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Satisfiability in a Temporal Multi-valueted Logic Based on $\mathbb{Z}$

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Abstract. In this paper we continue the series of papers by V. V. Rybakov devoted to properties of multi-valueted logics and where he propose a new approach for modelling knowledge and reasoning of agents in a multi-agent system. We prove that the satisfiability problem is decidable in a temporal multi-valueted logic based on $\mathbb{Z}$.

Keywords: temporal logic, multi-agent logic, epistemic modal logic, multi-valueted logic, satisfiability, decidability in logic, knowledge representation and reasoning, multi-agent systems.

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Introduction

In our paper we continue the research of V. V. Rybakov devoted to properties of multi-valueted logics ([17–20]). We consider a temporal multi-valueted logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$ and prove that the satisfiability problem in this logic is decidable.

Multi-valuation is a new approach developed by V. V. Rybakov for modelling and interpreting knowledge and reasoning of agents in a multi-agent system.

Multi-agent systems are one of the demanded directions in data science and artificial intelligence. They are used in online commerce, computer games, mobile and network technologies, geographic information systems, decision-making systems for emergency response, in medicine and the defense industry. Multi-agent systems have a complex architecture; a variety of software and mathematical tools are used in their design, including methods of mathematical logic. For example, using statements in various logical systems, one can set reasoning about agents, express their so-called mental properties: knowledge, beliefs, goals, etc.

Logical formalization of multi-agent systems is important since many of mental properties of agents should have a consistent description. At the beginning, first-order logic was used to formalize statements about agents. But many properties of agents are inexpressible in terms of the first-order predicate calculus, and first-order logics that underlie the description of properties of agents, are undecidable. The language of multimodal logics and logics of knowledge (which are a part of multimodal logics) turned out much more flexible and promising, especially the language of multimodal temporal logics since such logics have means for describing properties associated with real time, which is especially important in view of dynamic nature of an agent's

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and a multi-agent system's functioning. Therefore, a large number of studies concerning the logical properties of multi-agent systems lies in the field of multimodal temporal logics. Since the 1960s and since the first major work of Jaakoo Hintikka "Knowledge and belief: an introduction to the logic of the two notions" [11] on logics of knowledge, many works have been published on modal, epistemological and temporal logics and their applications in multi-agent systems and on related to the theory of such systems – game theory ([1, 4, 8, 9, 13–16, 21]). For example, the Nobel laureate in economics Robert Aumann applied a logic of knowledge in his research on the game theory for the analysis of economic systems, [1].

Now such scientists as M. Wooldridge and W. van der Hoek [12], F. Baader [2], R. Fagin, J. Halpern, Y. Moses and M. Vardi ([9, 10, 22, 23]), A. Perea ([14, 15]), J. van Benthem [7], V. V. Rybakov ([3, 5, 6, 17–20]) and others are working in this direction.

In the language of logics of knowledge and the language of multimodal logics with several modal operators $\Box_i$, in the language of temporal logics with several temporary operators of the same type, as well as in logics that simultaneously use logical operators of all three listed logics, it is possible to model reasoning about knowledge and behaviour of agents in multi-agent systems interpreting an index i of logical operators in a formula as knowledge of the i-th agent in a multi-agent system.

V. V. Rybakov proposed in his papers another way of constructing reasoning of agents, namely, through formulas of multi-valueted logics.

The main difference between multi-valueted logics and logics, the language of which contains operators of logics of knowledge, multimodal logics and k operators of one type of temporary logics, is the interpretation of the i-th logical operator of each type in formulas of such logics. In multi-valueted logics, each index i is associated not with a binary relation R_i and (or) with one of several binary relations Next_i, $\leqslant_i$, etc. on Kripke frame, but with the i-th valuation V_i in a selected model.

Examples of formulas in multi-valueted logics and their interpretation from the point of view of agents can be found in the papers of V. V. Rybakov mentioned above. In this article, we also provide examples of agent's reasoning for our logic.

1. Logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$

In Section 1 we introduce a temporal multi-valueted logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$, for which we investigate the satisfiability problem in this paper. The logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$ is defined as the set of formulas that are valid on certain relational multi-valueted models.

The alphabet of the language $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$ consists of a countable set of propositional variables $Prop = \{p_1, \ldots, p_i, \ldots\}$, brackets (,), Boolean logical operators $\{\neg, \wedge, \vee, \rightarrow\}$, modal operators $\{\Box_1, \Box_2, \ldots, \Box_k\}$ ("it is necessary that …"), and temporal operators $\{\mathcal{N}_i, \mathcal{S}_i, \mathcal{U}_i \mid i = 1, \ldots, k\}$ (that is, unary operators "NextTime" or "Tomorrow", and binary operators "Since" and "Untill").

Recall, that each modal operator $\Diamond_i$ ("it is possible that …") is defined by means of the modal operator $\Box_i$ in the following away: $\Diamond_i = \neg\Box_i\neg$, $i = 1, \ldots, k$.

Now we give an inductive definition of a formula in the language of $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$.

1. Any propositional variable $p \in Prop$ is a formula.
2. If A is a formula, then $\neg A$ is a formula.
3. If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are formulas.
4. If A is a formula, then $\Box_i A$ are formulas, $i = 1, \dots, k$.
5. If A is a formula, then $\mathcal{N}_i A$ is a formula, $i = 1, \dots, k$.
6. If A and B are formulas, then $(A\, \mathcal{S}_i\, B)$, $(A\, \mathcal{U}_i\, B)$ are formulas, $i = 1, \dots, k$.

There are no other formulas in the language of the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$. We omit the outer parentheses of the formulas in what follows.

Let there be given a non-empty set W, binary relations $R_1, \dots, R_k$ on this set: $R_i \subseteq W^2$, and a set $Prop$ of propositional variables.

Definition 1.1. *A relational multi-valueted model is a model*

$$\mathcal{M} = \langle W, R_1, \dots, R_k, V_1, \dots, V_t \rangle, \tag{1}$$

where $p \in Prop$: $V_i(p) \subseteq W$.

Definition 1.2. *A relation* Next *is a binary relation such that*

$$a \text{ Next } b \quad \textit{if and only if} \quad b = a + 1.$$

Further, for convenience, instead of a Next b we write $\mathrm{Next}(a) = b$.
Let $\leqslant$ be a standard linear order on $\mathbb{Z}$.

Definition 1.3. *Let $\mathcal{M}_{\mathbb{N}}$ be a following relational linear multi-valueted model:*

$$\mathcal{M}_{\mathbb{N}} = \langle \mathbb{N}, \leqslant, \mathrm{Next}, V_1, \dots, V_k \rangle. \tag{2}$$

Definition 1.4. *Let $\mathcal{M}_{\mathbb{Z}}$ be a following relational linear multi-valueted model:*

$$\mathcal{M}_{\mathbb{Z}} = \langle \mathbb{Z}, \leqslant, \mathrm{Next}, V_1, \dots, V_k \rangle. \tag{3}$$

Denote by $\mathcal{K}(\mathcal{M}_{\mathbb{N}})$ a set of all possible models $\mathcal{M}_{\mathbb{N}}$, and by $\mathcal{K}(\mathcal{M}_{\mathbb{Z}})$, consequently, a set of all possible models $\mathcal{M}_{\mathbb{Z}}$.
If $a, b \in \mathcal{K}(\mathcal{M}_{\mathbb{N}})$ or $a, b \in \mathcal{K}(\mathcal{M}_{\mathbb{Z}})$, then the fact when $a \leqslant b$ and $a \neq b$ we denote by $a < b$.
Let $\mathcal{M} \in \mathcal{K}(\mathcal{M}_{\mathbb{N}})$ or $\mathcal{M} \in \mathcal{K}(\mathcal{M}_{\mathbb{Z}})$. Define the truth of formulas in the model $\mathcal{M}$.

