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On the Uniqueness of the Classical Solutions of the Radial Viscous Fingering Problems in a Hele-Shaw Cell

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Abstract. In [9, 10] we established the existence of classical solutions to two-phase and one-phase radial viscous fingering problems, respectively, in a Hele-Shaw cell by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in a parabolic equation. In this paper we show the uniqueness of such solutions to the respective problems.

Keywords: classical solution, unique existence, radial viscous fingering.

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Dedicated to the memory of Yu. Ya. Belov (1944–2019)

1. Introduction and preliminaries

Viscous fingering occurs in the flow of two immiscible, viscous fluids between the plates of a Hele-Shaw cell ([3]). Due to pressure gradients or gravity, the initially planar interface separating the two fluids undergoes a Saffman–Taylor instability ([5]), and develops finger-like structure (see also [4] and the literatures therein).

In [9,10] we established the existence of solutions belonging to the standard Hölder spaces for two-phase and one-phase radial viscous fingering problems in a Hele–Shaw cell, without surface tension effect, by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in parabolic equations (cf. [1, 2]). However, our results in [9, 10] are only the existence of the solutions because of the sub-sequential limiting procedure.

The aim of this paper is to prove the uniqueness of such solutions to the respective problems.

This paper consists of three sections. In the rest of this section, we give a brief formulation of the problem in the two-phase case that we discuss. In Section 2, we give a proof of the uniqueness of the classical solution to the two-phase problem, and in Section 3 to the one-phase problem.

1.1. Formulation of the two-phase problem

The motion of a slow quasistationary displacement of a fluid by another fluid in a Hele–Shaw cell is described by

$$\nabla \cdot \mathbf{v}_i = 0, \quad \mathbf{v}_i = -M_i \nabla p_i \quad \text{in} \ \ \Omega_i(t), \ t > 0 \ (i = 1, 2). \tag{1.1}$$

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Here $M_i = b^2/12\mu_i$ is mobility; μ_i is the fluid viscosity; b is the width of two plates; $\mathbf{v}_i$ is the velocity vector field in the fluid and p_i is the pressure ($i = 1$ and 2 for the displacing and the displaced fluid, respectively). For a radial fingering problem it is sufficient to consider (1.1) under the following geometric situation:

$$\Omega_1(t) = \left\{ x \in \mathbb{R}^2 \;\middle|\; R_* < |x| < R(t) + \zeta\left(\frac{x}{|x|}, t\right) \right\},$$
$$\Omega_2(t) = \left\{ x \in \mathbb{R}^2 \;\middle|\; R(t) + \zeta\left(\frac{x}{|x|}, t\right) < |x| < R^* \right\},$$

where R_* is the radius of the hole through which the displacing fluid is injected or driven by suction at a flow rate $Q(t)$, R^* is the radius of the Hele–Shaw cell occupied by the displaced fluid, $R(t)$ is the time-dependent unperturbed radius satisfying

$$\pi R(t)^2 = \pi R_0^2 + \int_0^t Q(\tau)\,\mathrm{d}\tau, \quad R_0 \equiv R(0) > R_*,$$

and ζ is the perturbed radius.

The boundary and initial conditions for (1.1) are as follows:

$$\begin{cases} \mathbf{v}_1 \cdot \mathbf{n} = \dfrac{Q(t)}{2\pi R_*} \quad \text{on } \Gamma_*,\ t > 0, \qquad p_2 = p_e \quad \text{on } \Gamma^*,\ t > 0, \\ \mathbf{v}_1 \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n} = V_n, \quad p_1 = p_2 \quad \text{on } \Gamma(t),\ t > 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \mathbf{v}_i|_{t=0} = \mathbf{v}_i^0, \quad p_i = p_i^0 \quad \text{on } \ \Omega_i(0) \equiv \Omega_i \quad (i = 1, 2), \\ \zeta|_{t=0} = \zeta^0 \in (R_* - R_0, R^* - R_0) \quad \text{on } \ \Gamma(0) \equiv \Gamma, \end{cases} \tag{1.3}$$

where $\Gamma_* = \{x \in \mathbb{R}^2 \,|\, |x| = R_*\}$, $\Gamma(t) = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = R(t) + \zeta(x/|x|, t)\right\}$, $\Gamma^* = \{x \in \mathbb{R}^2 \,|\, |x| = R^*\}$; V_n is the normal velocity of the interface $\Gamma(t)$; $\mathbf{n}$ is the unit normal vectors, outward to Γ_* or to $\Gamma(t)$ in the direction from $\Omega_1(t)$ to $\Omega_2(t)$; p_e is the surface pressure acting on Γ^*.

