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On Error Estimates in S_p for Cubature Formulas Exact for Haar Polynomials

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Abstract. On the spaces S_p, an upper and lower estimates for the norm of the error functional cubature formulas possessing the Haar d-property are obtained for the n-dimensional case.

Keywords: Haar d-property, error estimates for cubature formulas, function spaces S_p.

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Introduction

The problem of constructing and analyzing cubature formulas that are exact for a given set of functions was earlier considered primarily as applied to the computation of integrals exact for algebraic and trigonometric polynomials. For example, the approximate integration formulas of algebraic accuracy can be found in [1, 2]. The cubature formulas exact for trigonometric polynomials in particular were studied in [3–7].

The approximate integration formulas exact for the system of Haar functions can be found in the monograph [8]. The accuracy of approximate integration formulas for finite Haar sums was used in [8] to derive error estimates for these formulas.

A description of all minimal weighted quadrature formulas possessing the Haar d-property, i.e., formulas exact for Haar functions of groups with indices not exceeding a given number d, was given in [9]. The error estimates for quadrature formulas possessing the Haar d-property in the case of the weight function $g(x) \equiv 1$ were obtained in [10]. In particular, in the mentioned paper the upper estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was found for the quadrature formulas having the Haar d-property:

$$\|\delta_N\|_{S_p^*} \leqslant (2^d)^{-\frac{1}{p}},$$

and the lower estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was obtained for the quadrature formulas exact for constants:

$$\|\delta_N\|_{S_p^*} \geqslant 2^{-\frac{1}{p}} N^{-\frac{1}{p}}.$$

The problem of constructing cubature formulas possessing the Haar d-property, i.e., formulas exact for Haar polynomials of degree at most d, was solved in the two-dimensional case in

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[11–15] under the condition that the weight function $g(x_1, x_2) \equiv 1$. The error estimates for these cubature formulas was derived in [16]. In particular, in [16] the upper estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was obtained for the mentioned cubature formulas:

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}} (2^d)^{-\frac{1}{p}}.$$

In the present paper the error estimates of cubature formulas with arbitrary positive coefficients at the nodes, similar to the estimates given above for the one- and two-dimensional cases, are derived in the n-dimensional case. As a result, we find the upper estimates for the error functional δ_N of the cubature formulas possessing the Haar d-property:

$$|\delta_N[f]| \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} \|f\|_{S_p}, \qquad \|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}},$$

and we obtain the lower estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ for the cubature formulas exact for any constant:

$$\|\delta_N\|_{S_p^*} \geqslant \left(2^{n+1} - n - 1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}.$$

1. Basic definitions

In this paper, we use the original definition of the functions $\chi_{m,j}(x)$ introduced by A. Haar [17].

The binary intervals of rank m are the intervals $l_{m,1} = \left[0, \dfrac{1}{2^{m-1}}\right)$, $l_{m,2^{m-1}} = \left(\dfrac{2^{m-1}-1}{2^{m-1}}, 1\right]$, $m = 2, 3, \ldots$, and $l_{m,j} = \left(\dfrac{j-1}{2^{m-1}}, \dfrac{j}{2^{m-1}}\right)$, $m = 3, 4, \ldots$, $j = 2, \ldots, 2^{m-1}-1$. By a binary interval of the 1st rank we will consider the interval $l_{1,1} = [0, 1]$. The binary segments of rank m are the closed intervals $\overline{l_{m,j}} = \left[\dfrac{j-1}{2^{m-1}}, \dfrac{j}{2^{m-1}}\right]$, $m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$.

The left and right halves of $l_{m,j}$ (without its midpoint) are denoted by $l_{m,j}^-$ and $l_{m,j}^+$, respectively. Obviously, $l_{m,j}^- = l_{m+1,2j-1}$, $l_{m,j}^+ = l_{m+1,2j}$.

In [17], the Haar functions $\chi_{m,j}(x)$ are defined by:

$$\chi_{m,j}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & x \in l_{m,j}^-, \\ -2^{\frac{m-1}{2}}, & x \in l_{m,j}^+, \\ 0, & x \in [0,1] \setminus \overline{l_{m,j}}, \\ \left\{\chi_{m,j}(x-0) + \chi_{m,j}(x+0)\right\}/2, & x \text{ is an interior discontinuity point}, \end{cases} \tag{1}$$

$m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$.

Thus, the Haar system of functions is constructed in groups: the mth group contains 2^{m-1} functions $\{\chi_{m,j}(x)\}$, where $m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$. The Haar system of functions includes the function $\chi_1(x) \equiv 1$ too, which is outside of any group.

In the one-dimensional case, the Haar polynomials of degree d are by definition the functions

$$P_d(x) = a_0 + \sum_{m=1}^{d} \sum_{j=1}^{2^{m-1}} a_m^{(j)} \chi_{m,j}(x),$$

where $d = 1, 2, \ldots,\ a_0,\ a_m^{(j)} \in \mathbb{R},\ m = 1, \ldots, d,\ j = 1, \ldots, 2^{m-1}$, and

$$\sum_{j=1}^{2^{d-1}} \left\{a_d^{(j)}\right\}^2 \neq 0.$$

By the 0-degree Haar polynomials we will consider real constants.

In the n-dimensional case, the Haar polynomials of degree d are the functions

$$P_d(x_1, \ldots, x_n) = a_0 +$$
$$+ \sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1 + \ldots + m_s \leqslant d} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\chi_{m_1,j_1}(x_{i_1}) \ldots \chi_{m_s,j_s}(x_{i_s}),$$

where $d = 1, 2, \ldots,\ \ a_0,\ a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s) \in \mathbb{R},\ 1 \leqslant i_1 < \ldots < i_s \leqslant n,\ \ m_1 + \ldots + m_s \leqslant d$, $s = 1, \ldots, n,\ j_k = 1, \ldots, 2^{m_k-1},\ k = 1, \ldots, s$, and

$$\sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1 + \ldots + m_s = d} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \left\{a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\right\}^2 \neq 0.$$

The same way as in the one-dimensional case, by 0-degree Haar polynomials we will consider real constants.

Consider the following cubature formula

$$I[f] = \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \approx \sum_{k=1}^{N} C_k f\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) = Q_N[f], \tag{2}$$

where $\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) \in [0,1]^n$ are the nodes, the coefficients C_k at the nodes are real, $k = 1, \ldots, N$.

