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On periodic groups of Shunkov with the Chernikov centralizers of involutions

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Abstract. The group is called layer-finite if the set of its elements of any given order is finite. Layer-finite groups for the first time appeared without a name in the article of S. N. Chernikov (1945), and then in his subsequent publications received the name layer-finite groups. Almost layer-finite groups are extensions of a layer-finite groups by finite groups. The class of almost layer-finite groups is wider than the class of layer-finite groups, it includes all Chernikov groups, whereas it is easy to give examples of Chernikov groups that are not layer-finite. The author develops the direction of characterizations of known well studied classes of groups in other classes of groups with some additional (rather weak) finiteness conditions. In this paper almost layer-finite groups receive characterization in the class of periodic Shunkov groups. Shunkov group is a group G in which for any of its finite subgroup K in the factor group $N_G(K)/K$ any two conjugate elements of prime order generate a finite subgroup. We study periodic Shunkov's groups with the condition: normalizer of any finite non-unit subgroup is almost layer-finite. It is proved that if in such group the centralizers of involutions are Chernikov, then the group is almost layer-finite.

Keywords: infinite group, finitness condition, Shunkov group, Chernikov group, involution.

1. Introduction

In this paper almost layer-finite groups receive characterization in the class of periodic Shunkov groups.

Theorem. *Let G be a Shunkov periodic group, centralizers of each involution is a Chernikov one. If the normalizer of any non-trivial finite subgroup of the group G is an almost layer-finite, then G is an almost layer-finite group.*

The author has previously proved a similar theorem for groups with the minimality condition for non-almost layer-finite subgroups [1; 2].

2. Proof of the Theorem

Let G be a Shunkov periodic group that is not almost layer-finite. Additionally we assume that the centralizers of all involutions in the group G is Chernikov and the normalizer of any non-trivial finite subgroup of the group G is almost layer-finite.

We need the following auxiliary lemmas.

Lemma 1. *In a maximal almost layer-finite group V of the group G all involutions with infinite centralizers in V generate an Abelian subgroup of order not greater than four.*

Proof is similar to the proof of Lemma 8 from [3].

Lemma 2. *In a maximal almost layer-finite subgroup V of G there is no elementary Abelian 8th order subgroup with an almost regular involution in V.*

Lemma 2 is immediately follows from Lemma 1.

Lemma 3. *In an almost layer-finite group there is only a finite number of non-conjugate finite solvable subgroups of a given order (Lemma 10 from [4]).*

Lemma 4. *Let V be a maximal almost layer-finite subgroup of G containing involutions. Then*

1) all involutions with infinite centralizers in V are conjugate in V;

2) if k is an involution of V and $C_V(k)$ is finite, then k induces an automorphism in some Abelian normal subgroup of the finite index in V, which translates each element of this subgroups in reverse.

Proof of Lemma 4 is similar to the proof of Lemma 12 from [4].

If the product of all normal layer-finite subgroups of a group is layer-finite, then it is called a *layer-finite radical* of the group.

Lemma 5. *Let F, M be two distinct infinite maximal almost layer-finite subgroups of the group G, $R(F)$ and $R(M)$ are their layer-finite radicals. Then $R(F) \cap R(M) = 1$.*

Proof of Lemma 5 is similar to the proof of Lemma 4 from [4].

Lemma 6. *If some Sylow 2-subgroup of G is finite, then all Sylow 2-subgroups in G are finite and conjugate.*

The proof of the lemma is similar to the proof of Lemma 3.1 from [5].

Recall that a group is called *infinitely isolated* if it contains the centralizer of any of its non-unit elements, if the intersection of this centralizer with a subgroup is infinite.

Lemma 7. *Any maximal almost layer-finite subgroup of G is an infinitely isolated subgroup.*

Proof of Lemma 7 is similar to the proof of Lemma 6 from [6].

By S we denote some Sylow 2-subgroup of G, i be the central involution from S or from the intersection of the center and the complete part of S if it is infinite (if S is infinite, then it is Chernikov by Lemma 1 from [4], by the properties of infinite Chernikov primary groups, in them the intersection of the complete part with the center is non-trivial), H be a maximal almost layer-finite subgroup of the group G containing the infinite centralizer $C_G(i)$, which is almost layer-finite by assumption. Such a maximal subgroup exists by Zorn's lemma and by theorem 1 from [10]. The centralizer $C_G(i)$ is infinite, since otherwise, by Proposition 7 of [12], the group G would be locally finite and, by by theorem 1 from [10], is almost layer-finite, that contradicts our assumption about the group G.

By the theorem from [7], we can assume that H is not strongly embedded subgroup of G. *Strongly embedded* is called a proper subgroup containing involutions that intersects with conjugate subgroups by subgroups without involutions. From here, by Lemma 7 it immediately follows that H has an almost regular involution. If S is finite, then we can choose this involution from S due to Lemma 6, but if S is infinite, then by Theorem 2 of [4] it contains infinitely many involutions, among which by Lemma 1 there is an almost regular in H involution. Fix for this involution notation j. We denote by $R(H)$ a layer-finite radical of the group H.

Lemma 8. *Suppose that for the maximal almost layer-finite subgroup H of a group G the difference $H \setminus R(H)$ does not possess involutions, conjugated with i in G. Let V be a subgroup conjugate with H in G, h be a non-trivial p-element from $D = H \cap V$. If $C_V(h)$ is infinite, then $C_H(h)$ is infinite and vice versa.*

Proof of Lemma 8 is similar to the proof of Lemma 11 from [3].

Lemma 9. *Let b be an element of prime order and the intersections $C_G(b) \cap W$, $C_G(b) \cap W^g$ are infinite, where W is a maximal almost layer-finite subgroup of the group G. Then $W = W^g$.*

Proof of Lemma 9 is similar to the proof of Lemma 6 from [4].

Let K be a subgroup of H generated by all involutions with infinite centralizers in H.

Lemma 10. *If $H \setminus R(H)$ does not contain involutions conjugate with i in G, then $H = C_G(i)$ and Sylow 2-subgroups in $R(H)$ are locally cyclic or generalized quaternion groups.*

Proof. If $K = \langle i \rangle$, then $H = C_G(i)$ and according to the definition of group K in $R(H)$ there is only one involution. But then by Shunkov theorem from

[11] Sylow 2-subgroups in $R(H)$ are locally cyclic or generalized quaternion groups.

Let $K \neq \langle i\rangle$ and S be a finite group. By Lemma 1 K is an elementary fourth-order Abelian subgroup. Then H has an element h of odd order, strictly real with respect to involution t from $G \setminus H$ conjugate with i in G. The group $C_G(h)\lambda\langle t\rangle$ is almost layer-finite. In $C_G(h)$ we take the Sylow 2-subgroup Q. Since S is finite subgroup, then Q is a finite subgroup by Lemma 6. By Frattini Lemma [13] we can assume that Q is a t-invariant, that is, $P = Q\lambda\langle t\rangle$. Suppose that Q has an Abelian subgroup R of fourth order. By Lemma 6, $P^c \leq S$, where c is an element of G. If R^c is a Klein group of orger 4, then by theorem from [**?**] there is an involution in it r with infinite centralizer in H. Then $h^c \in C_G(r) \leq H$, besides $t^c \in H$. Since involutions t^c, i are conjugate in G, then by the condition of the lemma $t^c \in K$. The order of K is four and $t^c \in K \lhd H$. At the same time, from the strict reality of h with respect to t we have $t^c h^c t^c = (h^c)^{-1}$. But it is impossible, since $h^c \in H$. Therefore, R^c is a cyclic group and $h^c \in G \setminus H$.