Definition 1.5. *For any points $a, b, c \in \mathcal{M}$ the following holds.*

$$\forall p \in Prop: \; a \Vdash_{V_j} p \iff a \in V_j(p),$$
$$a \Vdash_{V_j} \neg\varphi \iff a \nVdash_{V_j} \varphi,$$
$$a \Vdash_{V_j} (\varphi \wedge \psi) \iff a \Vdash_{V_j} \varphi \quad u \quad a \Vdash_{V_j} \psi,$$
$$a \Vdash_{V_j} \mathcal{N}_i\, \varphi \iff \forall b \left[(a \text{ Next } b) \Rightarrow b \Vdash_{V_j} \varphi\right],$$
$$a \Vdash_{V_j} (\varphi\, \mathcal{U}_i\, \psi) \iff \exists b \left[(a \leqslant b) \wedge \left(b \Vdash_{V_j} \psi\right) \wedge \forall c \left[(a \leqslant c < b) \Rightarrow \left(c \Vdash_{V_j} \varphi\right)\right]\right],$$
$$a \Vdash_{V_j} (\varphi\, \mathcal{S}_i\, \psi) \iff \exists b \left[(b \leqslant a) \wedge \left(b \Vdash_{V_j} \psi\right) \wedge \forall c \left[(b < c \leqslant a) \Rightarrow \left(c \Vdash_{V_j} \varphi\right)\right]\right],$$
$$a \Vdash_{V_j} \Box_i\, \varphi \iff \forall b \left[(a \leqslant b) \Rightarrow \left(b \Vdash_{V_j} \varphi\right)\right],$$
$$a \Vdash_{V_j} \Diamond_i\, \varphi \iff \exists b \left[(a \leqslant b) \wedge \left(b \Vdash_{V_j} \varphi\right)\right].$$

It is easy to see that $\Diamond_i \varphi = \top \mathcal{U}_i \varphi$, that is, modal operators $\Box_i$ and $\Diamond_i$ can be expressed by means the operator $\mathcal{U}_i$ in the considerable models.

Definition 1.6. *A formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is called satisfiable in $\mathcal{K}(\mathcal{M}_\mathbb{Z})$ if there is a model $\mathcal{M}_\mathbb{Z} \in \mathcal{K}(\mathcal{M}_\mathbb{Z})$ and a point $a \in \mathcal{M}_\mathbb{Z}$ such that $a \Vdash_{V_j} \varphi$ for some $j = 1, \ldots, k$.*

Definition 1.7. *A formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is called refutable in $\mathcal{K}(\mathcal{M}_\mathbb{Z})$ if there is a model $\mathcal{M}_\mathbb{Z} \in \mathcal{K}(\mathcal{M}_\mathbb{Z})$ and a point $a \in \mathcal{M}_\mathbb{Z}$ such that $a \nVdash_{V_j} \varphi$ for some $j = 1, \ldots, k$.*

Definition 1.8. *A formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is called valid in a model $\mathcal{M}_\mathbb{Z} \in \mathcal{K}(\mathcal{M}_\mathbb{Z})$ if for any point $a \in \mathcal{M}_\mathbb{Z}$: $a \Vdash_{V_j} \varphi$ for any $j = 1, \ldots, k$.*

Definition 1.9. *A formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is called unsatisfiable in this logic, if for any model $\mathcal{M}_\mathbb{Z} \in \mathcal{K}(\mathcal{M}_\mathbb{Z})$ and for any point $a \in \mathcal{M}_\mathbb{Z}$: $a \nVdash_{V_j} \varphi$ for any $j = 1, \ldots, k$.*

Definition 1.10. *The multi-valueted logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is the set of formulas in the language of this logic that are valid in any model from the class $\mathcal{K}(\mathcal{M}_\mathbb{Z})$.*

Formulas that belong to the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ are called theorems of this logic.
Definitions 1.8 and 1.10 yield that

Theorem 1.11. *A formula φ is a theorem of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ if and only if $\neg\varphi$ is an unsatisfiable formula in this logic.*

In the paper by V. V. Rybakov [17] similarly it is defined the multi-valueted logic $\mathcal{L}(\mathcal{K})$. The language of $\mathcal{L}(\mathcal{K})$ is defined in the same away as the language of $\mathcal{L}(\mathcal{M}_\mathbb{Z})$, but without the temporal operators $\mathcal{S}_i$. The logic $\mathcal{L}(\mathcal{K})$ is defined as the set of formulas that are valid in any model from the class $\mathcal{K}(\mathcal{M}_\mathbb{N})$.

Thus, the fundamental difference between the logics $\mathcal{L}(\mathcal{M}_\mathbb{Z})$, $\mathcal{L}(\mathcal{K})$, and others multi-valueted logics from multi-modal logics with k modal operators $\Box_i$, temporal logics with k temporal operators of the same type, as well as logics using both modal and temporal operators, is in the interpretation of the i-th modal or temporal operators of each type in formulas of such logics. In multi-valueted logics, every index i is connected not with a binary relation R_i on Kripke-frame and (or) with one of several binary relations Next_i, $\leqslant_i$ and etc. on a temporal frame, but with the i-th valuation on a selected model.

As we write in the introduction, in the language of multimodal logics with several modal operators $\Box_i$ and the language of temporal logic with several temporal operators of the same type, as well as in logics that use both modal and temporal operators, one can model reasoning about the knowledge and behaviour of agents in multi-agent systems, interpreting an index i of modal and temporal operators in a formula as knowledge of the i-th agent in a multi-agent system.

In the papers by V.V. Rybakov ([17–20]), there was proposed another approach to modelling such reasoning, namely, by means formulas of a multi-valueted logic. Examples of formulas in multi-valueted logics and their interpretation from the point of view of agents can be found in his cited works.

Let us give two more examples.

1. Consider two agents: i и j. A situation when the agent i believes that starting from the event ψ, the event φ will always be true in the future, while the agent j believes that starting from the event ψ, the event φ is possible in the future, we can describe by the following formula:

$$\Big((\Box_i \varphi) \ \mathcal{S}_i \psi \Big) \wedge \Big((\Diamond_j \neg\varphi) \ \mathcal{S}_j \psi \Big).$$

Note that this formula is unsatisfiable in logics with the usual interpretation of indices by means of various reachability relations, while in a model $\mathcal{M}_{\mathbb{Z}}$, with a certain φ and ψ and not coinciding with each other V_i and V_j, the formula can be satisfiable.

2. Consider three agents: i, j и k. Let the agent i believes that, starting from the event ψ, the event φ is possible in the future, and the agent j believes that, starting from the event ψ, the event $\neg\varphi$ is possible in the future. In this case, the agent k concludes that both events are possible. This situation can be described by the following formula:

$$\Big(((\Diamond_i \varphi) \ \mathcal{S}_i \psi) \wedge ((\Diamond_j \neg\varphi) \ \mathcal{S}_j \psi) \Big) \to \Big(\Diamond_k \varphi \wedge \Diamond_k \neg\varphi \Big).$$

2. Satisfiability in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$

In Section 2., to decide the problem of satisfiability in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$, we introduce a class of some special finite models $\mathcal{K}(\mathcal{M}_\Delta)$ and prove that the formula in the language of our logic is satisfiable if and only if it is satisfiable in some model $\mathcal{M}_C$ from the class $\mathcal{K}(\mathcal{M}_\Delta)$.