The cubature formula (2) is said to possess the Haar d–property (or just the d-property) if it is exact for any Haar polynomial $P(x_1, \ldots, x_n)$ of degree at most d, i. e., $Q_N[P] = I[P]$. Such a formula with the least possible number of nodes is called a minimal cubature formula with the d-property.

We recall the definition of the linear normed space S_p in the n-dimensional case introduced by I. M. Sobol' [8].

Let p be a fixed number with $1 \leqslant p < +\infty$. The set of functions $f(x_1, \ldots, x_n)$ defined in the unit n-dimensional cube $[0,1]^n$ and representable as a Fourier–Haar series

$$f(x_1, \ldots, x_n) = c_0 +$$
$$+ \sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\chi_{m_1,j_1}(x_{i_1}) \ldots \chi_{m_s,j_s}(x_{i_s}) \tag{3}$$

with real coefficients $c_0,\ c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)$ $(1 \leqslant i_1 < \ldots < i_s \leqslant n,\ \ m_1, \ldots, m_s = 1, 2, \ldots,$ $s = 1, \ldots, n,\ j_k = 1, \ldots, 2^{m_k-1},\ k = 1, \ldots, s)$ satisfying the conditions

$$A_p^{(i_1,\ldots,i_s)}(f) =$$
$$= \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} 2^{\frac{m_1-1}{2} + \ldots + \frac{m_s-1}{2}} \left\{\sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \left|c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\right|^p\right\}^{\frac{1}{p}} \leqslant A_{i_1,\ldots,i_s}, \tag{4}$$

(where $A_{i_1,\dots,i_s}$ are real constants, $1 \leqslant i_1 < \ldots < i_s \leqslant n$, $1 \leqslant s \leqslant n$) is called the class $S_p(A_1,\dots,A_n,\dots,A_{i_1,\dots,i_s}\dots,A_{1,\dots,n})$.

It was proved in [8] that the set of functions $f(x_1,\dots,x_n)$ belonging to all the classes $S_p(A_1,\dots,A_n,\dots,A_{i_1,\dots,i_s}\dots,A_{1,\dots,n})$ (with all possible $A_1,\dots,A_n,\dots, A_{i_1,\dots,i_s},\dots,A_{1,\dots,n}$, while p being fixed) equipped with the norm

$$\|f\|_{S_p} = \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} A_p^{(i_1,\dots,i_s)}(f), \tag{5}$$

forms a linear normed space, which is denoted by S_p. All the functions $f(x_1,\dots,x_n)$ that differ by constant terms are regarded as a single function.

The coefficients c_0, $c^{(j_1,\dots,j_s)}_{m_1,\dots,m_s}(i_1,\dots,i_s)$ $(1 \leqslant i_1 < \ldots < i_s \leqslant n,\ m_1,\dots,m_s = 1,2,\dots,$ $s=1,\dots,n,\ j_k = 1,\dots,2^{m_k-1},\ k=1,\dots,s)$ in the representation of the function $f(x_1,\dots,x_n)$ as a series (3) are called the Fourier–Haar coefficients of this function.

In [8] it was proved that the series (3) converges absolutely and uniformly.

2. Derivation of estimates for the norm of the error functional of cubature formulas in S_p

Let (2) be a cubature formula with the coefficients C_k at the nodes satisfying the inequalities $C_k > 0$, $k = 1,2,\dots,N$. We denote the error functional of the cubature formula (2) by $\delta_N[f]$ so that

$$\delta_N[f] = I[f] - Q_N[f] = \int_0^1 \ldots \int_0^1 f(x_1,\dots,x_n)\,dx_1\ldots dx_n - \sum_{k=1}^{N} C_k f\left(x_1^{(k)},\dots,x_n^{(k)}\right), \tag{6}$$

where the function $f \in S_p$, $p>1$. It was shown in [8] that any such function is continuous at all points which coordinates are not binary rational numbers. Hence the integral $\int\limits_0^1\ldots\int\limits_0^1 f(x_1,\dots,x_n)\,dx_1\ldots dx_n$ exists not only in the Lebesgue sense, but also in the Riemann sense.

Let

$$\Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) = 2^{-\frac{m_1-1}{2}-\ldots-\frac{m_s-1}{2}} \left\{ \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \left| \sum_{k=1}^{N} C_k\, \chi_{m_1,j_1}\!\left(x_{i_1}^{(k)}\right)\ldots\chi_{m_s,j_s}\!\left(x_{i_s}^{(k)}\right)\right|^q \right\}^{\frac{1}{q}}, \tag{7}$$

where $q>1$, $1 \leqslant i_1 < \ldots < i_s \leqslant n$, $m_1,\dots,m_s = 1,2,\dots$, $s=1,\dots,n$.

Lemma 1. *If the cubature formula (2) is exact for any constant and $f\in S_p$, then for the absolute value of the error functional satisfies the inequality*

$$\begin{aligned} \left|\delta_N[f]\right| \leqslant & \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty}\ldots\sum_{m_s=1}^{\infty} \Bigg\{ 2^{\frac{m_1-1}{2}+\ldots+\frac{m_s-1}{2}} \times \\ & \times \left[\sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \left| c^{(j_1,\dots,j_s)}_{m_1,\dots,m_s}(i_1,\dots,i_s)\right|^p \right]^{\frac{1}{p}} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) \Bigg\}. \end{aligned} \tag{8}$$

Proof. The series (3) is substituted into (6). Since the series (3) converges uniformly and since the cubature formula (2) is exact for any constant, we have:

$$\delta_N[f] = -\sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \Big\{ c^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(i_1,\ldots,i_s)\times \\ \times \sum_{k=1}^{N} C_k \chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\Big\}. \tag{9}$$

Since the series in (3) is absolutely convergent, it follows that the series in (9) also absolutely converges. Applying the triangle inequality to the expression on the right-hand side of (9), we obtain:

$$\left|\delta_N[f]\right| \leqslant \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \Big| c^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(i_1,\ldots,i_s)\times \\ \times \sum_{k=1}^{N} C_k \chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\Big|. \tag{10}$$

Now we apply the Hölder inequality to the sums over $j_1,\ldots,j_s$ on the right-hand side of (10). Taking into account (7), we obtain the inequality (8). □

It was shown in [9] that there exist Haar polynomials of one variable of degree m that satisfy the equality:

$$\kappa_{m,j}(x) = \begin{cases} 2^m, & x\in l_{m+1,j}, \\ 2^{m-1}, & x\in \overline{l_{m+1,j}}\setminus l_{m+1,j}, \\ 0, & x\in[0,1]\setminus\overline{l_{m+1,j}}, \end{cases} \tag{11}$$

where $m=1,2,\ldots$ and $j=1,2,\ldots,2^m$. It was also proved in [9] that the functions $\kappa_{m,1}(x),\ldots,\kappa_{m,2^m}(x)$ form a basis in the linear space of Haar polynomials of degree at most m.