We will show that this is impossible. Indeed, if $R^c \cap K \neq 1$, then by Lemma 7 H we get a contradiction with the fact that $h^c \in G \setminus H$. Let $R^c \cap K = 1$ and v be an involution of R^c. Since K is a Klein group of orger 4, normal in H, $K < C_G(v)$ and $|C_H(v)| < \infty$, and then we obtain a contradiction with Lemma 2.

So in Q there are no fourth order subgroups, therefore, Q is a subgroup of the second order. If $p > 3$, then $K < C_G(H)$. However, this is impossible as we just noticed. Therefore, $p = 3$. Note that $h \notin R(H)$, since in this case $N_G(\langle h\rangle) \cap H$ is infinite and by Lemma 5 $t \in N_G(\langle h\rangle) \leq H$ contrary to the choose $t \in G \setminus H$. By Lemma 4 all involutions of K are conjugate to each other in H and since K is a Klein group of orger 4, then $H/C_G(K) = (\overline{b})\lambda(\overline{v})$, where $|\overline{b}| = 3, |\overline{v}| = 2$ and $\overline{v}\overline{b}\overline{v} = \overline{b}^{-1}$. Based on the properties of almost layer-finite groups and on the representation of the factor group $H/C_G(K)$ we get that H has a pair of elements b, v such, that b is a 3-element, v is 2-element, $v^{-1}bv = b^{-1}$ and $bC_G(K) = \overline{b}, vC_G(K) = \overline{v}$.

We will prove that v is an involution. If not, then $\langle v\rangle \cap C_G(K)$ possesses an involution x. If $x \notin K$, then $C_H(x)$ is finite and we obtain a contradiction with Lemma 2. So $x \in K$ and $x \in C_G(b)$, where b is 3-element. Since K is a Klein group of orger 4, then $K < C_H(b)$. However, the image $\overline{b}$ of the element b is not unit in $H/C_G(K)$, therefore $|v| = 2$.

Now we will prove that $b \in R(H)$. By Lemma 4, there is a normal Abelian subgroup L of finite index in H, on which v acts strictly real. Those.

$$v^{-1}dv = d^{-1} \quad (d \in L), \quad v^{-1}bv = b^{-1}.$$

Further, $b^{-1}db \in L$ and $b^{-1}d^{-1}b^{-1} = v^{-1}(b^{-1}db)v = bd^{-1}b^{-1}$ or $d^{-1} = b^{-2}d^{-1}b^2$, and since b is an element of odd order, then $b \in \langle b^2\rangle) < C_G(L)$, i.e. element b has a finite index centralizer in H, so the element b itself belong to $R(H)$.

Let A be a Sylow 3-subgroup of H containing the element h. The Sylow 3-subgroups in H are conjugate [14] and how we just showed $R(H)$ has a 3-element. Consequently, $L = A \cap R(H) \neq 1$ and $L \lhd A$. But then in $Z(A)$ exists third order element s. And since $h \in A$, then $s \in C_G(h)$. Take in $C_G(h)$ some Sylow 3-subgroup Q containing elements h, s. As shown above, the Sylow 2-subgroup of $C_G(h)$ can only be a group of order two. From here and from the Brauer-Suzuki theorem [15; 16] it follows that $C_G(h) = V\lambda\langle y\rangle$, where $|y| \leq 2$. Obviously $V \lhd M = C_G(h)\lambda\langle t\rangle$. There is an involution t_1 in $N_M(Q)$, conjugate with t in M by Frattini Lemma [13]. If $t_1 \in H$, then being conjugate with i in G involution $t_1 \in K$ by the condition of the lemma, and this contradicts the equalities $t_1^{-1}ht_1 = h^{-1}, |h| = 3$ and almost regularity h in H. This means $t_1 \notin H$.

Consider the intersection $D = Q \cap H$. If Q does not lie in H, then by properties of nilpotent groups $N_G(D) \neq Q$. Take the element l from the difference $N_G(D) \setminus D$ and consider the intersection $H \cap H^l$. It contains an element of prime order s with an infinite centralizer in H. By Lemma 8 it will have an infinite centralizer in H^l, but then $H = H^l$ by Lemma 9 and $l \in H$. So our the assumption is wrong and Q is contained in H. Repeating this reasoning for the involution t_1 instead of the element l and the subgroup Q instead of D we get the inclusion $t_1 \in H$ in contradiction with what was proved above. This means that K is not a Klein group of orger 4 and $H = C_G(i)$.

Now suppose S be infinite. By Theorem 2 of [4] S is an extension of a quasi-cyclic 2-group by a reversing automorphism. Since R is a layer-finite group, $S \cap R$ is a quasi-cyclic 2-group. □

Lemma 11. *At least one of statements is valid:*

1) S is a 8th order dihedral group, and i, j are conjugate in G;

2) $H = C_G(i)$ and Sylow 2-subgroups from $R(H)$ are locally cyclic or generalized quaternion groups.

Proof. If $H \setminus R(H)$ does not possess involutions, conjugate with i in G, then by Lemma 10 $H = C_G(i)$ and the Sylow 2-subgroups from $R(H)$ are locally cyclic or generalized quaternion groups. The same is true if we assume that $|K| = 2$.

Let $K = \langle i\rangle \times \langle t\rangle$. By Lemma 2, the maximal elementary Abelian subgroup R in S has an order 4, and since $|C_G(i)| = \infty$, then $t \notin R$ (since $t \notin C_G(j)$).

Suppose that $j = g^{-1}ig$ and $D = H \cap H^g$ Let V be a Sylow 2-subgroup of D and $R \leq V$, P, Q are Sylow 2-subgroups from H, H^g, respectively, and $V = P \cap Q$. Obviously $R \leq Z(V)$ (since $i \in Z(V)$, so we select $j \in Z(V^g)$ also, V, V^g are conjugate in D, hence $V^g = V^h, i^h \neq j$ and R is a maximal subgroup in V).

Since $K < P$ and $t \notin C_G(j)$, then $V \neq P$, similarly to $V \neq Q$. Hence from the normalizer condition in nilpotent groups $N_G(V)$ does not lie in H. Obviously $R \lhd L = N_G(V)$.

If there was no element in L that induces an automorphism of 3-th order in R, then $L = C_L(R)(d)$, where $d \in P < H$ and $C_L(R) < C_G(i) \leq H$. Therefore, $L < H$, contrary to what was proved above. So in $N_G(V)$ there is an element that induces an automorphism of order 3 in R. If V had element of order 4, then it could be chosen in V so that $b^2 = j$, and since $|K| = 4, K \lhd H, b \in H$, then $b^2 = j$ implies $t \in K < C_G(j)$ contrary to what was proved above. This contradiction means that $R = V = C_P(j)$.