Class of models $\mathcal{K}(\mathcal{M}_\Delta)$

Let there be given a segment $\Delta = [d^-, d^+] \subset \mathbb{Z}$, where $d^- \leqslant -4$, $d^+ \geqslant 5$, and some points $c^-, c^+ \in [d^-, d^+]$, $c^+ \geqslant 4$, $c^- \leqslant -3$. Let $Prop(\varphi)$ be the set of propositional variables of a formula φ. Then,

$$\mathcal{M}_\Delta = \Big\langle \big[d^-, d^+\big], \preccurlyeq, \mathrm{Next}', V_1, \ldots, V_k \Big\rangle,$$

where the relation Next' on $[d^-, d^+)$ coincides with the relation Next, and at d^+ it is defined in the following away: we consider that $\mathrm{Next}'(d^+) = c^+ + 1$ and $\mathrm{Next}'(c^- - 1) = d^-$, that is, we determine the relation Next on the segment $[d^-, d^+]$.

The relation $\preccurlyeq$ on $\mathcal{M}_\Delta$ we determine in the following away:

1. if $x_1 \leqslant x_2$, then $x_1 \preccurlyeq x_2$,
2. on the segments $[d^-, c^- - 1]$ and $[c^+ + 1, d^+]$ the relation $\preccurlyeq$ is an equivalence relation.

If $x_1, x_2 \in [d^-, d^+]$, then the case, when $a \preccurlyeq b$ and $a \neq b$ we denote as $x_1 \prec x_2$.

Valuations $V_1, \ldots, V_k$ are the valuations of variables from the set $Prop(\varphi)$.

Rules for calculating the truth for any valuation V_i we determine in the same away as in the model $\mathcal{M}_{\mathbb{Z}}$ replacing in all items of Definition 1.5 the relation Next on Next', relation $\leqslant$ on $\preccurlyeq$, and the symbol $<$ on $\prec$.

Let $\mathcal{K}(\mathcal{M}_\Delta)$ be a class of all possible models $\mathcal{M}_\Delta$, that is, models obtained for all possible values of d^-, d^+, c^- and c^+, satisfied the conditions indicated above, and for all possible valuations of the variables from $Prop(\varphi)$.

Suppose, that there is some model $\mathcal{M}_{\mathbb{Z}}$ (3):

$$\mathcal{M}_{\mathbb{Z}} = \langle \mathbb{Z}, \leqslant, \mathrm{Next}, V_1, \ldots, V_k \rangle$$

and a point $a \in \mathcal{M}_{\mathbb{Z}}$ with a valuation V_j such that $a \Vdash_{V_j} \varphi$. Obviously, we can take $a = 1$, that is,

$$(\mathcal{M}_{\mathbb{Z}},\, 1) \Vdash_{V_j} \varphi.$$

Let us show that by transforming the model $\mathcal{M}_{\mathbb{Z}}$ in a certain way, we can construct for this formula some model $\mathcal{M}_C \in \mathcal{K}(\mathcal{M}_\Delta)$ such that

$$(\mathcal{M}_C,\, 1) \Vdash_{V_j} \varphi.$$

Model $\mathcal{M}_C$

First, we separately transform the positive part

$$\mathcal{M}^+ = \langle \mathbb{N},\, \leqslant,\, \mathrm{Next},\, V_1,\, \ldots,\, V_k \rangle$$

of the model $\mathcal{M}_{\mathbb{Z}}$ and obtain a model $\mathcal{M}_C^+$ that is a part of the model $\mathcal{M}_C$. We use in our proof the results from the paper by V. V. Rybakov [17] obtained for $\mathcal{M}_C^+$.

Then we completely construct the model $\mathcal{M}_C$.

I. The positive part of $\mathcal{M}_C^+$ is constructed in the paper by V. V. Rybakov [17] and has the following construction.

Let there be given points n, c^+ and d^+ from $\mathbb{N}$, and such that $4 \leqslant c^+ < c^+ + 1 < d^+$. Then,

$$\mathcal{M}_C^+ = \left\langle \left[1,\, d^+\right],\, \preccurlyeq,\, \mathrm{Next}',\, V_1,\, \ldots,\, V_k \right\rangle, \tag{4}$$

where the relation Next' on $[1,\, d^+)$ coincides the relation Next, and at d^+ it is defined in the following away: we consider $d^+ = c^+ + 1$, that is, $\mathrm{Next}'(d^+) = c^+ + 1$. Thus, we determine the relation Next on the finite segment $[1,\, d^+]$.

The relation $\preccurlyeq$ on $\mathcal{M}_C^+$ we determine in the following away:

1. if $x_1 \leqslant x_2$, then $x_1 \preccurlyeq x_2$;
2. on the segment $[c^+ + 1,\, d^+]$ the relation $\preccurlyeq$ is an equivalence relation.

If $x_1,\, x_2 \in [1,\, d^+]$, then the case when $a \preccurlyeq b$ and $a \neq b$ we denote as $x_1 \prec x_2$.

Valuations $V_1,\, \ldots,\, V_k$ on each point of $\mathcal{M}_C^+$ coincide with the valuations of the corresponding points of the model $\mathcal{M}_{\mathbb{Z}}$.

Now we describe in detail the main stages of constructing the model $\mathcal{M}_C^+$.

First, for every element $b \in \mathcal{M}_{\mathbb{Z}}$ we define the following values.

1. For any valuation V_i, $i \in [1, k]$,

$$Sub_i(b) = \{\alpha \in Sub(\varphi) \mid b \Vdash_{V_i} \alpha\}.$$

In fact, $Sub_i(b)$ defines formula valuations on $\mathcal{M}_C$. Obviously, there are exist $2^{\|Sub(\varphi)\|}$ such possible different sets and this number is finite.

2. Compose from the valuations one more set $Sub_i(b)$:

$$D(b) = \{Sub_i(b) \mid i \in [1,\, k]\}.$$

It is convenient to represent $D(b)$ as a column:

$$\begin{pmatrix} Sub_1(b) \\ Sub_2(b) \\ \ldots\ldots \\ Sub_k(b) \end{pmatrix}$$

Obviously, there are no more than

$$2^{\|Sub(\varphi)\|} \times \dots \times 2^{\|Sub(\varphi)\|} = 2^{k \cdot \|Sub(\varphi)\|} \tag{5}$$

such distinct columns. Denote this set as $\mathcal{D}$. The power of the set $\mathcal{D}$ is also a finite number.

3. We put in the model $\mathcal{M}^+$:

$$F(b) = \{D(c) \mid c \geqslant b\}.$$

Since $\mathcal{D}$ is a finite set and $\mathbb{N}$ is infinite, there is an element $c^+ \geqslant 4$ such that $\forall\, d \geqslant c^+$ and $\forall\, g \geqslant c^+$ the following holds: $F(d) = F(g)$ (the need to choose the number 4 is justified in Lemma 9 in [17] when estimating the power of the model $\mathcal{M}_C$).

4. For any $x \in \mathcal{M}^+$, a set of realizators is the minimum interval $R(x) = [x,\, y]$, where for any two subformulas $\varphi_1 \mathcal{U}_j \varphi_2$ and $\mathcal{N}_j\, \varphi_1$ from the set $Sub(\varphi)$ there is a valuation V_i such that the following holds:

$$\Big[\Big((x \Vdash_{V_i} \varphi_1 \mathcal{U}_j\, \varphi_2) \wedge (x \nVdash_{V_i} \varphi_2) \Big) \Rightarrow$$
$$\Rightarrow \bigg(\Big(\exists\, y \in R(x)\, (y \Vdash_{V_i} \varphi_2)\Big) \wedge \Big(\forall\, z\, (x \leqslant z < y)\, (z \Vdash_{V_i} \varphi_1)\Big) \bigg) \Big] \bigwedge$$
$$\bigwedge \Big[x \Vdash_{V_i} \mathcal{N}_j \varphi_1 \Rightarrow (x+1) \in R(x) \Big].$$

Denote by $Rls(x)$ the largest point y in $R(x)$. The minimum interval $R(x)$ can be large, but it exists.