The definition of the Haar functions (1) and relation (11) imply the following equalities:

$$\chi_{m,j}(x_i) = 2^{-\frac{m+1}{2}}\Big[\kappa_{m,2j-1}(x_i)-\kappa_{m,2j}(x_i)\Big], \tag{12}$$

$$\kappa_{m,2j-1}(x_i)+\kappa_{m,2j}(x_i) = 2\kappa_{m-1,j}(x_i), \tag{13}$$

$i=1,\ldots,n,\ m=1,2,\ldots,\ j=1,\ldots,2^{m-1}$.

Let

$$K^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(x_{i_1},\ldots,x_{i_s}) = \kappa_{m_1,j_1}(x_{i_1})\ldots\kappa_{m_s,j_s}(x_{i_s}), \tag{14}$$

$1\leqslant i_1<\ldots<i_s\leqslant n,\ m_1,\ldots,m_s=1,2,\ldots,\ s=1,\ldots,n,\ j_r=1,\ldots,2^{m_r-1},\ r=1,\ldots,s.$

Lemma 2. *For any ordered set $(i_1,\ldots,i_s)$, $1\leqslant i_1<\ldots<i_s\leqslant n$, $1\leqslant s\leqslant n$, and for any positive integer M there exists at least one ordered set $(M_1,\ldots,M_s)$ satisfying the inequality $M_1+\ldots+M_s\geqslant M$ such that*

$$\varSigma^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}(q) = \sup_{m_1+\ldots+m_s\geqslant M} \varSigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q). \tag{15}$$

Proof. For a fixed positive integer M, we choose $(\widetilde{m}_1,\ldots,\widetilde{m}_s)$ in accordance with condition that the sum $m_1+\ldots+m_s$ is minimum among all ordered sets $(m_1,\ldots,m_s)$ such that $m_1+\ldots+m_s\geqslant M$ and each of the closed s-dimensional binary parallelepipeds $\overline{l_{m_1+1,j_1}}\times\ldots\times\overline{l_{m_s+1,j_s}}$ contains at most one node of the cubature formula (2).

If the coordinates of the nodes of the cubature formula (2) $x_{i_r}^{(k)} \notin \{2^{-\widetilde{m}_r}(2j_r-1) : \ j_r = 1,\ldots,2^{\widetilde{m}_r-1}\}$, $k=1,\ldots,N$, then we set $\widehat{m}_r=\widetilde{m}_r$. Otherwise, we set $\widehat{m}_r = 1+\max\big\{m_r \in \mathbb{N} :$ there exists $x_r^{(K)} = 2^{-m_r}(2j_r^{(K)}-1)$, $1 \leqslant j_r^{(K)} \leqslant 2^{m_r-1}$, $1 \leqslant K \leqslant N\big\}$, $r=1,\ldots,s$.

Then, for all ordered sets $(m_1,\ldots,m_s)$ such that $m_1+\ldots+m_s \geqslant \widehat{m}_1+\ldots+\widehat{m}_s$ the following three conditions are satisfied:

– the inequality $m_1+\ldots+m_s \geqslant M$ holds;

– each of the closed s-dimensional binary parallelepipeds $\overline{l_{m_1+1,j_1}} \times \ldots \times \overline{l_{m_s+1,j_s}}$ contains at most one node of the cubature formula (2);

– the coordinates of every node of the cubature formula (2) differ from the points $\{2^{-m_r}(2j_r-1)\} = \operatorname{supp}\{\kappa_{m_r,2j_r-1}\} \cap \operatorname{supp}\{\kappa_{m_r,2j_r}\}$, $j_r=1,\ldots,2^{m_r-1}$, $r=1,\ldots,s$.

By virtue of (7), (12), we have:

$$\Sigma^{(i_1,\ldots,i_s)}_{\widehat{m}_1,\ldots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{j_1=1}^{2^{\widehat{m}_1-1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s-1}}\left|\sum_{k=1}^{N} C_k\times\right.\right.$$
$$\left.\left.\times\left[\kappa_{\widehat{m}_1,2j_1-1}\left(x_{i_1}^{(k)}\right)-\kappa_{\widehat{m}_1,2j_1}\left(x_{i_1}^{(k)}\right)\right]\cdots\left[\kappa_{\widehat{m}_s,2j_s-1}\left(x_{i_s}^{(k)}\right)-\kappa_{\widehat{m}_s,2j_s}\left(x_{i_s}^{(k)}\right)\right]\right|^q\right\}^{\frac{1}{q}}, \tag{16}$$

$1 \leqslant i_1 < \ldots < i_s \leqslant n,\ 1 \leqslant s \leqslant n.$

According to the choice of $(\widehat{m}_1,\ldots,\widehat{m}_s)$, the coordinates

$$x_{i_1}^{(k)},\ldots,x_{i_s}^{(k)} \quad (k=1,\ldots,N) \tag{17}$$

of every node of the cubature formula (2) differ from the points $\big\{2^{-\widehat{m}_r}(2j_r-1)\big\} = \operatorname{supp}\{\kappa_{\widehat{m}_r,2j_r-1}\} \cap \operatorname{supp}\{\kappa_{\widehat{m}_r,2j_r}\}$, $j_r=1,\ldots,2^{m_r-1}$, $r=1,\ldots,s$, and each of the closed s-dimensional binary parallelepipeds

$$\overline{l_{\widehat{m}_1+1,j_1}} \times \ldots \times \overline{l_{\widehat{m}_s+1,j_s}} \tag{18}$$

contains at most one node of the cubature formula (2) (by this fact every binary segment $\overline{l_{\widehat{m}_r+1,j_r}} = \operatorname{supp}\{\kappa_{\widehat{m}_r,j_r}\}$ contains a projection at most one of node of the cubature formula), $j_r=1,\ldots,2^{m_r}$, $r=1,\ldots,s$. Then the equality (16) can be rewritten as

$$\Sigma^{(i_1,\ldots,i_s)}_{\widehat{m}_1,\ldots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{j_1=1}^{2^{\widehat{m}_1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s}}\left[\sum_{k=1}^{N} C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)\right]^q\right\}^{\frac{1}{q}} =$$
$$= 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{k=1}^{N}\sum_{j_1=1}^{2^{\widehat{m}_1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s}}\left[C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)\right]^q\right\}^{\frac{1}{q}}, \tag{19}$$

