consequence of Theorem 3.1.

Theorem 3.2. *Let $p \in [1,\infty)$. Then $C^1_\nabla(\mathbb{T},\mathbb{R})$ is dense in $L^p_\nabla(\mathbb{T},\mathbb{R})$.*

Proof. Let $f \in L^p_\nabla(\mathbb{T},\mathbb{R})$ and $\varepsilon > 0$. From Theorem 3.1, we have $B_{\frac{\varepsilon}{4}}(f) \cap C_{ld}(\mathbb{T},\mathbb{R}) \neq 0$, where $B_\varepsilon(f) := \left\{ g \in L^p_\nabla(\mathbb{T},\mathbb{R}) : \|g-f\|_{L^p_\nabla} < \varepsilon \right\}$. Then there is $f_{\varepsilon,1} \in C_{ld}(\mathbb{T},\mathbb{R})$ such that $\|f_{\varepsilon,1} - f\|_{L^p_\nabla} < \frac{\varepsilon}{4}$. By Lemma 3.2, we conclude that there is $f_{\varepsilon,2} \in C(\mathbb{T},\mathbb{R})$ such that $\|f_{\varepsilon,2} - f_{\varepsilon,1}\|_{L^p_\nabla} < \frac{\varepsilon}{4}$. By Lemma 3.6, there exists is $f_{\varepsilon,3} \in C^1_{ld}(\mathbb{T},\mathbb{R})$ such that $\|f_{\varepsilon,2} - f_{\varepsilon,3}\|_{L^p_\nabla} < \frac{\varepsilon}{4}$. From Lemma 3.4, there exists is $f_{\varepsilon,4} \in C^1(\mathbb{T},\mathbb{R})$ so that $\|f_{\varepsilon,4} - f_{\varepsilon,3}\|_{L^p_\nabla} < \frac{\varepsilon}{4}$. Therefore, we find $\|f_{\varepsilon,4} - f\|_{L^p_\nabla} < \varepsilon$, which implies that $B_\varepsilon(f) \cap C^1_\nabla(\mathbb{T},\mathbb{R}) \neq 0$. Then $C^1_\nabla(\mathbb{T},\mathbb{R})$ is dense in $L^p_\nabla(\mathbb{T},\mathbb{R})$. This completes the proof. □

Remark 3.3. *Let E, F, G be three spaces such that $E \subset F \subset G$ and $(G, \mathcal{T})$ be a topological space. If E is dense in G, then F is dense in G.*

By the previous result, we deduce the following corollary.

Corollary 3.2. *Let $p \in [1,\infty)$. Then the spaces $C^1_{ld}(\mathbb{T},\mathbb{R}), C(\mathbb{T},\mathbb{R})$ and $C_{rd}(\mathbb{T},\mathbb{R})$ are dense in $L^p_\nabla(\mathbb{T},\mathbb{R})$.*

Proof. Let $p \in [1,\infty)$. We have $C^1_\nabla(\mathbb{T},\mathbb{R}) \subset C^1_{ld}(\mathbb{T},\mathbb{R}) \subset C(\mathbb{T},\mathbb{R}) \subset C_{ld}(\mathbb{T},\mathbb{R})$. From Theorem 3.1 and Remark 3.3, we conclude $C^1_{ld}(\mathbb{T},\mathbb{R}), C(\mathbb{T},\mathbb{R})$ and $C_{rd}(\mathbb{T},\mathbb{R})$ are dense in $L^p_\nabla(\mathbb{T},\mathbb{R})$. This completes the proof. □

The next result shows that the spaces $C^1_{ld}(\mathbb{T},\mathbb{R})$ and $C^1_\nabla(\mathbb{T},\mathbb{R})$ are dense in $C(\mathbb{T},\mathbb{R})$.