Consider the interval $R(c^+) = [c^+,\, Rls(c^+)]$.

By the definition of the point c^+, there is a point d such that $d \geqslant Rls(c^+) + 2$ and $D(c^+) = D(d)$. Take the least $d = d^+$ with this property and delete from $\mathcal{M}^+$ all points that are strictly larger than d^+. We put $\mathrm{Next}'(d^+) = c^+ + 1$ in the model $\mathcal{M}_C^+$.

Thus we obtain the positive part of the model $\mathcal{M}_C$ – the model $\mathcal{M}_C^+$.

Rules for calculating the truth for any valuation V_i in this part of the model we define in the same away as in the model $\mathcal{M}_\mathbb{Z}$ replacing in all items of Definition 1.5 the relation Next on Next', the relation $\leqslant$ on $\preccurlyeq$, and the symbol $<$ on $\prec$.

II. Construct now the model $\mathcal{M}_C$.

Let there be given the points d^+ и c^+, which we define above when construct the model $\mathcal{M}_C^+$, and points d^- and c^- from $\mathbb{Z} \setminus \mathbb{N}$ and such that $d^- + 1 < c^- \leqslant -3$ (the need to choose the number -3 is justified in Theorem 3.1 when estimating the power of the model $\mathcal{M}_C$). Then,

$$\mathcal{M}_C = \Big\langle \big[d^-,\, d^+\big],\, \preccurlyeq,\, \mathrm{Next}',\, V_1,\, \dots,\, V_k \Big\rangle,$$

where the relation Next' on $[d^-,\, d^+)$ coincides with the relation Next, and at d^+ is defined in the following away: we consider that $\mathrm{Next}'(d^+) = c^+ + 1$ and $\mathrm{Next}'(c^- - 1) = d^-$, that is, we determine the relation Next on $[d^-,\, d^+]$.

The relation $\preccurlyeq$ on $\mathcal{M}_C$ we determine in the following away:

1. if $x_1 \leqslant x_2$, then $x_1 \preccurlyeq x_2$,

2. on the segments $[d^-,\, c^- - 1]$ and $[c^+ + 1,\, d^+]$ the relation $\preccurlyeq$ is an equivalence relation.

If $x_1,\, x_2 \in [d^-,\, d^+]$, then the case when $a \preccurlyeq b$ and $a \neq b$ we denote as $x_1 \prec x_2$.

Valuations $V_1,\, \dots,\, V_k$ on each point of the models $\mathcal{M}_C$ coincide with valuations of the corresponding points of the model $\mathcal{M}_{\mathbb{Z}}$.

Now we describe in detail points d^- and c^- of the model $\mathcal{M}_C$.

Fist, we define the following values for every element $b \in \mathcal{M}_{\mathbb{Z}}$.

The first items 1 and 2 are the same as items 1 и 2 from I, where we define $Sub_j(b)$ and $D(b)$.

3. For each element $b \leqslant 0$ in $\mathcal{M}_C$, we put:

$$G(b) = \{D(c) \mid c \leqslant b\}.$$

Since $\mathcal{D}$ is a finite set and $\mathbb{Z} \setminus \mathbb{N}$ is infinite, then there is an element $c^- \leqslant 0$ such that $\forall\, d \leqslant c^-$ and $\forall\, g \leqslant c^-$ the equality $G(d) = G(g)$ is true.

4. For any $x \in \mathcal{M}_C$, $x \leqslant 0$, a set of realizators is the minimum interval $R(x) = [y,\, x]$, where for any two subformulas $\varphi_1 \mathcal{S}_j \varphi_2$ from the set $Sub(\varphi)$ there is a valuation V_i such that the following holds:

$$\Bigg[\Big(\,(x \Vdash_{V_i} \varphi_1\, \mathcal{S}_j\, \varphi_2) \,\wedge\, (x \nVdash_{V_i} \varphi_2)\Big) \Rightarrow \\ \Rightarrow \bigg(\Big(\exists\, y \in R(x)\,(y \Vdash_{V_i} \varphi_2)\Big) \,\wedge\, \Big(\forall\, z\,(y < z \leqslant x)\,(z \Vdash_{V_i} \varphi_1)\Big)\bigg)\Bigg]. \tag{6}$$

Denote by $Rls(x)$ the largest point y in $R(x)$. The minimum interval $R(x)$ can be large, but it exists.

Consider the interval $R(c^-) = [Rls(c^-),\, c^-]$.

By the definition of the point c^-, there is a point d such that $d < Rls(c^-)$ and $D(c^-) = D(d)$. Take the largest $d = d^-$ with this property and delete from $\mathcal{M}_C$ all points that are strictly least than d^-.

Thus we obtain the model $\mathcal{M}_C \in \mathcal{K}\,(\mathcal{M}_\Delta)$.

The rules for calculating the truth for any valuation V_i in this part of the model we define in the same away as in the model $\mathcal{M}_{\mathbb{Z}}$ replacing in all items the definitions 1.5 the relation Next on Next', the relation $\leqslant$ on $\preccurlyeq$ and the symbol $<$ on $\prec$.

Let there be given a formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$, a model $\mathcal{M}_{\mathbb{Z}}$ and constructed according to the given formula and the model – the model $\mathcal{M}_C$.

Lemma 2.1. *For any subformula $\psi \in Sub(\varphi)$, for any element $a \in \mathcal{M}_{\mathbb{Z}}$ and any valuation V_j the following holds:*

$$(\mathcal{M}_{\mathbb{Z}},\, a) \Vdash_{V_i} \psi \Longleftrightarrow (\mathcal{M}_C,\, a) \Vdash_{V_i} \psi.$$

Proof. We prove by induction on the length of the subformula $\psi \in Sub(\varphi)$.

I. Given the models $\mathcal{M}^+$ and $\mathcal{M}_C^+$, the statement of the lemma proved for formulas constructed from the propositional variables, Boolean logical operators, and temporal operators $\mathcal{N}_j$ and $\mathcal{U}_j$ by V. V. Rybakov [17] (Lemmas 7 and 8). As it is follows from the remark after Definition 1.5, we have that the lemma also holds for formulas with operators $\Box_j$ and $\Diamond_j$.

Moreover, based on the construction of the point d^+ and due to the fact that we did not change the model $\mathcal{M}_{\mathbb{Z}}$ from the point d^- to the point c^+, it is easy to see that for any element $a \in \mathcal{M}_{\mathbb{Z}}$ the proof of this lemma for the case when formulas consist of propositional variables, Boolean logical operators, and temporary operators $\mathcal{N}_j$ and $\mathcal{U}_j$, almost literally repeats the proofs from Lemmas 7 and 8 in [17].

Now we prove the case when $\psi = \varphi_1 \mathcal{S}_j \varphi_2$.

Suppose that the lemma holds for the formulas φ_1 and φ_2.

Necessity.

II. Let the following holds:

$$(\mathcal{M}_{\mathbb{Z}}, a) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2. \tag{7}$$

Show that the following also holds:

$$(\mathcal{M}_C, a) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2. \tag{8}$$

Formula (7) is equivalent to the truth of two statements at the same time:

$$\exists b \; (b \leqslant a) \; (\mathcal{M}_{\mathbb{Z}}, b) \Vdash_{V_i} \varphi_2 \quad \text{and} \tag{9}$$

$$\forall c \; ((b < c \leqslant a) \Rightarrow (\mathcal{M}_{\mathbb{Z}}, c) \Vdash_{V_i} \varphi_1). \tag{10}$$

If $b \in [d^-, d^+]$, then by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ on $\mathcal{M}_C$ we obtain that (9) yields

$$\exists b \; (b \preccurlyeq a) \; (\mathcal{M}_C, b) \Vdash_{V_i} \varphi_2, \tag{11}$$

and (10) yields

$$\forall c \; ((b \prec c \preccurlyeq a) \Rightarrow (\mathcal{M}_C, c) \Vdash_{V_i} \varphi_1). \tag{12}$$

But the truth of the statements (11) and (12) means the truth of the statement (8), therefore, for the case when $b \in [d^-, d^+]$, the necessity is proven.