$1 \leqslant i_1 < \ldots < i_s \leqslant n,\ 1 \leqslant s \leqslant n$. Here we use the fact that the sum

$$\sum_{k=1}^{N} C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)$$

contains at most one nonzero term for any ordered set $(j_1,\ldots,j_s)$.

Consider the coordinates (17) of nodes of the cubature formula (2) satisfying the equality

$$x_{i_r}^{(k)} = 2^{-\widehat{m}_r}j_r,\ 1 \leqslant j_r \leqslant 2^{\widehat{m}_r},\ 1 \leqslant r \leqslant s. \tag{20}$$

The following $(s+1)$ cases are possible for the quantity of such coordinates of the nodes.

1. Equality (20) does not hold for any of the coordinates (17) of the nodes (for definiteness, the numbers of such nodes are denoted by $k = 1, \dots, N_1$).

2. Only one coordinate in (17) satisfies equality (20) (let $k = N_1 + 1, \dots, N_2$ be the numbers of nodes whose coordinates satisfy this condition).

3. Exactly two coordinates in (17) satisfy equality (20) (to be specific, we assume that the coordinates of the nodes with numbers $k = N_2 + 1, \dots, N_3$ obey this condition).

. . . .

$s+1$. Equality (20) holds for all s coordinates (17) (let $k = N_s + 1, \dots, N$ be the numbers of nodes whose coordinates satisfy this condition).

Moreover, each of the nodes with the numbers $k = N_r+1, \dots, N_{r+1}$ belongs to exact 2^r closed s-dimensional binary parallelepipeds of the form (18), where $r = 0, 1, \dots, s$, $N_0 = 0$, $N_{s+1} = N$.

Given the above, as well as the equality (11), the relation (19) can be rewritten as

$$\Sigma^{(i_1,\dots,i_s)}_{\widehat{m}_1,\dots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\dots-\widehat{m}_s}\Bigg[\sum_{k=1}^{N_1}\left(2^{\widehat{m}_1+\dots+\widehat{m}_s}C_k\right)^q + 2\sum_{k=N_1+1}^{N_2}\left(2^{\widehat{m}_1+\dots+\widehat{m}_s-1}C_k\right)^q + \\ + 4\sum_{k=N_2+1}^{N_3}\left(2^{\widehat{m}_1+\dots+\widehat{m}_s-2}C_k\right)^q + \dots + 2^s\sum_{k=N_s+1}^{N}\left(2^{\widehat{m}_1+\dots+\widehat{m}_s-s}C_k\right)^q\Bigg]^{\frac{1}{q}} = \\ = \left[\sum_{k=1}^{N_1} {C_k}^q + 2^{1-q}\sum_{k=N_1+1}^{N_2} {C_k}^q + 2^{2(1-q)}\sum_{k=N_2+1}^{N_3} {C_k}^q + \dots + 2^{s(1-q)}\sum_{k=N_s+1}^{N} {C_k}^q\right]^{\frac{1}{q}}, \tag{21}$$

$1 \leqslant i_1 < \dots < i_s \leqslant n,\ 1 \leqslant s \leqslant n.$

Since this reasoning holds not only for $(\widehat{m}_1, \dots, \widehat{m}_s)$, but also for any ordered set $(m_1, \dots, m_s)$ such that $m_1 + \dots + m_s \geqslant \widehat{m}_1 + \dots + \widehat{m}_s$ it is true that the value $\Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q)$ does not depend on $m_1, \dots, m_s$ for all $(m_1, \dots, m_s)$ satisfying the inequality $m_1 + \dots + m_s \geqslant \widehat{m}_1 + \dots + \widehat{m}_s$. Therefore, $\sup\limits_{m_1+\dots+m_s \geqslant M}$ in the equality (15) reduces to $\max\limits_{M \leqslant m_1+\dots+m_s \leqslant \widehat{m}_1+\dots+\widehat{m}_s}$, whence we obtain the assertion of the lemma. □

Let q be a number related to p by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{22}$$

Let us prove the following theorem.

Theorem 1. *If the cubature formula (2) is exact for any constant, then its error functional satisfy the following relations:*

$$|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{m_1,\dots,m_s \in \mathbb{N}} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q),\ f \in S_p, \tag{23}$$

$$\|\delta_N\|_{S_p^*} = \sup_{m_1,\dots,m_s \in \mathbb{N}} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q). \tag{24}$$

If the cubature formula (2) possesses the Haar d-property, then

$$|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{m_1+\dots+m_s > d} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q),\ f \in S_p, \tag{25}$$

$$\|\delta_N\|_{S_p^*} = \sup_{m_1+\dots+m_s > d} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q). \tag{26}$$

Proof. Let the cubature formula (2) be exact for any constant. By virtue of (4), (5), the inequality (23) follows from (8). Using (23), we obtain:

$$\|\delta_N\|_{S_p^*} \leqslant \sup_{m_1,\dots,m_s\in\mathbb{N}} \Sigma_{m_1,\dots,m_s}^{(i_1,\dots,i_s)}(q).$$

In order to establish that this inequality can not be improved, we use the technique applied in [8]. For $M=s$, we fix the ordered set $(M_1,\dots,M_s)$, the existence of which was proved in Lemma 2. We introduce the following notation:

$$\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)} = 2^{-\frac{M_1-1}{2}-\dots-\frac{M_s-1}{2}} \sum_{k=1}^{N} C_k\, \chi_{M_1,j_1}\left(x_{i_1}^{(k)}\right)\dots\chi_{M_s,j_s}\left(x_{i_s}^{(k)}\right).$$