Further, P is a dihedral group or a semidihedral group [17] and $K \lhd P$. Therefore, P is a dihedral group of order 8. Then, in view of the conjugacy of Sylow subgroups in H, the same is valid for S. □

Remark 1. In view of the structure of a non-Chernikov almost Abelian almost layer-finite group B we assume that the number p chosen so that it does not divide the index $|B : L(B)|$, where $L(B)$ is a nilpotent radical of a groups B (this index is finite, and the set $\pi(B)$ is infinite by Theorem 1.1.6 from [2]).

In addition to choosing the number p, we can assume that it does not belongs to the set $\cup\pi(C_B(K))$, where K runs through all elementary Abelian subgroups of B having in B finite centralizers (Similar to the proof of Lemma 11 from [4], it is shown that the set of non-conjugate elementary Abelian subgroups of almost layer-finite group V with finite centralizers in V is finite) in the case of Chernikov group H and in the case of non-Chernikov H the number $p \notin \pi(C_H(K))$ for elementary Abelian subgroup K of H with finite centralizer in H.

We fix a notation. In the future we will talk about the element a from B or from H of prime order chosen according to the remark.

Consider groups of the form $L_n = \langle a, a^{s_n}, i\rangle$, where $i \in Z(S)$, $s_n \in C_G(i)$, $a \in G \setminus H$ is a strictly real element with respect to the involution i (if we consider the case of the Chernikov group H, then the element a is taken from the non-Chernikov group B and the choice of its order is unlimited; if H is a non-Chernikov group, then the element a can be chosen from subgroup conjugate to H and again to choose its order infinitely many variants).

Such groups as shown by A.N. Izmailov (see, for example, [5]) are finite, as soon as the groups $\langle a, a^{s_n}\rangle$ are finite, and the last groups are finite since G is a Shunkov group.

Denote the set of groups L_n by $\mathfrak{N}$.

The set $\mathfrak{N}$ is infinite, otherwise for some sequence of the elements $s_1, s_2, ..., s_n, ...$ from $C_G(i)$ $a^{s_1} = a^{s_2} = ... = a^{s_n} = ...$ and hence $s_n s_1^{-1} \in C_G(a), n = 1, 2, ...$ and $C_G(a) \cap C_G(i)$ is infinite, but then, by Lemma 7, $a \in H$ contrary to the choice of the element a.

To prove the theorem, we still need several lemmas.

Lemma 12. *The subgroups of the set $\mathfrak{N}$ are almost all semisimple.*

Proof. Suppose that Sylow 2-subgroups in G are cyclic or generalized quaternion groups.

Then the Sylow 2-subgroup of L_n is, by assumption, cyclic or a generalized quaternion group for any subgroup L_n of $\mathfrak{N}$ and according to the Brauer-Suzuki theorem [15; 16] $L_n = O_{2'}(L_n) \cdot C_{L_n}(i)$. If the element a does not belong to $O_{2'}(L_n)$, then the element $\overline{a} = aO_{2'}(L_n)$ is strictly real with respect to the involution $\overline{i} = iO_{2'}(L_n)$. But the involution $\overline{i}$ is contained in the center of the factor group $L_n/O_{2'}(L_n)$. Contradiction means the inclusion of the element a in $O_{2'}(L_n)$. Obviously the same is true for the element a^{s_n}. Then, in view of generating L_n by elements a, a^{s_n}, i, it has the structure $O_{2'}(L_n)\lambda\langle i\rangle$, that is, it is solvable by the Feit-Thompson theorem.

Suppose, that $L_1, L_2, ..., L_n, ...$ is an infinite sequence of different subgroups from $\mathfrak{N}$, where $L_n = \langle a, a^{s_n}, i\rangle$ and L_n has a non-trivial elementary Abelian subgroup V_n, normal in L_n, $n = 1, 2, ...$ We represent V_n as $V_n = Z_n \times F_n$, where $Z_n = C_G(i) \cap V_n$, and if $|F_n|$ are odd, then $F_n = \langle h \in V_n \mid h^i = h^{-1}\rangle$.

If in the set of subgroups of the form V_n, $n = 1, 2, ...$, there is only a finite set of different, it is obvious without breaking the generality of reasoning, we can assume that

$$V = V_1 = V_2 = ... = V_n = ...$$

Consider the maximal almost layer-finite subgroup M in G containing $N_G(V)$. By assumption, M is almost layer-finite and $L_n = \langle a, a^{s_n}, i\rangle < M$.

Consider two cases:

1) $C_M(i)$ is infinite. Since M is a maximal almost layer-finite in G subgroup, then by Lemma 7 $C_G(i)$ contained entirely in M. By Lemma 1, i is contained in a finite normal subgroup of M, and therefore, in a layer-finite radical $R(M)$. The element a^{s_n} is a strictly real relative to i and contained in M. From here we get

$$ia^{s_n}i = (a^{s_n})^{-1}, (a^{s_n})^{-1}ia^{s_n} \in R(M).$$

Comparing these relations, we note: $i(a^{s_n})^2 \in R(M)$. Then, $(a^{s_n})^2 \in R(M)$ and, taking into account the oddness of the order of the elements a^{s_n}, finally $a^{s_n} \in R(M)$. Due to the infinity of the set $\{a^{s_n}\}$, $n = 1, 2, ...$, we get contradiction with the definition of a layer-finite group.

2) $C_M(i)$ is finite. In this case, by Lemma 4 there is a normal in M subgroup U of finite index in M each whose element is strictly real with respect to i.

The element a is also strictly real with respect to i. From here following equality

$$aha^{-1} = ia^{-1}iih^{-1}iiai = ia^{-1}h^{-1}ai = a^{-1}ha$$

or $ha^2 = a^2h$ show that a is permuted with any element of U. So it belongs to a finite normal subgroup of the group M and belongs to its layer-finite radical $R(M)$. Similar we show $a^{s_n} \in R(M)$, but this cannot be due to the infinity of the set $\{a^{s_n}\}$ and by the definition of layer-finite radical.

Thus, both cases are impossible. Therefore not breaking the generality of reasoning, we can assume that the subgroups of the form V_n, $n = 1, 2, ...$, are different.

Let $Z_n \neq 1$ for any n. Suppose first that the set of primes p for p-subgroups Z_n is finite.

Without breaking the generality of reasoning, we can assume that all Z_n are p-groups by one prime number p. Due to the properties of almost layer-finite groups among them only a finite number of subgroups not conjugate in $C_G(i)$. Therefore, without breaking the generality of reasoning can be considered that

$$Z = Z_1^{t_1} = ... = Z_n^{t_n} = ...,$$

where $t_n \in C_G(i)$. The set of $\{t_n \mid n = 1, 2, ...\}$ can be as finite, so and infinite. However, the set $\{L_n^{t_n} \mid n = 1, 2, ...\}$ is always infinite due to the choice of the pair (a, i).

Consider $N_G(Z)$. By assumption, a maximal almost layer-finite subgroup X in G containing $N_G(Z)$ is almost layer-finite.

First, let the orders of the subgroups V_n, $n = 1, 2, ...$, are odd.