Suppose now that $b < d^-$. Hence, by the definition of the operator $\mathcal{S}_j$, it is easy to obtain that if $a \geqslant d^-$, then

$$\big(\mathcal{M}_{\mathbb{Z}}, d^-\big) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2.$$

By the construction of the points c^- and d^- we have $D(c^-) = D(d^-)$. Hence,

$$\big(\mathcal{M}_{\mathbb{Z}}, c^-\big) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2.$$

But d^- by its construction lies to the left of the interval of realizators $R(c^-)$, for which the condition (6) holds. Therefore, there is also a point $b_1 \in (d^-, c^-)$ such that

$$(\mathcal{M}_{\mathbb{Z}}, b_1) \Vdash_{V_i} \varphi_2. \tag{13}$$

Since $b_1 > b$, then by (10) we also have:

$$\forall c \; ((b_1 < c \leqslant a) \Rightarrow (\mathcal{M}_{\mathbb{Z}}, c) \Vdash_{V_i} \varphi_1). \tag{14}$$

But by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ on $\mathcal{M}_C$, (13) and (14) yield the truth of the following two statements:

$$\exists b \; (b \preccurlyeq a) \; (\mathcal{M}_C, b) \Vdash_{V_i} \varphi_2,$$

$$\forall c \; ((b \prec c \preccurlyeq a) \Rightarrow (\mathcal{M}_C, c) \Vdash_{V_i} \varphi_1),$$

from where we finally conclude that the statement (8) holds in the model $\mathcal{M}_C$.

Thus, the necessity is proven.

Sufficiency.

III. Let the following holds:

$$(\mathcal{M}_C,\, a) \Vdash_{V_i} \varphi_1\, \mathcal{S}_j\, \varphi_2. \tag{15}$$

Show that the following is also holds:

$$(\mathcal{M}_{\mathbb{Z}},\, a) \Vdash_{V_i} \varphi_1\, \mathcal{S}_j\, \varphi_2. \tag{16}$$

The statement (15) is equivalent to the truth of two statements:

$$\exists\, b\ (b \preccurlyeq a)\ (\mathcal{M}_C,\, b) \Vdash_{V_i} \varphi_2, \tag{17}$$

$$\forall\, c\ ((b \prec c \preccurlyeq a) \Rightarrow (\mathcal{M}_C,\, c) \Vdash_{V_i} \varphi_1)\,. \tag{18}$$

By the inductive hypothesis and by the definition of the relation $\preccurlyeq$ on $\mathcal{M}_C$, (17) and (18) yield

$$\exists\, b\ (b \leqslant a)\ (\mathcal{M}_{\mathbb{Z}},\, b) \Vdash_{V_i} \varphi_2,$$

$$\forall\, c\ ((b < c \leqslant a) \Rightarrow (\mathcal{M}_{\mathbb{Z}},\, c) \Vdash_{V_i} \varphi_1)\,,$$

which means that the condition (16) holds for the model $\mathcal{M}_{\mathbb{Z}}$.

Therefore, the sufficiency is also proven. □

Suppose, that in some model $\mathcal{M}_{\mathbb{Z}}$ there is a point a and a valuation V_j such that $(\mathcal{M}_{\mathbb{Z}},\, a) \Vdash_{V_i}$. Obviously, we can take $a = 1$, that is,

$$(\mathcal{M}_{\mathbb{Z}},\, 1) \Vdash_{V_i} \varphi.$$

Hence, Lemma 2.1 yields,

Lemma 2.2. *Let there be given a formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$, some model $\mathcal{M}_{\mathbb{Z}}$ and a model $\mathcal{M}_C$ that constructed by the given formula and the model. Then*

$$(\mathcal{M}_{\mathbb{Z}},\, 1) \Vdash_{V_i} \varphi \Longleftrightarrow (\mathcal{M}_C,\, 1) \Vdash_{V_i} \varphi.$$

Thus,

Theorem 2.3. *If a formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$ satisfiable in a model of the class $\mathcal{K}\,(\mathcal{M}_{\mathbb{Z}})$, then it satisfiable in a model of the class $\mathcal{K}\,(\mathcal{M}_{\Delta})$.*

Let us prove the converse statement.

Theorem 2.4. *If a formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$ satisfiable in a model of the class $\mathcal{K}\,(\mathcal{M}_{\Delta})$, then it satisfiable in a model of the class $\mathcal{K}\,(\mathcal{M}_{\mathbb{Z}})$.*

Proof. Let in some model $\mathcal{M}_{\Delta}$ we have

$$(\mathcal{M}_{\Delta},\, a) \Vdash_{V_i} \varphi, \tag{19}$$

where φ is a formula in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$.

Consider a model $\mathcal{M}_{\mathbb{Z}}$ with the following valuations $V_1,\, V_2,\, \ldots,\, V_k$ of its elements.

1. Let $a \in \mathcal{M}_{\mathbb{Z}}$ and $a \in [d^-, d^+]$.

Then the valuations $V_1, V_2, \ldots, V_k$ of any element of the model $\mathcal{M}_{\mathbb{Z}}$ on $[d^-, d^+]$ coincides with the valuations of the corresponded elements in the model $\mathcal{M}_{\Delta}$.

2. Let $a \in \mathcal{M}_{\mathbb{Z}}$ and $a \notin [d^-, d^+]$.

It is easy to see that if $a > d^+$, then we can express a as $a = c^+ + 1 + j + t \cdot (d^+ - c+)$, $t \geqslant 1$, where $c^+ + 1 + j \leqslant d^+$, $j \geqslant 0$.

Similarly, if $a < d^-$, then we can express a as $a = c^- - 1 - j - t \cdot (c^- - d^-)$, $t \geqslant 1$, where $c^- - 1 - j \geqslant d^-$, $j \geqslant 0$.

Thereat, the valuations of $c^+ + 1 + j + t \cdot (d^+ - c+) \in \mathcal{M}_{\mathbb{Z}}$, $t \geqslant 1$, coincide with the valuations of $c^+ + 1 + j \leqslant d^+$ from the model $\mathcal{M}_{\Delta}$, $j \geqslant 0$.

Similarly, the valuations of $c^- - 1 - j - t \cdot (c^- - d^-) \in \mathcal{M}_{\mathbb{Z}}$, $t \geqslant 1$, coincide with the valuations of $c^- - 1 - j \geqslant d^-$ from the model $\mathcal{M}_{\Delta}$, $j \geqslant 0$.

Taking into account the given relations on the model $\mathcal{M}_{\Delta}$, it is easy to obtain by induction on the length of the formula that for any element s of the model $\mathcal{M}_{\Delta}$, any valuation V_i and any subformula $\psi \in Sub(\varphi)$ the following holds:

$$(\mathcal{M}_{\Delta}, s) \Vdash_{V_i} \psi \Longleftrightarrow (\mathcal{M}_{\mathbb{Z}}, s) \Vdash_{V_i} \psi. \tag{20}$$

Thereat, (20) yields

$$(\mathcal{M}_{\mathbb{Z}}, a) \Vdash_{V_i} \varphi. \tag{21}$$

Thus, (19) yields (21), that is, the theorem is proven. □

3. Decidability of the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$

In Section 3 we prove that the satisfiability problem in the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$ is decidable, that is, there is an algorithm that for the final steps determines whether the formula φ is satisfiable in the language of the given logic on some model from the class $\mathcal{K}(\mathcal{M}_{\mathbb{Z}})$ or not. To prove the decidability for any formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$, we construct a finite number of finite models, which sufficient for checking the satisfiability of the formula φ. These models have the similar construction as the models $\mathcal{M}_{\Delta}$ from Section 2.