Then, according to Lemma 2, we have

$$\sup_{m_1+\dots+m_s\geqslant M} \Sigma_{m_1,\dots,m_s}^{(i_1,\dots,i_s)}(q) = \Sigma_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}(q) = \left[\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left|\sum_{k=1}^{N}\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^q\right]^{\frac{1}{q}}. \tag{27}$$

Consider the function

$$f_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}(x_1,\dots,x_n) = \sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}} \operatorname{sign}\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^{q-1}\chi_{M_1,j_1}(x_{i_1})\dots\chi_{M_s,j_s}(x_{i_s}),$$

$1\leqslant i_1<\dots<i_s\leqslant n,\ 1\leqslant s\leqslant n$. For this function, the Fourier–Haar coefficients are given by

$$c_0=0,\quad c_{m_1,\dots,m_s}^{(j_1,\dots,j_s)}(i_1,\dots,i_s) = \begin{cases} \operatorname{sign}\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^{q-1}, & m_1=M_1,\dots,m_s=M_s,\\ 0 & \text{otherwise.}\end{cases}$$

Then, taking into account the relation (4) and the equality $(q-1)p=q$, which follows from (22), we have:

$$A_p^{(i_1,\dots,i_s)}\left(f_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}\right) = 2^{\frac{M_1-1}{2}+\dots+\frac{M_s-1}{2}}\left[\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^q\right]^{\frac{1}{p}}. \tag{28}$$

At the same time, according to (9),

$$\delta_N\left[f_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}\right] = -\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left[\operatorname{sign}\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^{q-1}\times\right.$$
$$\left.\times\sum_{k=1}^{N}C_k\,\chi_{M_1,j_1}\left(x_{i_1}^{(k)}\right)\dots\chi_{M_s,j_s}\left(x_{i_s}^{(k)}\right)\right] = -2^{\frac{M_1-1}{2}+\dots+\frac{M_s-1}{2}}\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^q.$$

The last relation, combined with (27) and (28), shows that

$$\left|\delta_N\left[f_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}\right]\right| = 2^{\frac{M_1-1}{2}+\dots+\frac{M_s-1}{2}}\left[\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^q\right]^{\frac{1}{p}}\times$$
$$\times\left[\sum_{j_1=1}^{2^{M_1-1}}\dots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta_{j_1,\dots,j_s}^{(i_1,\dots,i_s)}\right|^q\right]^{\frac{1}{q}} = A_p^{(i_1,\dots,i_s)}\left(f_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}\right)\Sigma_{M_1,\dots,M_s}^{(i_1,\dots,i_s)}(q).$$

Note that $A_p^{(k_1,\ldots,k_s)}\left(f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right) = 0$ for all ordered sets $(k_1,\ldots,k_s) \neq (i_1,\ldots,i_s)$. Then $\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\|_{S_p} = A_p^{(i_1,\ldots,i_s)}\left(f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right)$, and

$$\left|\delta_N\left[f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right]\right| = \Sigma_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(q)\left\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right\|_{S_p},$$

which implies the equality (24).

If the cubature formula (2) possesses the Haar d-property, then by virtue of its accuracy for Haar polynomials of degree at most d, the equality (9) becomes

$$\delta_N[f] = -\sum_{s=1}^{n}\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\sum_{m_1+\ldots+m_s>d}\sum_{j_1=1}^{2^{m_1-1}}\cdots\sum_{j_s=1}^{2^{m_s-1}}\left\{c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\times\right.$$
$$\left.\times\sum_{k=1}^{N}C_k\chi_{m_1,j_1}\left(x_{i_1}^{(k)}\right)\ldots\chi_{m_s,j_s}\left(x_{i_s}^{(k)}\right)\right\}.$$

Hence, the inequality (8) can be written as

$$\left|\delta_N[f]\right| \leqslant \sum_{s=1}^{n}\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\sum_{m_1+\ldots+m_s>d}\left\{2^{\frac{m_1-1}{2}+\ldots+\frac{m_s-1}{2}}\times\right.$$
$$\left.\times\left[\sum_{j_1=1}^{2^{m_1-1}}\cdots\sum_{j_s=1}^{2^{m_s-1}}\left|c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\right|^p\right]^{\frac{1}{p}}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q)\right\}.$$

Then the inequality (23) becomes (25). Proceeding as in the proof of the equality (24), we construct the function $f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(x_1,\ldots,x_n)$ such that

$$\left|\delta_N\left[f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right]\right| = \left\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right\|_{S_p}\sup_{m_1+\ldots+m_s>d}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q), \tag{29}$$

where the ordered set $(M_1,\ldots,M_s)$ satisfies the following conditions:

$$M_1+\ldots+M_s>d,$$

$$\Sigma_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(q) = \sup_{m_1+\ldots+m_s>d}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q).$$

This ordered set exists by virtue of Lemma 2, which is used for $M = d+1$.

The equality (26) follows from (25) and (29). □

Lemma 3. *For positive integer $m_1,\ldots,m_s$ satisfying the inequality*

$$m_1+\ldots+m_s \leqslant d, \tag{30}$$

it is true that

$$Q_N\left[K_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(x_{i_1},\ldots,x_{i_s})\right] = I\left[K_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(x_{i_1},\ldots,x_{i_s})\right] = 1, \tag{31}$$

where $1\leqslant i_1<\ldots<i_s\leqslant n$, $s=1,\ldots,n$, $j_r=1,\ldots,2^{m_r-1}$, $r=1,\ldots,s$.

Proof. Since each of the functions $\kappa_{m_1,j_1}(x_{i_1}),\ldots,\kappa_{m_s,j_s}(x_{i_s})$ is a Haar polynomial of one variable and the degrees of these polynomials are $m_1,\ldots,m_s$ respectively, then it follows from (14) that for $m_1,\ldots,m_s$ satisfying the condition (30), the function $K^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(x_{i_1},\ldots,x_{i_s})$ is a Haar polynomial of degree $m_1+\ldots+m_s\leqslant d$ of variables $x_{i_1},\ldots,x_{i_s}$. Then, by virtue of the accuracy of the cubature formula (2) for the Haar polynomials of degree at most d, the first equality in (31) holds true.