If $C_X(i)$ is infinite, then by Lemma 1 the involution i belongs to a finite class of conjugate involutions in X. If there are infinitely many subgroups $F_n^{t_n}$, we find the various elements

$$f_1, f_2, ..., f_n, ...$$

such that $f_n^{-1} i f_n = f_k^{-1} i f_k$. Then from equality $f_n^{-1} f_k = i f_n f_k^{-1} i = f_n f_k^{-1}$ followed $f_k^2 = f_n^2$, but f_i have odd prime orders. Contradiction.

If $C_X(i)$ is finite, then as above $a^{t_n}, a^{s_n t_n} \in R(X)$, but there are infinitely many such different elements. Contradiction with layer-finiteness of $R(X)$.

Then, in any case, we can assume, without breaking the generality of reasoning, that

$$F = F_1^{t_1} = ... = F_n^{t_n} = ...$$

and that means

$$V = V_1^{t_1} = ... = V_n^{t_n} = ...$$

By including $N_G(V)$ in the maximal almost layer-finite subgroup W of G, we get that i, $L_n^{t_n} < W$, $n = 1, 2, ...$

If the involution i belongs to the layer-finite radical $R(W)$ of the group W, its centralizer in W is infinite and by Lemmas 5, 7 $W = H$ and a^{t_n} belongs to H together with the element a (t_n taken from H).

If $i \notin R(W)$, then by Lemmas 1 and 4 there is in W an infinite Abelian normal subgroup of finite index in W, consisting of strictly real elements with respect to i. In this case as above, we obtain in the layer-finite radical $R(W)$ infinitely many elements $a^{t_n}, a^{s_n t_n}$ of the same order.

A contradiction means that if the orders of subgroups of the form V_n are odd, then $Z_n = 1$ for almost all numbers n. But then for these n subgroups $V_n\langle a\rangle\lambda\langle i\rangle$ are Frobenius groups with the complement $\langle i\rangle$. In this case, by the properties of Frobenius groups, $V_n\langle a\rangle$ are Abelian groups.

Without breaking the generality of reasoning, we can assume that i, $V_n < N_G(\langle a\rangle) = D$, $n = 1, 2, ...$ According to the above, all elements of V_n are strictly real with respect to i, and by the assumption D is an almost layer-finite group. Since there are infinitely many subgroups V_n and these subgroups are consist of elements strictly real with respect to i, then there is an infinite subgroup of finite index in D, which is elementwise permutable with V_n, $n = 1, 2, ...$ Then, having finite index centralizers in D, subgroups V_n will be contained into the layer-finite radical $R(D)$ of the group D, and this cannot be due to its layer-finiteness, the infinity of the set $\{V_n\}$ and finiteness of $\pi(\{V_n\})$.

Since Sylow 2-subgroups in G are cyclic or generalized groups quaternions, then V_n cannot be a 2-group $n = 1, 2, ...$ After all, then in L_n there would be the only involution i centralizing the element a, and we chose it strictly real with respect to i.

Thus, we have the case: the set of prime numbers p for p-subgroups Z_n is infinite. Then there is infinitely many prime divisors of orders of subgroups V_n.

All Z_n will be contained into H, and moreover, almost all Z_n lie in $R(H)$ since we chose them by different p, and $\pi(H \setminus R(H))$ is a finite set. Then they are in view of the infinite isolation H with their centralizers tighten in H subgroups F_n, $n = 1, 2, ...$ Again, almost all F_n, and therefore V_n contain into $R(H)$. Then using Lemma 5 and the properties of layer-finite groups we get the inclusion $a \in H$, which is impossible. If in this case $Z_n \neq 1$ for any n, then again as above we obtain $V_n < N_G(\langle a\rangle) = D < U$, where U is a maximal almost layer-finite group containing D, $V_n < N_G(\langle a^s\rangle) = D_s < U_s$, where U_s is a maximal almost layer-finite group containing D_s. Since V has a normal Abelian subgroup of finite index consisting of strictly real elements with respect to i, then due to the infinity of $\pi(\{V_n\})$ for almost all s $U = U_s$ and hence $\{a^s\} < R(U)$ for the infinite set $\{a^s\}$. Contradiction.

It remains to consider the case when V_n is a 2-group.

Recall that we assume that the lemma is false and

$$L_1, L_2, ..., L_n, ...$$

is an infinite sequence of subgroups from $\mathfrak{N}$ and L_n has non-trivial elementary Abelian subgroup V_n, normal in L_n, $n = 1, 2, ...$ We represent V_n as $V_n = Z_n \times F_n$, where $Z_n = C_G(i) \cap V_n$. Above it is shown that without breaking the generality of reasoning can be considered

$$Z = Z_1^{t_1} = Z_2^{t_2} = ... = Z_n^{t_n} = ...$$

for some elements $t_n \in C_G(i)$.

Let V_n be a 2-group. We show that V_n does not contain involutions with infinite centralizers in H. Indeed, if $j \in V_n$ and $C_H(j)$ is infinite, then by Lemma 7 $C_G(j) < H$ and $|N_G(V_n) : C_G(j) \cap N_G(V_n)| < \infty$ implies $N_G(V_n) < H$ together with L_n, which is impossible. In this way, V_n contains only almost regular involutions in H, and by theorem from [**?**] and Lemma 7 $V_n \cap H$ is a cyclic group, that is, $V_n \cap H = (j)$.

By Lemma 2, we have $|V_n| \leq 4$. If $|V_n| = 2$, then $V_n = Z_n, \quad n = 1, 2, ...$, and this case has already been considered above and we proved its impossibility. So $|V_n| = 4$ and $V_n^{t_n} = \langle j \rangle \times \langle k_n \rangle$, where k_n is an involution for suitable elements t_n. As we have shown, the set $\{k_n\}$ is infinite. Hence we get the infinity of the set $\{ik_n\}$.

From the structure of the group $V_n \lambda \langle i \rangle$ we conclude that the order of the element ik_n may be equal to only 4 (it cannot be equal to 2 because of Lemma 2).

Recall that by X we denote the maximal almost layer-finite subgroup of G, which contains $N_G(Z)$.

If i is an almost regular involution in X, then by Lemma 4 in X there is a normal subgroup L_1 of finite index on which i acts strictly real. By Lemma 1 almost all involutions t_n are almost regular in X, then we can assume without breaking the generality of reasoning that they are all almost regular in X. Again, by Lemma 4, using the conjugacy of t_n in X (see Lemma 3), we find the normal subgroup L_2 of finite index in X on which all t_n act strictly real. Taking the intersection of $L_1 \cap L_2$ we get the subgroup L_3 also normal in X and of finite index in it whose centralizer contain elements it_n. But in the almost layer-finite group X this situation is impossible (all elements it_n have the same order 4) since this contradicts to Lemma 3 and to theorem on the power of classes of conjugate elements.

If i has an infinite centralizer in X, then using of Lemma 5 we get the inclusion $X \subseteq H$ and, therefore, $C_H(j)$ is infinite, this is contrary to the choice of j. □

Let L be a semisimple group, that is, does not have a soluble normal subgroup. Following [5] we denote by $F(L)$ the socle of L, i.e. normal subgroup of the highest order of L, which is direct product of simple groups.

Lemma 13. *Subgroups from the set $\mathfrak{N}$ have a socle, isomorphic to $PSL(2, q)$, where $q > 3$ is odd.*

The statement of the lemma is proved in view of Lemma 12 in exactly the same way as the theorem in [8].