V.V. Rybakov in Lemma 9 in [17] prove, that for any formula φ in the language of the logic $\mathcal{L}(\mathcal{K})$ the following holds:

$$\left(\mathcal{M}_C^+, 1\right) \Vdash_{V_i} \varphi \Longleftrightarrow (\mathcal{M}_C^{'}, 1) \Vdash_{V_i} \varphi,$$

where $\mathcal{M}_C^{'}$ is a model, obtained in a special way from the model $\mathcal{M}_C^+$ and having the power at most $f(\varphi) = 2^{k \cdot \|Sub(\varphi)\|} + 3$ elements, where k is the number of different valuations in the model $\mathcal{M}_{\mathbb{Z}}$.

Let us prove a similar result for our models $\mathcal{M}_C$.

Theorem 3.1. *Let there be given a formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_{\mathbb{Z}})$. Then,*

$$(\mathcal{M}_C, 1) \Vdash_{V_i} \varphi \Longleftrightarrow (\mathcal{M}_C^{'}, 1) \Vdash_{V_i} \varphi,$$

where $\mathcal{M}_C^{'}$ is a model obtained in a special way from the model $\mathcal{M}_C$ and that has a power no more than $f(\varphi) = 2 \cdot \left(2^{k \cdot \|Sub(\varphi)\|}\right) + 5$ elements, where k is a number of different valuations in the model $\mathcal{M}_{\mathbb{Z}}$.

Proof. Let

$$(\mathcal{M}_C,\, 1) \Vdash_{V_i} \varphi. \tag{22}$$

I. First we transform the "positive" part of the model $\mathcal{M}_C$ by deleting from it certain intervals $[x,\, y)$, where $x > 1$.

Recall that $c^- \geqslant 4$. Consider the largest $s_1 \in [2,\, c^+ - 1]$ such that

$$D(2) = D(s_1).$$

If $s_1 \neq 2$, then delete the interval $[2,\, s_1)$ from the model $\mathcal{M}_C$ and denote the obtained model as $\mathcal{M}_1$. In this case, we assume that $\mathrm{Next}'(1) = s_1$ and that for any $x_1,\, x_2 \notin [2,\, s_1)$ the following holds:

$$x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_1 \iff x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_C.$$

Valuations $V_1,\, \ldots,\, V_k$ for all elements included in the model $\mathcal{M}_1$, we leave the same as they were for these elements in the model $\mathcal{M}_C$.

If $s_1 = 2$, then we do not change anything and assume that $\mathcal{M}_1 = \mathcal{M}_C$.

Denote in ascending order the elements of the model $\mathcal{M}_1$

$$\ldots,\, -2,\, -1,\, 0,\, 1,\, s_1,\, a_2,\, a_3,\, \ldots\,,$$

where a_i is the $i+1$-th element of the model $\mathcal{M}_1$.

Let $s_1 \neq 2$. Prove that (22) is holds if and only if the following holds:

$$(\mathcal{M}_1,\, 1) \Vdash_{V_i} \varphi. \tag{23}$$

The statement (23) is equivalent to the statement that for any element $s \in \mathcal{M}_1$, any subformula $\psi \in Sub(\varphi)$ and any valuation V_i the following holds:

$$(\mathcal{M}_C,\, s) \Vdash_{V_i} \psi \iff (\mathcal{M}_1,\, s) \Vdash_{V_i} \psi. \tag{24}$$

We prove this statement by induction on the length of the formula ψ.

The truth of the statement (24) for Boolean operators is obvious. The proof for the operators $\mathcal{N}_j$ and $\mathcal{U}_j$ repeats the corresponding proof for these operators in Lemma 9 in [17].

Suppose that (24) is proved for φ_1 and φ_2. Let us prove that it is true for $\varphi_1 \mathcal{S}_j \varphi_2$.

By the definition of $\mathcal{S}_j$, the statement

$$(\mathcal{M}_C,\, s) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2 \tag{25}$$

is equivalent to

$$\exists b\, [(b \preccurlyeq s)\ (\mathcal{M}_C,\, b) \Vdash_{V_i} \varphi_2 \wedge \forall c\, ((b \prec c \preccurlyeq s) \Rightarrow (\mathcal{M}_C,\, c) \Vdash_{V_i} \varphi_1)]\,, \tag{26}$$

and the definition

$$(\mathcal{M}_1,\, s) \Vdash_{V_i} \varphi_1 \mathcal{S}_j \varphi_2 \tag{27}$$

is equivalent to

$$\exists b\, [(b \preccurlyeq s)\ (\mathcal{M}_1,\, b) \Vdash_{V_i} \varphi_2 \wedge \forall c\, ((b \prec c \preccurlyeq s) \Rightarrow (\mathcal{M}_1,\, c) \Vdash_{V_i} \varphi_1)]\,. \tag{28}$$

1. Let $b \notin [2,\, s_1)$.

Any element $x \notin [2,\, s_1)$ of $\mathcal{M}_C$ also belongs to $\mathcal{M}_1$, hence, by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_1$, (26) yields (28), and hence, yields (27).

2. Let $b \in [2,\, s_1)$.

The $s \geqslant s_1$. From the definition of the operator $\mathcal{S}_j$ it is easy to obtain that

$$(\mathcal{M}_1,\, s_1) \Vdash_{V_i} \varphi_1 \,\mathcal{S}_j\, \varphi_2.$$

Since $D(s_1) = D(2)$, we have

$$(\mathcal{M}_1,\, 2) \Vdash_{V_i} \varphi_1 \,\mathcal{S}_j\, \varphi_2.$$

If

$$(\mathcal{M}_1,\, 2) \Vdash_{V_i} \varphi_2,$$

then, since $D(s_1) = D(2)$, we have

$$(\mathcal{M}_1,\, s_1) \Vdash_{V_i} \varphi_2.$$

Consequently, (25) is also equivalent to the statement

$$\exists\, b_1\ [(b_1 \preccurlyeq s)\ (\mathcal{M}_C,\, b_1) \Vdash_{V_i} \varphi_2 \,\wedge\, \forall\, c\ ((b_1 \prec c \preccurlyeq s) \Rightarrow (\mathcal{M}_C,\, c) \Vdash_{V_i} \varphi_1)]\,, \tag{29}$$

where $b_1 < 2$ or $b_1 = s_1$. But such elements belong $\mathcal{M}_1$. Then by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_1$, (29) yields

$$\exists\, b_1\ [(b_1 \preccurlyeq s)\ (\mathcal{M}_1,\, b_1) \Vdash_{V_i} \varphi_2 \,\wedge\, \forall\, c\ ((b_1 \prec c \preccurlyeq s) \Rightarrow (\mathcal{M}_1,\, c) \Vdash_{V_i} \varphi_1)]\,, \tag{30}$$

which is equivalent to the statement (27).

3. Since all elements of the model $\mathcal{M}_1$ also belong to the model $\mathcal{M}_C$, using the inductive assumption and taking into account the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_1$, it is easy to carry out the proof in the opposite direction, that is, to obtain that (27) yields (25).

Thus, the statement (24), and, consequently, the statement (23), holds.

Now we transform the model $\mathcal{M}_1$ in the same way.