The second equality in (31) follows from the relations (14) and (11), which define the functions $K^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(x_{i_1},\ldots,x_{i_s})$ and $\kappa_{m_1,j_1}(x_{i_1}),\ldots,\kappa_{m_s,j_s}(x_{i_s})$. □

Lemma 4. *For positive integer l, the following inequality holds:*

$$\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q)\leqslant 2^{-m_1-\ldots-m_s+ls}\left\{\sum_{j_1=1}^{2^{m_1-l}}\ldots\sum_{j_s=1}^{2^{m_s-l}}\left\{Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s})\right]\right\}^q\right\}^{\frac{1}{q}},\quad (32)$$

where $1\leqslant i_1<\ldots<i_s\leqslant n,\ m_1,\ldots,m_s=1,2,\ldots,\ s=1,\ldots,n.$

Proof. Inequality (32) is proved by induction on l.

Applying the triangle inequality, and also taking into account the equality (12) and the positivity of the coefficients at the nodes of the cubature formula (2), we obtain:

$$\left|\sum_{k=1}^{N}C_k\,\chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant 2^{-\frac{m_1+1}{2}-\ldots-\frac{m_s+1}{2}}\times$$
$$\times\sum_{k=1}^{N}C_k\left|\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)-\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right|\ldots\left|\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)-\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right|.\quad (33)$$

The nonnegativity of the functions $\kappa_{m,j}(x)$ implies the inequality

$$\left|\kappa_{m_r,2j_r-1}\left(x^{(k)}_{i_r}\right)-\kappa_{m_r,2j_r}\left(x^{(k)}_{i_r}\right)\right|\leqslant\kappa_{m_r,2j_r-1}\left(x^{(k)}_{i_r}\right)+\kappa_{m_r,2j_r}\left(x^{(k)}_{i_r}\right),$$

$r=1,\ldots,s,\ \ k=1,\ldots,N$. Then, by virtue of the equalities (13) and (14), it is true that

$$\left|\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)-\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right|\ldots\left|\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)-\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant$$
$$\leqslant\left[\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)+\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right]\ldots\left[\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)+\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right]=$$
$$=2^s\kappa_{m_1-1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\kappa_{m_s-1,j_s}\left(x^{(k)}_{i_s}\right)=2^sK^{(j_1,\ldots,j_s)}_{m_1-1,\ldots,m_s-1}\left(x^{(k)}_{i_1},\ldots,x^{(k)}_{i_s}\right).$$

Combining this with (33) yields

$$\left|\sum_{k=1}^{N}C_k\,\chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant 2^{-\frac{m_1+1}{2}-\ldots-\frac{m_s+1}{2}+s}Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-1,\ldots,m_s-1}(x_{i_1},\ldots,x_{i_s})\right],$$

which implies (32) for $l=1$.

Based on the induction hypothesis that

$$\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q)\leqslant 2^{-m_1-\ldots-m_s+ls-s}\times$$
$$\times\left\{\sum_{j_1=1}^{2^{m_1-l+1}}\ldots\sum_{j_s=1}^{2^{m_s-l+1}}\left\{Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s})\right]\right\}^q\right\}^{\frac{1}{q}},\quad (34)$$

we prove (32). The sum on the right-hand side of the inequality (34) can be written as

$$\sum_{j_1=1}^{2^{m_1-l+1}} \cdots \sum_{j_s=1}^{2^{m_s-l+1}} \left\{ Q_N \left[K^{(j_1,\ldots,j_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q =$$
$$= \sum_{j_1=1}^{2^{m_1-l}} \cdots \sum_{j_s=1}^{2^{m_s-l}} \sum_{J_1=2j_1-1}^{2j_1} \cdots \sum_{J_s=2j_s-1}^{2j_s} \left\{ Q_N \left[K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q . \tag{35}$$

Using inequality

$$\sum_{i=1}^{M} a_i^q \leqslant \left\{ \sum_{i=1}^{M} a_i \right\}^q \quad (a_i \geqslant 0,\ i=1,\ldots,M,\ q>1)$$

and equality (13), we have:

$$\sum_{J_1=2j_1-1}^{2j_1} \cdots \sum_{J_s=2j_s-1}^{2j_s} \left\{ Q_N \left[K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q \leqslant$$
$$\leqslant \left\{ Q_N \left[\sum_{J_1=2j_1-1}^{2j_1} \cdots \sum_{J_s=2j_s-1}^{2j_s} K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q =$$
$$= \left\{ Q_N \left[\sum_{J_1=2j_1-1}^{2j_1} \cdots \sum_{J_s=2j_s-1}^{2j_s} \varkappa_{m_1-l+1,J_1}(x_{i_1}) \ldots \varkappa_{m_s-l+1,J_s}(x_{i_s}) \right] \right\}^q =$$
$$= \left\{ Q_N \left[\left(\varkappa_{m_1-l+1,2j_1-1}(x_{i_1}) + \varkappa_{m_1-l+1,2j_1}(x_{i_1}) \right) \ldots \left(\varkappa_{m_s-l+1,2j_s-1}(x_{i_1}) + \varkappa_{m_s-l+1,2j_s}(x_{i_s}) \right) \right] \right\}^q =$$
$$= \left\{ Q_N \left[2^s \varkappa_{m_1-l,j_1}(x_{i_1}) \ldots \varkappa_{m_s-l,j_s}(x_{i_s}) \right] \right\}^q = \left\{ 2^s Q_N \left[K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q .$$

In view of the equality (35) and the last relations, it follows from (34) that the inequality (32) holds true. □

Lemma 5. *If the cubature formula (2) possesses the Haar d-property, then*

$$\sup_{m_1+\ldots+m_s>d} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} . \tag{36}$$

Proof. Let $(m_1,\ldots,m_s)$ be an arbitrary fixed set of indices for which the inequality $m_1+\ldots+m_s > d$ holds true. We denote by l the minimal number among all integers L satisfying the condition

$$m_1+\ldots+m_s - Ls \leqslant d. \tag{37}$$

Then the following equality holds:

$$m_1+\ldots+m_s - ls = d - r, \text{ where } r \in \{0,1,\ldots,s-1\} . \tag{38}$$