Lemma 14. *Set* $\mathfrak{A} = \{L \in \mathfrak{C} \mid$ *all involutions of* $H \cap L$ *are contained in* $R(H)\}$ *is finite.*

Proof. Suppose that $\mathfrak{A}$ is infinite and

$$L_1, L_2, ..., L_n, ...$$

is an infinite sequence of different semisimple subgroups of $\mathfrak{A}$. Then $H_n = L_n \cap H$ is a strongly embedded subgroup in L_n for any n. (If H_n were not strongly embedded, i.e. $H_n \cap H_n^b$ would contain involutions for $b \in L_n \setminus H_n$, then in $H \cap H^b$ would contain the involution k from $R(H)$ the first, since $\mathfrak{a}$ so defined, secondly, k is the image of an involution from $R(H)$ in H^b, hence $k \in R(H^b)$, and therefore $H = H^b$ and $L_n < H$, which is impossible.)

Take in H_n some Sylow 2-subgroup Q_n and $i \in Z(Q_n)$. If S were not a dihedral group of order 8, then by Lemma 11 Q_n was be a cyclic group or generalized quaternion group. It is not difficult to see from the infinite isolation of H and from the properties of subgroups from $\mathfrak{A}$ that the subgroup Q_n is a Sylow group in L_n.

Then repeating the reasoning from the beginning of the proof of the lemma when considering the case when Sylow 2-subgroups in G are cyclic or generalized groups of quaternions, replacing the conditions imposed on the Sylow 2-subgroups of the group G on the condition for Sylow 2-subgroups from L_n, we get the impossibility this situation.

Consequently S is a dihedral group of order 8 and by Lemma 1 Q_n is an elementary Abelian group of order 4, moreover $Q_n \lhd H_n$ and $Q_1 = Q_2 = ... = Q_n = ...$

By Suzuki theorem [18] $L_n = \langle a, Q_n \rangle$ are isomorphic to $SL(2, Q)$, over the field Q of characteristics 2, but this contradicts Lemma 13. □

Lemma 15. *Every involution in a simple non-Abelian subgroup* U *of the group* G *is contained in a maximal elementary Abelian subgroup of order 4 of* U*.*

Proof. Let U be a simple non-Abelian group. Then, by the Brauer-Suzuki theorem [15; 16], any of its involutions is contained in the elementary Abelian subgroup of order not less than four, but by Lemmas 1 and 2 the order of this elementary Abelian subgroup cannot be greater than four. □

Lemma 16. *Orders of factor groups* $L/F(L)$, $\quad L \in \mathfrak{N}$ *limited in aggregate.*

Proof. Suppose that the lemma is false. In this case there is a sequence

$$L_1, L_2, ..., L_n, ...$$

for which $|L_n/F(L_n)|$ grows with the number n.

Let S_n be a Sylow 2-subgroup of L_n and $i \in S_n$. By Lemma 2.15 from [5] $Q_n = S_n \cap F(L_n)$ is a Sylow 2-subgroup in $F(L_n)$ and $L_n = N_{L_n}(Q_n)F(L_n), \quad n = 1, 2, ...$ Because $L_n = \langle a, a^{s_n}, i\rangle$, where $s_n \in C_G(i), \quad iai = a^{-1}$, then $i \notin F(L_n)$. (If $i \in F(L_n)$, then $a \notin F(L_n)$ otherwise the order of the group $L_n/F(L_n)$ did not grow. At that same time, the normality of $F(L_n)$ in L_n follows $a^{-1}ia = a^{-2}i \in F(L_n)$, but $a^{-2} \notin F(L_n)$, since a is an element of odd order and from $a^{-2} \in F(L_n)$ would get $a \in F(L_n)$, which is impossible.)

By Lemmas 1 and 2, the lower layer R_n of the center $Z(Q_n)$ is a subgroup of order ≤ 4. If $|R_n| = 4$, then by Lemma 15 all involutions of Q_n are contained in $R_n \lhd N_{L_n}(Q_n)$. And since the orders of the factor groups $N_{L_n}(Q_n)$ grows by virtue of the isomorphism theorem together with the number n, then S_n is a dihedral group of order 8 and $Q_n = R_n, \quad n \geq q$ for some number q. The subgroup $N_{F(L_n)}(R_n)$ is strongly embedded in $F(L_n)$ and according to Suzuki theorem [18] $F(L_n)$ is isomorphic to $SL(2, Q)$, where Q is a field of characteristic 2, $\quad n \geq q$. But this contradicts Lemma 13.

If $|R_n| = 2$, then by Lemma 13 the Sylow 2-subgroup of $F(L_n)$ is a dihedral group of order at least eight. If the orders of these dihedral groups are not limited in aggregate, then on the properties of linear groups unlimited the number of prime divisors in the set $\pi(C_G(k))$ for some involution k. But then $C_G(k)$ is not Chernikov and we get a contradiction with the condition of the theorem. So the order of the Sylow 2-subgroups of $F(L_n)$ are bounded in aggregate. From here according to Brouwer-Feit Theorem [19], the orders $F(L_n)$ are also limited in aggregate. But due to the semisimplicity of L_n, this implies that the index $|L_n : F(L_n)|$ is bounded.

Thus, we obtain the boundedness of orders of factor groups of the form $L_n/F(L_n), \; n = 1, 2, ...$ Therefore, the orders of the factor groups $L/F(L), \; L \in \mathfrak{C}$, are bounded in aggregate. □

Let, for the involution t $R_t = \langle t\rangle \times \langle k\rangle$ be a Klein group of orger 4 and A_t be a maximal almost layer-finite subgroup of the group G containing $C_G(t)$. Obviously $R_t < A_t$ and t belongs to the layer-finite radical $R(A_t)$ of the group A_t. If $C_{A_t}(k)$ is finite, then R_t is called *highlighted*. If the involution t is unique in $R(A_t)$, then $t \in C_{A_t}(R(A_t))$.

Lemma 17. *The highlighted subgroup is a Chernikov one.*

Proof. Let t be an involution, A_t be a maximal almost layer-finite subgroup of G containing $C_G(t)$ (recall that, by the condition of the theorem, involutions centralizers in G are Chernikov). As shown earlier, $C_G(t)$ is an infinite group, then A_t has a non-trivial normal in A_t subgroup K generated by all involutions with centralizers that are infinite in A_t. Since K is a finite group, the index $|N_{A_t}(K) : C_{A_t}(K)|$ is finite. By the inclusion $C_{A_t}(t) \leq C_{A_t}(K)$ we see that the index $|N_{A_t}(K) : C_{A_t}(t) \cap N_{A_t}(K)|$ is finite. Then A_t is a

Chernikov group as a finite extension of the Chernikov group (see Lemma 2.3 from [5]). □

Let

$$L_1, L_2, ..., L_n, ...$$

be an infinite sequence of different subgroups from $\mathfrak{N}$ (semisimple groups and $L_n \cap R(H)$ by Lemma 13 have almost regular involution j_n of $C_H(i)$.

There is an infinite set of elements $\{c_1, c_2, ..., c_n, ...\}$ in $R(H)$ such that $\{M_1, M_2, ...M_n, ...\}$, where $M_n = L_n^{c_n}, \quad n = 1, 2, ...$ consists of various subgroups and in $\cap M_n$ contains the same highlighted subgroup R_i.