Consider the maximum $s_2 \in [a_2,\, c^+ - 1]$ such that

$$D(a_2) = D(s_2).$$

If $s_2 \neq a_2$, then delete the interval $[a_2,\, s_2)$ from the model $\mathcal{M}_C$ and denote the obtained model as $\mathcal{M}_2$. In this case, we assume that $\mathrm{Next}'(s_1) = s_2$ and that for any $x_1,\, x_2 \notin [a_2,\, s_2)$ the following holds:

$$x_1 \preccurlyeq x_2 \ \text{ in } \ \mathcal{M}_2 \iff x_1 \preccurlyeq x_2 \ \text{ in } \ \mathcal{M}_1.$$

Valuations $V_1,\, \ldots,\, V_k$ of all elements included in the model $\mathcal{M}_2$ we leave the same as they were for these elements in the model $\mathcal{M}_1$.

If $s_2 = x_2$, then we do not change anything and assume that $\mathcal{M}_2 = \mathcal{M}_1$.

We denote in ascending order the elements of the model $\mathcal{M}_2$:

$$\ldots,\, -2,\, -1,\, 0,\, 1,\, s_1,\, s_2,\, b_3,\, \ldots\,,$$

where b_i is the $i+1$-th element of $\mathcal{M}_2$.

Further, in a similar way, we obtain from the model $\mathcal{M}_2$ the model $\mathcal{M}_3$ and so on. When we obtain at the k-th step that $s_k = c^+ - 1$, then we skip the point c^+, and on the $k+1$-th step we consider the interval $[c^+ + 1,\, d^+)$, and then we continue the transformation in the same way.

A proof that for any element $s \in \mathcal{M}_k$, any subformula $\psi \in Sub(\varphi)$, and any valuation V_i the following holds:

$$(\mathcal{M}_{k-1},\, s) \Vdash_{V_i} \psi \Longleftrightarrow (\mathcal{M}_k,\, s) \Vdash_{V_i} \psi \tag{31}$$

is similar to those given above by induction on the length of the formula ψ. Consequently, we obtain that (22) holds if and only if the following statement holds:

$$(\mathcal{M}_k,\, 1) \Vdash_{V_i} \varphi.$$

The number t of steps than we can take is equal to the number of the different sets $D(x)$, $x \in [2,\, d^+]$, where $x \notin \{1,\, c^+,\, d^+\}$, that is, does not exceed the value $2^{k\cdot\|Sub(\varphi)\|}$ (5).

Thereat, the number of positive elements in the model $\mathcal{M}_t$, obtained at the last step, obviously does not exceed the value

$$2^{k\cdot\|Sub(\varphi)\|} + 3. \tag{32}$$

II. After we obtain the model $\mathcal{M}_t$, we similarly continue transformation of the "negative part" of this model, deleting from it certain intervals $(y,\, x]$, where $x < 0$.

Recall, that $c^- \leqslant -3$. Consider the least $r_1 \in [c^- + 1,\, 0]$ such that

$$D(0) = D(r_1).$$

If $r_1 \neq 0$, then we delete the interval $(r_1,\, 0]$ from the model $\mathcal{M}_t$ and denote the obtained model $\mathcal{M}_{-1,t}$. In this case, we assume that $\text{Next}'(r_1) = 1$ and for any $x_1,\, x_2 \notin (r_1,\, 0]$ the following holds:

$$x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_{-1,t} \iff x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_t.$$

Valuations $V_1,\, \ldots,\, V_k$ of all elements included in the model $\mathcal{M}_{-1,t}$ we leave the same as they were in these elements in the model $\mathcal{M}_t$.

If $x = 0$, then we do not change anything and assume that $\mathcal{M}_t = \mathcal{M}_{-1,t}$.

The elements of the model $\mathcal{M}_{-1,t}$ we denote as:

$$\ldots,\, t_3,\, t_2,\, r_1,\, 1,\, s_1,\, a_2,\, a_3,\, \ldots\,.$$

Now we transform the model $\mathcal{M}_{-1,t}$ in the same way.

Consider the least $r_2 \in [c^- + 1,\, t_2]$ such that

$$D(t_2) = D(r_2).$$

If $r_2 \neq t_2$, then we delete the interval $(r_2,\, t_2]$ from the model $\mathcal{M}_{-1,t}$ and denote the obtained model $\mathcal{M}_{-2,t}$. In this case, we assume that $\text{Next}'(r_2) = r_1$ and for any $x_1,\, x_2 \notin (r_2,\, t_2]$ the following holds:

$$x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_{-2,t} \iff x_1 \preccurlyeq x_2 \text{ in } \mathcal{M}_{-1,t}.$$

Valuations $V_1, \ldots, V_k$ of all elements included in the model $\mathcal{M}_{-2,t}$ we leave the same as they were in these elements in the model $\mathcal{M}_{-1,t}$.

If $r_2 = t_2$, then we do not change anything and assume that $\mathcal{M}_{-2,t} = \mathcal{M}_{-1,t}$.

The elements of the model $\mathcal{M}_{-2,t}$ we denote as:

$$\ldots, u_4, u_3, r_2, r_1, 1, s_1, s_2, b_3, \ldots .$$

Further, in a similar way, moving from larger negative numbers to smaller ones, we obtain from the model $\mathcal{M}_{-2,t}$ the model $\mathcal{M}_{-3,t}$, from $\mathcal{M}_{-3,t}$ the model $\mathcal{M}_{-4,t}$ and so on. When on the l-th we obtain that $r_l = c^- + 1$, then we skip the point c^- and on the $l+1$-th step we consider the interval $(d^-, c^- - 1]$, and then continue the transformations in the same way.

The number$m$ of steps than we can take is equal to the number of different sets $D(x)$, $x \in [d^-, 0]$, where $x \notin \{c^-, d^-\}$, that is, does not exceed the value $2^{k \cdot \|Sub(\varphi)\|}$ (5).

Then the number of positive elements in the model $\mathcal{M}_{-m,t}$, obtained at the last step, obviously does not exceed the value

$$2^{k \cdot \|Sub(\varphi)\|} + 2. \tag{33}$$

From estimations (33) and (32) we have

$$|\mathcal{M}_{-m,t}| \leqslant 2 \cdot \left(2^{k \cdot \|Sub(\varphi)\|}\right) + 5. \tag{34}$$

A proof that for any element $s \in \mathcal{M}_{-m,t}$, any subformula $\psi \in Sub(\varphi)$, and any valuation V_i the following holds:

$$(\mathcal{M}_C, s) \Vdash_{V_i} \psi \Longleftrightarrow (\mathcal{M}_{-m,t}, s) \Vdash_{V_i} \psi \tag{35}$$

is similar to the proof in Part I.

First, we prove by induction on the length of the subformula ψ that

$$(\mathcal{M}_t, s) \Vdash_{V_i} \psi \Longleftrightarrow (\mathcal{M}_{-1,t}, s) \Vdash_{V_i} \psi, \tag{36}$$

then we prove at each step in a similar way that

$$(\mathcal{M}_{-l,t}, s) \Vdash_{V_i} \psi \Longleftrightarrow \left(\mathcal{M}_{-(l+1),t}, s\right) \Vdash_{V_i} \psi, \tag{37}$$

and thus we arrive at the statement (35).

Let us prove the truth of the statement (36). We prove it by induction on the length of the formula ψ. For Boolean operators, the proof is obvious. Then, we make the inductive assumption that the statement is proved for the formulas φ_1 and φ_2.