Theorem 3.3. *The spaces $C^1_{ld}(\mathbb{T},\mathbb{R})$ and $C^1_\nabla(\mathbb{T},\mathbb{R})$ are dense $C(\mathbb{T},\mathbb{R})$.*

Proof. Let $f \in C(\mathbb{T},\mathbb{R})$ and $\varepsilon > 0$. By Lemma 3.5, we have $B_{\frac{\varepsilon}{2}}(f) \cap C^1_{ld}(\mathbb{T},\mathbb{R}) \neq 0$, where $B_\varepsilon(f) = \{g \in C(\mathbb{T},\mathbb{R}) : \|f-g\|_\infty < \varepsilon\}$. Then there is $g_\varepsilon \in C^1_{ld}(\mathbb{T},\mathbb{R})$ such that $\|g_\varepsilon - f\|_\infty < \frac{\varepsilon}{2}$. By Lemma 3.7, we conclude that there is $h_\varepsilon \in C^1(\mathbb{T},\mathbb{R})$ such that $\|h_\varepsilon - g_\varepsilon\|_{L^p_\nabla} < \frac{\varepsilon}{2}$. Therefore $\|h_\varepsilon - f\|_{L^p_\nabla} < \varepsilon$ and hence, we conclude that $B_\varepsilon(f) \cap C^1_\nabla(\mathbb{T},\mathbb{R}) \neq 0$. This completes the proof. □

4. Conclusion and application

Use of density properties : To show some results concerning a given function f, it is sometimes useful to look at the problem with hindsight by placing yourself in a suitable functional spaces and using density properties of certain function subclasses. Thus, we are led to demonstrate the desired property for simpler functions. We give an application that can be attacked in the following way.

Lemma 4.1. *If $f \in L^1_\nabla(\mathbb{T},\mathbb{R})$ is such that*

$$\int_{\mathbb{T}} f(t)\,\varphi(t)\,\nabla t = 0, \qquad \textit{for } \varphi \in C_{ld}(\mathbb{T},\mathbb{R}), \tag{12}$$

then $f(t) = 0$, $\nabla-a.e$ in $\mathbb{T}$.

Proof. Let $f \in C_{ld}(\mathbb{T},\mathbb{R})$ be such that $\int\limits_{\mathbb{T}} f(t)\,\varphi(t)\,\nabla t = 0$. For $\varphi \in C_{ld}(\mathbb{T},\mathbb{R})$, take $\varphi = f$. We obtain $\|f\|^2_{L^2_\nabla(\mathbb{T},\mathbb{R})} = \int\limits_{\mathbb{T}} |f(t)|^2\,\nabla t = 0$, which implies $f(t) = 0$, $\nabla-$a.e in $\mathbb{T}$. So, the property (12) is verified for $f \in C_{ld}(\mathbb{T},\mathbb{R})$. As $C_{ld}(\mathbb{T},\mathbb{R})$ is dense in $L^1_\nabla(\mathbb{T},\mathbb{R})$, if $f \in L^1_\nabla(\mathbb{T},\mathbb{R})$ and $\varepsilon > 0$, then there is $f_\varepsilon \in C_{ld}(\mathbb{T},\mathbb{R})$ such that $\lim\limits_{\varepsilon\to 0} f_\varepsilon = f$ in $L^1_\nabla(\mathbb{T},\mathbb{R})$ and $\int\limits_{\mathbb{T}} f_\varepsilon(t)\,\varphi(t)\,\nabla t \to 0$, as $\varepsilon \to 0$. This completes the proof. ∎

We give the results found in the paper in the following diagram density between some of the functional spaces on time scales.

$$\begin{array}{ccccc}
C^1_{ld}(\mathbb{T},\mathbb{R}) & \to & C_{ld}(\mathbb{T},\mathbb{R}) & & \\
\downarrow & \searrow & \downarrow & & \\
C_{ld}(\mathbb{T},\mathbb{R}) & \to & L^p_\nabla(\mathbb{T},\mathbb{R}) & \to & L^1_\nabla(\mathbb{T},\mathbb{R}) \\
\uparrow & \nearrow & \uparrow & & \\
C^1_\nabla(\mathbb{T},\mathbb{R}) & \longrightarrow & C_{rd}(\mathbb{T},\mathbb{R}). & &
\end{array} \tag{13}$$

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Проблема плотности некоторых функциональных пространств для изучения динамических уравнений на временных масштабах

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Аннотация. В этой статье мы изучаем некоторые свойства топологической плотности некоторых функциональных пространств на временных масштабах и их отношения с пространствами Лебега в смысле ∇-интегралов на временных масштабах. Наши результаты снабжены приложениями.

Ключевые слова: шкала времени, плотность, мера.