Lemma 18. *If R_t is a highlighted subgroup of $V = \cap M_n, \quad n = 1, 2, ...,$ then*

1) the set $\mathfrak{B}_t = \{C_{M_n}(x) \mid x \in R_t \setminus 1, \quad n = 1, 2, ...\}$ is finite;

2) the orders of subgroups of the set $\mathfrak{N}$ are bounded in aggregate.

Proof. First we prove 1). Let $\{C_{M_n}(t) \mid n = 1, 2, ...\}$ be infinite. Let us prove that in this case the orders of the subgroups $M_n \cap R(A_t)$ are not bounded in aggregate.

Suppose that the orders of the subgroups $R(A_t) \cap M_n$ are bounded in aggregate. From here and from the inclusion $\langle t \rangle < C_{A_t}(R(A_t))$ obviously follows boundedness in aggregate of orders $|C_{M_n}(t)|, n = 1, 2, ...,$ and since $R_t < C_G(t) < A_t$ and R_t is a highlighted subgroup, then some involution u of R_t induces in a normal subgroup B_t of finite index in $R(A_t)$ automorphism, which translates any element from B_t to the inverse (Lemma 4). We represent $A_t = B_t Q$, where Q is a finite subgroup from A_t and $R_t < Q$. From here due to the boundedness of the orders $|C_{M_n}(t)|, n = 1, 2, ...,$ infinity of $\{C_{M_n}(t) \mid n = 1, 2, ...\}$, layer-finiteness of $R(A_t)$ and the finiteness of the index $|R(A_t) : B_t|$ implies the existence of such the numbers q such that $C_{M_q}(t)$ has an element d representable as $d = br$, where $b \in B_t$, $r \in Q, |b^{2|Q|}| > |C_{M_q}(t)|$ and $b^u = b^{-1}$.

Based on such a representation of the element d, we write $d^u d^{-1} = b^{-1} r^u r^{-1} b^{-1}$. Obviously, $r^u r^{-1} \in C_{A_t}(B_t)$ and $b^{-1} \in R(A_t), r \in Q, u \in R_t < Q$, and therefore $d^u d^{-1} = b^{-2} r^u r^{-1} \in C_{M_q}(t)$ and $r^u r^{-1} \in Q \cap C_{A_t}(B_t)$. But then $(d^u d^{-1})^{|Q|} = b^{-2|Q|} \in C_{M_q}(t)$, which is impossible.

Hence $|R(A_t) \cap M_n|$ is not limited in aggregate. Choose in $R(A_t)$ Abelian normal subgroup C_t of finite index, moreover, $t \in C_t$ (such a subgroup exists by Lemma 4). It is obvious that $|C_t \cap M_n|$ is also not limited in aggregate.

Since A_t by Lemma 17 is a Chernikov group, then C_t satisfies the minimal condition and $\{M_n\}$ has an infinite subset $\mathfrak{C}_1$, and A_t has such a subgroup X_1, that for any subgroup $L \in \mathfrak{C}_1 \quad X_1$ coincides with the subgroup from $C_t \cap L$ generated by all elements of prime order from $C_t \cap L$. For the same reasons $\mathfrak{C}_1$ has such an infinite subset $\mathfrak{C}_2$, and C_t such a subgroup $X_2 > X_1$, that for any subgroup $L \in \mathfrak{C}_2$ the factor group X_2/X_1

coincides with the subgroup generated by all elements of prime orders from $(C_t \cap L)/X_1$. Thus we build a chain

$$\mathfrak{C}_1 \geq \mathfrak{C}_2 \geq ...$$

and accordingly a chain

$$X_1 < X_2 < ...$$

Let k be an involution from $R_t \setminus \langle t \rangle$. By the condition $A_k \neq A_t$ and t is an almost regular in A_k. Then, by Lemma 4, t induces in some normal subgroup C_k of finite index from A_k an automorphism, translates all elements into inverse (we can assume that C_k is chosen from $R(A_k)$).

Let's assume that the set $\{C_L(k) \mid L \in \mathfrak{C}_n\}$ is infinite for any n. Then the orders of the subgroups $C_k \cap L$ as we showed in the case of t and B_t not limited in aggregate. Since A_k is Chernikov there is a finite subgroup in C_t of odd order $W \neq 1$, which starting with some number q is contained in a certain subgroup from $\mathfrak{C}_n$ for any n. Obviously the subgroup $T = (X, R_t, W)$ is locally finite.

In particular, we have $tct = t^{-1}$ for any element $c \in W < C_k$. Since $\langle X, R_t \rangle < A_t \cap T$, then by Lemma 5 $T < A_t$ and, hence $W < A_t$. Since any element t from W has an odd order and is strictly real with respect to t, then taking into account the normality of the layer-finite radical $R(A_t)$ of the group A_t and the uniqueness of the involution t in $R(A_t)$ we get for any element a from $R(A_t)$: $c^{-1}ac = t^{-1}ctac = t^{-1}cac^{-1}t$ or $c^{-2}ac = a$, which implies since the order of the element c is odd, its hit in the centralizer of the layer-finite radical $R(A_t)$, and it means, into the layer-finite radical itself. Then $W < R(A_t) \cap R(A_k)$ and by Lemma 5 we get $A_t = A_k$. Contradiction means that for an involution k starting from some number q set $\{C_L(k) \mid L \in \mathfrak{C}_n, \quad n \geq q\}$ is finite.

Let E_1 be some subgroup from $\mathfrak{C}_q$, E_2 be some subgroup from $\mathfrak{C}_{q+1}$ etc. such that $X_n < E_n$. By the above, $\{C_{E_n}(x) \mid x \in R_t \setminus \langle t \rangle, n = 1, 2, ...\}$ is finite. It can be considered without breaking the generality of reasoning, that $E_n = M_n$, i.e. $\mathfrak{B} = \{C_{M_n}(x) \mid x \in R_t \setminus \langle t \rangle\}$ is finite, moreover $R_t < M_n$ by assumption, $X_t < M_n$ by the construction of the chain $\{M_n\}$.

Let $F(M_n)$ be a socle of M_n. By Lemma 13 all $F(M_n)$ contain involutions. In addition, $F(M_n) \cap R_t \neq 1$. If not, and if there is a q such that $F(M_q) \cap R_t = 1$, then due to the properties of primary groups and normality of $F(M_q)$ in M_q, the subgroup $F(M_q)$ is non-trivial intersects with the center of a Sylow 2-subgroup that contains R_t, in particular, there is an involution z centralizing R_t. But then $R_t < C_G(z), \quad r \notin R_t$ means that in $C_G(z)$ there is an elementary Abelian subgroup of order 8, one of the involutions of which is almost regular in a maximal almost layer-finite subgroup in G, containing $C_G(z)$ (R_t is chosen so), and this contradicts Lemma 2.