Applying Lemmas 4 and 3 (by virtue of (37), the condition of Lemma 3 for the lower indices of the Haar polynomial $K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s})$ is satisfied) and taking into account (22) yields

$$\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \leqslant 2^{-m_1-\ldots-m_s+ls} \left\{ \sum_{j_1=1}^{2^{m_1-l}} \cdots \sum_{j_s=1}^{2^{m_s-l}} 1 \right\}^{\frac{1}{q}} =$$
$$= 2^{-m_1-\ldots-m_s+ls} \left(2^{m_1+\ldots+m_s-ls}\right)^{\frac{1}{q}} = \left(2^{m_1+\ldots+m_s-ls}\right)^{-\frac{1}{p}} . \tag{39}$$

The relations (39) and (38) imply

$$\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \leqslant \left(2^{d-r}\right)^{-\frac{1}{p}} = 2^{\frac{r}{p}}\left(2^d\right)^{-\frac{1}{p}} \leqslant 2^{\frac{s-1}{p}}\left(2^d\right)^{-\frac{1}{p}} \leqslant 2^{\frac{n-1}{p}}\left(2^d\right)^{-\frac{1}{p}},$$

whence we obtain the inequality (36). □

Lemma 6. *If the cubature formula (2) is exact for any constant, then*

$$\sup_{m_1,\ldots,m_s\in\mathbb{N}} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \geqslant \left(2^{n+1}-n-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}. \tag{40}$$

Proof. Consider the function

$$\varphi(C_1,\ldots,C_N) = \sum_{k=1}^{N_1} {C_k}^q + 2^{1-q}\sum_{k=N_1+1}^{N_2} {C_k}^q + 2^{2(1-q)}\sum_{k=N_2+1}^{N_3} {C_k}^q + \ldots + 2^{s(1-q)}\sum_{k=N_s+1}^{N} {C_k}^q, \tag{41}$$

where the constants $N_1,\ldots,N_s$ are defined in the proof of Lemma 2. By virtue of (21), the equality

$$\Sigma^{(i_1,\ldots,i_s)}_{\widehat{m}_1,\ldots,\widehat{m}_s}(q) = \left[\varphi\left(C_1,C_2,\ldots,C_N\right)\right]^{\frac{1}{q}} \tag{42}$$

holds true.

If the cubature formula (2) satisfies the condition

$$C_1 + C_2 + \ldots + C_N = 1 \ (C_i > 0,\ i = 1,2,\ldots,N),$$

which follows from the accuracy of (2) for any constant, it is easy to show that the function (41) attains its infimum, which is equal to

$$\left[N_1 + 2\left(N_2-N_1\right) + 2^2\left(N_3-N_2\right) + \ldots + 2^s\left(N-N_s\right)\right]^{1-q} =$$

$$= \left[N + \left(2^1-1\right)\left(N_2-N_1\right) + \left(2^2-1\right)\left(N_3-N_2\right) + \ldots + \left(2^s-1\right)\left(N-N_s\right)\right]^{1-q},$$

when

$$C_1 = C_2 = \ldots = C_{N_1} = \left[N_1 + 2\left(N_2-N_1\right) + 2^2\left(N_3-N_2\right) + \ldots + 2^s\left(N-N_s\right)\right]^{-1},$$

$$C_{N_1+1} = C_{N_1+2} = \ldots = C_{N_2} = 2\left[N_1 + 2\left(N_2-N_1\right) + 2^2\left(N_3-N_2\right) + \ldots + 2^s\left(N-N_s\right)\right]^{-1},$$

$$C_{N_2+1} = C_{N_2+2} = \ldots = C_{N_3} = 2^2\left[N_1 + 2\left(N_2-N_1\right) + 2^2\left(N_3-N_2\right) + \ldots + 2^s\left(N-N_s\right)\right]^{-1},$$

....

$$C_{N_s+1} = C_{N_s+2} = \ldots = C_{N} = 2^s\left[N_1 + 2\left(N_2-N_1\right) + 2^2\left(N_3-N_2\right) + \ldots + 2^s\left(N-N_s\right)\right]^{-1}.$$

Then, taking into account (22), we derive from (42)

$$\Sigma^{(i_1,\ldots,i_s)}_{\widehat{m}_1,\ldots,\widehat{m}_s}(q) \geqslant \left[N + \left(2^1-1\right)\left(N_2-N_1\right) + \left(2^2-1\right)\left(N_3-N_2\right) + \ldots + \left(2^s-1\right)\left(N-N_s\right)\right]^{-\frac{1}{p}} \geqslant$$

$$\geqslant \left[N + \left(2^1-1\right)N + \left(2^2-1\right)N + \ldots + \left(2^s-1\right)N\right]^{-\frac{1}{p}} = \left(2^{s+1}-s-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}} \geqslant$$

$$\geqslant \left(2^{n+1}-n-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}},$$

where $(\widehat{m}_1,\ldots,\widehat{m}_s)$ is the ordered set chosen in the proof of Lemma 2 (in this case $M = s$, where M is the parameter from the conditions of Lemma 2). This yields the inequality (40). □

Theorem 2. *For the cubature formula (2) exact for any constants, the norm of the error functional satisfies the inequality*

$$\|\delta_N\|_{S_p^*} \geqslant \left(2^{n+1} - n - 1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}. \tag{43}$$

If the cubature formula (2) possesses the Haar d-property, then

$$|\delta_N[f]| \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} \|f\|_{S_p}, \tag{44}$$

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}}. \tag{45}$$

Inequality (43) follows from Theorem 1 and Lemma 6, while inequalities (44), (45) follow from Theorem 1 and Lemma 5.

Remark 1. In [9] one considered the following weighted quadrature formulas possessing the Haar d-property:

$$\int_0^1 g(x) f(x)\,dx \approx \sum_{k=1}^{N} C_k f\left(x^{(k)}\right), \tag{46}$$

where $x^{(k)} \in [0,1]$ are the nodes of a formula; C_k are the coefficients of the formula at the nodes (real numbers); and $k = 1, \ldots, N$. If the weight function $g(x) \equiv 1$, then the number N of nodes of the quadrature formula (46) satisfies the inequality $N \geqslant 2^{d-1}$. The last inequality follows from a lower estimate for the number of nodes of the quadrature formula (46) possessing the Haar d-property, where $g(x)$ is an arbitrary weight function (see [9]).