Let $\psi = \varphi_1 \mathcal{U}_j \varphi_2$ and

$$(\mathcal{M}_t, s) \Vdash_{V_i} \varphi_1 \mathcal{U}_j \varphi_2, \tag{38}$$

that is

$$\exists\, b\left[(s \preccurlyeq b) \wedge (b \Vdash_{V_i} \varphi_2) \wedge \forall\, c\left[(s \preccurlyeq c \prec b) \Rightarrow (c \Vdash_{V_i} \varphi_1)\right]\right].$$

If $b \notin (r_1, 0]$, then, as we have in I.1 and I.3, (36) follows from the inductive assumption and from the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_{-1,t}$.

Let $b \in (r_1, 0]$. Thereat, we obtain from the properties of the temporal operator $\mathcal{U}_j$ that

$$(\mathcal{M}_t, r_1) \Vdash_{V_i} \varphi_1 \mathcal{U}_j \varphi_2,$$

and hence, since $D(r_1) = D(0)$,

$$(\mathcal{M}_t,\, 0) \Vdash_{V_i} \varphi_1 \mathcal{U}_j\, \varphi_2.$$

If

$$(\mathcal{M}_t,\, 0) \Vdash_{V_i} \varphi_2,$$

then, since $D(r_1) = D(0)$,

$$(\mathcal{M}_t,\, r_1) \Vdash_{V_i} \varphi_2,$$

But then there is a point $b_1 \geqslant 0$ or $b_1 = r_1$ such that

$$\exists\, b_1 \left[(s \preccurlyeq b_1) \wedge ((\mathcal{M}_t,\, b_1) \Vdash_{V_i} \varphi_2) \wedge \forall\, c \left[(s \preccurlyeq c \prec b_1) \Rightarrow ((\mathcal{M}_t,\, c) \Vdash_{V_i} \varphi_1)\right]\right].$$

Hence, by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ in the $\mathcal{M}_{-1,t}$, we have that now in the model $\mathcal{M}_{-1,t}$ the following holds:

$$\exists\, b_1 \left[(s \preccurlyeq b_1) \wedge ((\mathcal{M}_{-1,t},\, b_1) \Vdash_{V_i} \varphi_2) \wedge \forall\, c \left[(s \preccurlyeq c \prec b_1) \Rightarrow ((\mathcal{M}_{-1,t},\, c) \Vdash_{V_i} \varphi_1)\right]\right],$$

which is equivalent to the statement

$$(\mathcal{M}_{-1,t},\, s) \Vdash_{V_i} \varphi_1 \mathcal{U}_j\, \varphi_2. \tag{39}$$

Consequently, (38) yields (39).

Further, since all elements of the model $\mathcal{M}_{-1,t}$ belong to the model $\mathcal{M}_t$, then as in Part I, by the inductive hypothesis and by the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_{-1,t}$, it is easy to obtain that (39) yields (38).

Proofs for the operators $\mathcal{N}_j$ and $\mathcal{S}_j$ are similar. In the case when the corresponding point $b \notin (r_1,\, 0]$, the necessity and sufficiency follow from the inductive hypothesis and from the definition of the relation $\preccurlyeq$ in the model $\mathcal{M}_{-1,t}$. And in the case when $b \in (r_1,\, 0]$, we use the condition $D(r_1) = D(0)$.

Thus, the statement (36) holds. In $m-1$ steps, we get the statement (35).

Denote the model $\mathcal{M}_{-m,t} = \mathcal{M}_C^{'}$. Obviously, the statement (35) is equalent to the fact that for any formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$ and any valuation V_i the following holds:

$$(\mathcal{M}_C,\, 1) \Vdash_{V_i} \varphi \Longleftrightarrow (\mathcal{M}^{'},\, 1) \Vdash_{V_i} \varphi.$$

We estimate the power of the $\mathcal{M}_C^{'}$ in (34). Thus, the theorem is proved. □

Lemma 2.2 and Theorem 3.1 together give us the following result.

Lemma 3.2. *Let there be given a formula φ in the language of the logic $\mathcal{L}\,(\mathcal{M}_{\mathbb{Z}})$, some model $\mathcal{M}_{\mathbb{Z}}$ and a model $\mathcal{M}_C^{'}$ that constructed by the given formula and the model. Then*

$$(\mathcal{M}_{\mathbb{Z}},\, 1) \Vdash_{V_i} \varphi \Longleftrightarrow (\mathcal{M}_C^{'},\, 1) \Vdash_{V_i} \varphi.$$

For a given formula φ, we denote

$$f(\varphi) = 2 \cdot \left(2^{k \cdot \|Sub(\varphi)\|}\right) + 5,$$
$$f_1(\varphi) = 2^{k \cdot \|Sub(\varphi)\|} + 2,$$
$$f_2(\varphi) = 2^{k \cdot \|Sub(\varphi)\|} + 3.$$

Model $\mathcal{M}_{C,\varphi}$

Let us describe one more class of models constructed for a formula φ – class of models $\mathcal{M}_{C,\varphi}$, a special case of which are the models $\mathcal{M}_C^{'}$.

Let there be given a formula φ.

The class $\mathcal{K}(\mathcal{M}_{C,\varphi})$ is the class of models $\mathcal{M}_\Delta$ with

$$\Delta = [-f_1(\varphi),\ f_2(\varphi)]\,.$$

Obviously, that $\mathcal{K}(\mathcal{M}_{C,\varphi})$ is a finite set of finite models and $\mathcal{M}_C^{'} \in \mathcal{K}(\mathcal{M}_{C,\varphi})$, where $\mathcal{M}_C^{'}$ is a model obtained in Theorem 3.1.

Now it is easy to get the main result.

Theorem 3.3. *The problem of satisfiability in the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is decidable.*

Proof. Let there be given a formula φ in the language of the logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$. In a finite number of steps, we can check its satisfiability at any point of any model from the class $\mathcal{K}(\mathcal{M}_{C,\varphi})$, which contains the model $\mathcal{M}_C^{'}$.

If the formula φ is satisfiable in some model from the class $\mathcal{K}(\mathcal{M}_{C,\varphi})$, then by Theorem 2.4, it is satisfiable in some model from the class $\mathcal{K}(\mathcal{M}_\mathbb{Z})$.

If the formula φ is refutable in some model from the class $\mathcal{K}(\mathcal{M}_{C,\varphi})$, then it is refutable in the model $\mathcal{M}_C^{'}$, and hence, by Lemma 3.2, φ is refutable in some model from the class $\mathcal{K}(\mathcal{M}_\mathbb{Z})$. □

Besides, if the formula φ is valid in all models from the class $\mathcal{K}(\mathcal{M}_{C,\varphi})$, then there are no models in the class $\mathcal{K}(\mathcal{M}_\mathbb{Z})$ where φ is refutable. Indeed, suppose the converse, that is, there exists $\mathcal{M}_\mathbb{Z}$ such that φ is refutable at a. As we wrote above, we can take $a = 1$. But by Lemma 3.2, the formula φ is refutable in the model $\mathcal{M}_C^{'} \in \mathcal{K}(\mathcal{M}_{C,\varphi})$.

Finally,

Theorem 3.4. *The logic $\mathcal{L}(\mathcal{M}_\mathbb{Z})$ is decidable.*

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Выполнимость во временной логике с мультиозначиванием, основанной на $\mathbb{Z}$

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Аннотация. Статья продолжает серию работ В. В. Рыбакова, посвященных свойствам логик с мультиозначиванием и в которых предложен новый подход для моделирования знаний и рассуждений агентов в мультиагентной среде. В нашей работе доказано, что проблема выполнимости во временной логике с мультиозначиванием, основанной на $\mathbb{Z}$, разрешима.

Ключевые слова: временная логика, мультиозначивание, логика знаний, мультиагентная логика, проблема выполнимости в логике, разрешающие алгоритмы, представления знаний, мультиагентные системы.