Thus, $F(M_n) \cap R_t \neq 1$. Since $F(M_n)$ is simple and the orders of the factor groups $M_n/F(M_n)$ are limited in aggregate (Lemmas 13, 16), then we can assume that $X_n < F(M_n)$ and $|F(M_n)|$ grow with n. If the set $\{F(M_n) \mid M \in \mathfrak{C}, t \notin F(M)\}$ is infinite, then $F(M) \cap R_t \neq 1$ implies unboundedness in the aggregate of orders of subgroups of the set

$$\mathfrak{B}_k = \{F(M) \mid M \in \mathfrak{N}, \quad t \notin F(M), \quad k \in F(M)\},$$

where k is an involution from $R_t \setminus \langle t \rangle$. And since $\mathfrak{B}$ is a finite set, then $|C_{F(M)}(k)|$ $(F(M) \in \mathfrak{B}_{\mathfrak{k}})$ limited in aggregate. But then $|F(M)|$ is also limited in aggregate by Brouwer-Fowler Theorem [17] and by Lemma 13. Contradiction. Therefore, without breaking the generality of reasoning, we can assume that $t \in F(M_n)$, and the rest involutions of R_t are not contained in $F(M_n)$.

Let Q_n be a k-invariant Sylow 2-subgroup of $F(M_n)$ containing t (this can be found due to conjugacy of primary Sylow subgroups in M_n and Frattini argument [13]). In the center $Q_n \lambda(k)$ obviously there is an involution. It cannot be different from t, since this would contradict Lemma 2. Consequently, $t \in Z(Q_n)$. At the same time, in view of Lemma 15, t contains in $P_n = \langle t \rangle \times \langle v_n \rangle < Q_n$, where v_n is an involution, $n = 1, 2, ...$ If in $\{P_n\}$ exists infinitely many non-highlighted subgroups, it would be possible to consider that P_n is non-highlighted for any n. Then, by Lemma 11, the Sylow 2-subgroups from M_n are a dihedral groups of order 8 and $Q_n = P_n$, moreover, the subgroups $N_{F(M_n)}(P_n)$ are strongly embedded respectively in $F(M_n)$. Hence by Suzuki theorem [18] we obtain a contradiction with Lemma 13. We assume that P_n is a highlighted subgroup for any n. Since $v_n \in C_G(t) \leq A_t$, then by Lemma 3 we will assume that $v = v_1^{b_1} = ... = v_n^{b_n} = ...$, where $b_n \in C_t < C_G(t)$. However, $X_n < C_t$, which means $X_n < M_n^{b_n} = U_n$. By the sequence $\{M_n\}$, the orders of these subgroups grow infinitely. Therefore the sequence

$$U_1, U_2, ..., U_n, ... \tag{2.1}$$

consists of various subgroups and their intersection contains a highlighted subgroup $P = \langle v \rangle \times \langle t \rangle$.

Repeat for P and for the sequence $\{U_n\}$ the same reasoning with respect to R_t and to the set $\mathfrak{N}$. Based on this reasoning, let us prove the existence in the sequence 2.1 of an infinite subset $\mathfrak{K}$ such that the set $\{C_U(v) \mid U \in \mathfrak{K}\}$ is finite. And since $v \in F(U)$, and $F(U)$ is a simple group by Lemma 13, then, by Brouwer-Fowler Theorem [17] and Lemma 16, the orders of the subgroups from $\mathfrak{K}$ are bounded in aggregate. But then, given the equalities $|U_n| = |M_n|, \quad n = 1, 2, ...$, we obtain a contradiction with unboundedness in aggregate of orders of subgroups from the set $\{M_n\}$. This contradiction means that the set $\{C_{M_n}(x) \mid x \in R_t \setminus \{1\}, \quad n = 1, 2, ...\}$ is finite and assertion 1 is proved.

Let us prove 2. Let the orders of subgroups from $\mathfrak{M}_\mathfrak{n}$ be unbounded in aggregate. As elements of the set $\mathfrak{L}$ we choose subgroups of $\mathfrak{N}$ for whose orders the equality holds

$$|M_1| < |M_2| < ...$$

and the intersection $\cap M_n$ has the highlighted subgroup $R = \langle i\rangle \times \langle j\rangle$.

By 1 $\mathfrak{B}_i = \{C_{M_n}(x) \mid x \in R_t \setminus \{1\}, \quad n = 1, 2, ...\}$ is finite and, as shown above, $F_{M_n} \cap R \neq 1, \quad n = 1, 2, ...$ But then without breaking the generality of reasoning, we will assume that in all M_n the same involution k from R lies. Since $\mathfrak{B}_i$ is a finite set, then by Brouwer-Fowler Theorem [17] and Lemma 16 imply limitation of orders of subgroups from $\mathfrak{N}$ contrary to the choice of the set $\mathfrak{L}$ from $\mathfrak{N}$. Thus 2 is proved. □

Proof of the theorem. We first prove that the set $\mathfrak{N}$ has so infinite subset $\mathfrak{L}$ such that $V = \cap M, \quad M \in \mathfrak{L}$, is a strongly embedded subgroup in each subgroup of $\mathfrak{L}$.

Let $\mathfrak{A}_1$ be an arbitrary infinite subset of $\mathfrak{N}, \quad V_1 = \cap B_1, \quad T_1 = N_{B_1}(V_1), \quad B_1 \in \mathfrak{A}_1, \quad Q_1$ be a Sylow 2-subgroup of V_1 containing $R = \langle i\rangle \times \langle j\rangle$. By Lemma 2, the intersection $R \cap Z(Q_1)$ has the involution t_1. Let A_1 be a maximal almost layer-finite subgroup of G containing $C_G(t_1), \quad Y_1 = A_1 \cap B_1, B_1 \in \mathfrak{A}_1, P_1 = (t_1) \times (z_1)$ is a subgroup of order 4 from Q_1. Since R is a highlighted subgroup by Lemma 18 the set

$$\{C_{B_1}(x) \mid x \in R \setminus (1), \quad B_1 \in \mathfrak{A}_1\} \tag{2.2}$$

is finite. Based on Lemmas 4, 11 it is easy to get an idea to represent the subgroup A_1 in the following form $A_1 = C_{A_1}(t_1)C_{A_1}(k)$, where k is an involution from $R \setminus (t_1)$. From here and from finiteness of the set 2.2 implies the finiteness of the set

$$\{Y_1 \mid B_1 \in \mathfrak{A}_1\}. \tag{2.3}$$

If P_1 is a non-highlighted subgroup, then by Lemma 11 $C_{B_1} \leq Y_1, \quad x \in P_1 \setminus (1)$. If P_1 is a highlighted subgroup, then, by Lemma 18, the set $\{C_{B_1}(x) \mid x \in P_1 \setminus (1), \quad B_1 \in \mathfrak{A}_1\}$ is finite. From here and from the finiteness of the set 2.3 it follows finiteness of a set

$$\{C_{B_1}(x) \mid x \in P_1 \setminus (1), \quad B_1 \in \mathfrak{A}_1\} \tag{2.4}$$

for any subgroup of the form P_1 from Q_1. On the Frattini argument $T_1 = N_{T_1}(Q_1)V_1$, and since $R \leq Q_1$, then by Lemma 14 $N_G(Q_1)$ is finite. Hence the set

$$\{N_{B_1}(Q_1), T_{B_1} \mid B_1 \in \mathfrak{A}_1\}. \tag{2.5}$$

is finite.