Moreover, in [9] all minimal weighted quadrature formulas possessing the d-property were described. In the case of the weight function $g(x) \equiv 1$, it was proved that the minimal formula is unique: the number of its nodes is $N = 2^{d-1}$, the nodes of this formula are $x^{(k)} = 2^{-d}(2k-1)$, and the node coefficients are $C_k = 2^{-d+1}$ for $k = 1, 2, \ldots, 2^{d-1}$. The norm of the error functional of this formula satisfies the equality (see [10])

$$\|\delta_N\|_{S_p^*} = 2^{-\frac{1}{p}} N^{-\frac{1}{p}}, \tag{47}$$

which also follows from the inequalities (43) and (45) for $n = 1$; a number d related to N by $N = 2^{d-1}$.

Remark 2. In [12], one constructed the minimal cubature formulas possessing the Haar d-property for $d \geqslant 5$:

$$\int_0^1 \int_0^1 f(x_1, x_2)\,dx_1\,dx_2 \approx \sum_{k=1}^{N} C_k f\left(x_1^{(k)}, x_2^{(k)}\right), \tag{48}$$

where $\left(x_1^{(k)}, x_2^{(k)}\right) \in [0,1]^2$ are the nodes of a formula; C_k are the coefficients of the formula at the nodes (real numbers); and $k = 1, \ldots, N$. The number N of nodes of such formulas satisfies the equality

$$N = \begin{cases} 2^d - 3 \cdot 2^{\frac{d-1}{2}} + 2, & d \text{ is odd}, \\ 2^d - 2^{\frac{d}{2}+1} + 2, & d \text{ is even}, \end{cases} \tag{49}$$

where $d = 5, 6, 7, \ldots$ Then, the norm of the error functional of the minimal cubature formulas (48) possessing the Haar d-property satisfies the inequality

$$\|\delta_N\|_{S_p^*} \leqslant E_N, \tag{50}$$

$$\text{where } E_N = \begin{cases} 2^{\frac{1}{p}} \left(N + \frac{3\sqrt{2}}{2}\sqrt{N - \frac{7}{8}} + \frac{1}{4}\right)^{-\frac{1}{p}}, & d \text{ is odd}, \\ 2^{\frac{1}{p}} \left(N + 2\sqrt{N-1}\right)^{-\frac{1}{p}}, & d \text{ is even}. \end{cases} \tag{51}$$

The inequality (50) follows from the estimate

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}} (2^d)^{-\frac{1}{p}},$$

which was obtained in [16] for the norm of the error functional of arbitrary cubature formulas (48) having the Haar d-property. The number N of nodes of these cubature formulas is defined by (49).

The relations (50), (51) also follows from (45) for $n = 2$; a number d related to N by (49).

3. Conclusions

In [8], the cubature formulas

$$\int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \approx \frac{1}{N} \sum_{k=1}^{N} f\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) \tag{52}$$

with nodes $\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) \in [0,1]^n$ $(k = 1, \ldots, N)$ were considered that form P_τ-nets, i.e., nets that consist of $N = 2^\nu$ nodes and satisfy the following condition: each binary parallelepiped of volume $2^{\tau-\nu}$ contains 2^τ net points $(\nu > \tau)$. For such formulas with a function f from S_p, the following upper estimate for the norm of the error functional was proved in [8]:

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1+\tau}{p}} N^{-\frac{1}{p}}. \tag{53}$$

It is easy to see that for $n = 1$ and $n = 2$ P_τ-nets with an arbitrarily large number $N = 2^\nu$ of nodes exist for any $\tau = 0, 1, 2, \ldots$ Therefore, in the one- and two-dimensional cases, the constant multiplier on the right-hand side of (53) takes the least value at $\tau = 0$, and estimate (53) for the cubature formulas (52) with nodes forming P_0-nets in the one-dimensional case is written as

$$\|\delta_N\|_{S_p^*} \leqslant N^{-\frac{1}{p}}, \tag{54}$$

while in the two-dimensional case this estimate is written as

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}} N^{-\frac{1}{p}}. \tag{55}$$

It was proved in [8] that cubature formulas (52) with 2^d nodes forming P_0-nets have the Haar d-property. Therefore, the estimate (45), which is obtained in the present paper, is a generalization of the estimate (53) to the case of arbitrary cubature formulas possessing the Haar d-property.

Moreover, for any cubature formula (52) with a function $f \in S_p$, it was established in [8] that the norm of the error functional satisfies the lower estimate

$$\|\delta_N\|_{S_p^*} \geqslant N^{-\frac{1}{p}}.$$

Hence, the cubature formulas (52) with the nodes forming P_τ-nets have the best convergence rate of δ_N in the norm, which is equal to $N^{-\frac{1}{p}}$ as $N \to \infty$.

The relations (43), (47), (50), (51) imply that for minimal formulas possessing the Haar d-property in the one- and two-dimensional cases $\|\delta_N\|_{S_p^*} \asymp N^{-\frac{1}{p}}$ as $N \to \infty$.

Comparing the values on the right-hand sides of the relations (47) and (54), as well as (50) and (55), we conclude that the upper bounds for the $\|\delta_N\|_{S_p^*}$ in the case of minimal quadrature formulas (46) with the weight function $g(x) \equiv 1$ and the minimal cubature formulas (48) with the d-property are less than the upper bounds for this value in the inequalities (54) and (55), respectively, i.e., the upper bounds for the norm of the error functional of formulas with nodes forming the P_0-net in the one- and two-dimensional cases.

In addition, the quadrature formula (46) with the weight function $g(x) \equiv 1$ and the number $N = 2^{d-1}$ of nodes, as well as the cubature formula (48) with the number N of nodes satisfying the equality (49), being the minimal formulas of approximate integration, provide the best pointwise convergence of $\delta_N[f]$ to zero as $N \to \infty$.

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Об оценках погрешности на пространствах S_p кубатурных формул, точных для полиномов Хаара

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Аннотация. Получены верхняя и нижняя оценки нормы функционала погрешности обладающих d-свойством Хаара кубатурных формул на пространствах S_p в n-мерном случае.

Ключевые слова: d-свойство Хаара, погрешность кубатурной формулы, пространства S_p.