If at least one of the subgroups belonging to finite sets 2.2–2.5 not contained in any subgroup of some infinite subset of $\mathfrak{A}_1$, then obviously in $\mathfrak{A}_1$ exists such an infinite subset of $\mathfrak{A}_2$ that

$$V_2 = \cap B_2 \neq V_1, \quad B_2 \in \mathfrak{A}_2, \quad V_1 < V_2.$$

Let $T_2 = N_{B_2}(V_2)$, $B_2 \in \mathfrak{A}_2$, Q_2 be a Sylow 2-subgroup of V_2 and $Q_1 \leq Q_2$. According to Lemmas 1, 2 the intersection of $R \cap Z(Q_2)$ has an involution t_2. Let also A_2 be a maximal almost layer-finite subgroup of G, containing $C_G(t_2)$, $Y_2 = A_2 \cap B_2$, $B_2 \in \mathfrak{A}_2$, $P_2 = (t_2) \times (z_2)$ is a subgroup of order 4 from Q_2. Using the same arguments used in justifying of the finiteness of sets 2.2–2.5, we prove the finiteness of the sets

$$\{C_{B_2}(x) \mid x \in R \setminus (1), \quad B_2 \in \mathfrak{A}_2\} \tag{2.6}$$

$$\{Y_2 \mid B_2 \in \mathfrak{A}_2\} \tag{2.7}$$

$$\{C_{B_2}(x) \mid x \in P_2 \setminus (1), \quad B_2 \in \mathfrak{A}_2\} \tag{2.8}$$

$$\{N_{B_2}(Q_2), T_{B_2} \mid B_2 \in \mathfrak{A}_2\} \tag{2.9}$$

Regarding the set of $\mathfrak{A}_2$ and subsets 2.6–2.9 reason like the previous case, etc. As a result, we get in G strictly increasing chain of subgroups $V_1 < V_2 < ... < V_r < ...$ and, accordingly, the chain $Q_1 \leq Q_2 \leq ... \leq Q_r \leq ...$

Since, by Lemma 18, the orders of subgroups from $\mathfrak{N}$ are bounded in aggregate then the specified chains will terminate at the finite number r, that is, the set $\mathfrak{A}_r$ is the last member of a strictly decreasing series $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset ... \supset \mathfrak{A}_r$ has such an infinite subset of $\mathfrak{L}$, for subgroups whose claims are true:

1) $V = \cap M$, $M \in \mathfrak{L}$ and $N_M(V) = V$, $M \in \mathfrak{L}$;

2) if Q is a Sylow 2-subgroup of V, then $N_V(Q) = N_M(Q)$, $M \in \mathfrak{L}$;

3) if P is a Klein subgroup of orger 4 from V, in particular, $P = R$, then $C_V(x) = C_M(x)$, $x \in P \setminus \{1\}$, $M \in \mathfrak{L}$.

Now, based on assertions 1–3, we prove that V is a strongly embedded subgroup in any subgroup of $\mathfrak{L}$. Let E be some subgroup from $\mathfrak{L}$. By assertion 1 $N_E(V) = V$ and assume that for some element g of $E \setminus V$ the intersection of $V \cap V^g$ has an involution z. Let Q be a Sylow 2-subgroup of V^g and $z \in Q$. As Chernikov p-group is ZA-group and satisfies normalization condition [20] by assertion 3 it is easy to prove the inclusion $Q < V \cap V^g$. Since by the Sylow's theorem [9] Sylow 2-subgroups are conjugate in V, then in V there exists the element h such that $Q^{hg} = Q$, and, therefore, $hg \in N_E(Q)$. But on to assertion 2 $hg \in N_E(Q) = N_V(Q) \leq V$ and $g \in V$ contrary to the assumption $g \in E \setminus V$. Therefore, V is a strongly embedded subgroup in any subgroup of $\mathfrak{L}$ and existence of the sets $\mathfrak{L}$ is proved.

Let the set $\mathfrak{L}$ consist of subgroups $C_1, C_2, ..., C_n, ...$ such that $C_n = \langle a^{t_n}, a^{r_n}, i\rangle$, $t_n, r_n \in C_G(i)$.

By the definition of the set $\mathfrak{N}$, we can assume that all $a^{t_1}, ..., a^{t_n}, ...$ are different. As in a group G with H its strongly embedded subgroup and some involution i from H with the condition $\langle i, i^g\rangle$, $g \in G \setminus H$ is finite any element g of $G \setminus H$ has a representation $g = hj$, where $h \in H$, j is an involution of $G \setminus H$ [21], $a^{t_n} = h_n i_n$, where $h_n \in V$, i_n, $n = 1, 2, ...$ are involutions from $C_n \setminus V$, $D_n = V \cap V^{i_n}$ is a group of odd order. Since V is a finite group, then we assume that $h = h_1 = h_2 = ...$, $D = D_1 = D_2 = ...$

Consider the group $U = N_G(D)$. As proven above, $i_n \in U$ and the set $\{i_n \mid n = 1, 2, ...\}$ is infinite. By the conditions of the theorem the group U is almost layer-finite. Involutions $i_1, i_2, ..., i_n, ...$ in view of Lemma 1 can be considered not belonging to $R(U)$. Further, $R(U)$ has a finite index in U, and the set $\{i_n \mid n = 1, 2, ...\}$ is infinite, then we can assume that all this set is selected from one adjacent class $R(U)i_1$. Then from $i_1 = r_n i_n$ follows $i_1 i_n = r_n i_n i_n = r_n \in R(U)$. This means in view of the layer-finiteness $R(U)$ unboundedness in aggregate of the orders of the elements $i_1 i_n$. Then the orders of the elements $a_1^{-t_1} a_n^{t_n} = i_1 h^{-1} h i_n = i_1 i_n$ is also unlimited in aggregate. Hence the orders of the groups $\langle a^{t_1}, a^{t_n}, i\rangle$ is also unbounded in aggregate contrary Lemma 18. The obtained contradiction proves the theorem. The theorem is proved.

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О периодических группах Шункова с черниковским централизатором инволюции

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Аннотация. Группа называется слойно конечной, если множество ее элементов любого заданного порядка конечно. Слойно конечные группы впервые появились без названия в статье С. Н. Черникова (1945), а затем в его последующих публикациях получили название слойно конечных групп. Почти слойно конечные группы являются расширениями слойно конечных групп при помощи конечных групп. Класс почти слойно конечных групп шире, чем класс слойно конечных групп, он включает в себя все группы Черникова, в то время как легко привести примеры групп Черникова, которые не являются слойно конечно. Автор развивает направление характеризации известных хорошо изученных классов групп в других классах групп с некоторыми дополнительными (довольно слабыми) условиями конечности. В данной работе почти слойно конечные группы получают характеризацию в классе периодических групп Шункова. Группа Шункова - это группа G, в которой для любой ее конечной подгруппы K в фактор-группе $N_G(K)/K$ любые два сопряженных элемента простого порядка порождают конечную подгруппу. Мы изучаем периодические группы Шункова с условием: нормализатор любой конечной неединичной подгруппы почти слойно-конечен. Доказано, что если в такой группе централизаторами инволюций являются черниковскими, то группа почти слойно конечна.

Ключевые слова: Бесконечная группа, условие конечности, группа Шункова, группа Черникова, инволюция.

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