Our two-phase problem is to find $(\mathbf{v}_i, p_i)$ $(i = 1, 2)$ and ζ satisfying (1.1)–(1.3), which is reduced to find (p_1, p_2) and ζ satisfying

$$\begin{cases} \Delta p_i = 0 \quad \text{in } \Omega_i(t),\ t > 0\ (i = 1, 2), \\ -M_1 \nabla p_1 \cdot \mathbf{n} = \dfrac{Q(t)}{2\pi R_*} \quad \text{on } \Gamma_*,\ t > 0, \quad p_2 = p_e \quad \text{on } \Gamma^*,\ t > 0, \\ -M_1 \nabla p_1 \cdot \mathbf{n} = -M_2 \nabla p_2 \cdot \mathbf{n} = V_n, \quad p_1 = p_2 \quad \text{on } \Gamma(t),\ t > 0, \\ p_i \,\big|_{t=0} = p_i^0 \quad \text{on } \Omega_i\ (i = 1, 2), \quad \zeta \,\big|_{t=0} = \zeta^0 \quad \text{on } \Gamma. \end{cases} \tag{1.4}$$

As the compatibility conditions p_1^0 and p_2^0 are assumed to satisfy

$$\begin{cases} \Delta p_i^0 = 0 \quad \text{in } \ \Omega_i \quad (i = 1, 2), \\ -M_1 \nabla p_1^0 \cdot \mathbf{n} = \dfrac{Q(0)}{2\pi R_*} \quad \text{on } \ \Gamma_*, \quad p_2^0 = p_e|_{t=0} \quad \text{on } \ \Gamma^*, \quad p_1^0 = p_2^0 \quad \text{on } \ \Gamma. \end{cases} \tag{1.5}$$

In polar coordinates (r,θ) problem (1.4) is written as

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial p_1}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 p_1}{\partial \theta^2}=0 \quad (r\in(R_*,R(t)+\zeta),\ \theta\in[0,2\pi),\ t>0)\,,\\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial p_2}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 p_2}{\partial \theta^2}=0 \quad (r\in(R(t)+\zeta,R^*),\ \theta\in[0,2\pi),\ t>0)\,,\\ M_1\dfrac{\partial p_1}{\partial r}\Big|_{r=R_*}=-\dfrac{Q(t)}{2\pi R_*} \quad p_2\Big|_{r=R^*}=p_e \quad (\theta\in[0,2\pi),\ t>0)\,,\\ -\dfrac{\partial}{\partial t}(R(t)+\zeta)=M_1\left(\dfrac{\partial p_1}{\partial r}-\dfrac{1}{r^2}\dfrac{\partial \zeta}{\partial\theta}\dfrac{\partial p_1}{\partial\theta}\right)=M_2\left(\dfrac{\partial p_2}{\partial r}-\dfrac{1}{r^2}\dfrac{\partial \zeta}{\partial\theta}\dfrac{\partial p_2}{\partial\theta}\right), \quad p_1=p_2\\ \qquad\qquad (r=R(t)+\zeta,\ \theta\in[0,2\pi),\ t>0)\,,\\ p_1\Big|_{t=0}=p_1^0 \quad \big(r\in(R_*,R_0+\zeta^0),\ \theta\in[0,2\pi)\big)\,,\\ p_2\Big|_{t=0}=p_2^0 \quad \big(r\in(R_0+\zeta^0(\theta),R^*),\ \theta\in[0,2\pi)\big)\,,\\ \zeta\Big|_{t=0}=\zeta^0 \quad (\theta\in[0,2\pi))\,. \end{cases} \tag{1.6}$$

Now let us transform the free boundary problem (1.6) into the problem on fixed domains. Introduce the transformations from $\Omega_1(t)=\{R_*<r<R(t)+\zeta(\theta,t),\ 0\leqslant\theta<2\pi\}$ onto $\Omega_1=\big\{R_*<r'<R_0+\zeta^0(\theta'),\ 0\leqslant\theta'<2\pi\big\}$ by the change of the variables $r'=\dfrac{R_0+\zeta^0-R_*}{R+\zeta-R_*}\times$ $\times\,(r-R_*)+R_*,\ \theta'=\theta,\ t'=t$, and $\Omega_2(t)=\{R(t)+\zeta(\theta,t)<r<R^*,\ 0\leqslant\theta<2\pi\}$ onto $\Omega_2=\big\{R_0+\zeta^0(\theta')<r'<R^*,\ 0\leqslant\theta'<2\pi\big\}$ by $r'=\dfrac{R_0+\zeta^0-R^*}{R+\zeta-R^*}(r-R^*)+R^*,\ \theta'=\theta,\ t'=t$. Moreover, by letting $p_i(r,\theta,t)=p_i'(r',\theta',t')$ $(i=1,2)$, $\zeta(\theta,t)=\zeta'(\theta',t')$, and by omitting the *primes* for simplicity, problem (1.6) takes the form

$$\begin{cases} \mathcal{L}^i_\zeta p_i=0 \quad \text{in } \ \Omega_i,\ t>0 \quad (i=1,2),\\ \dfrac{\partial p_1}{\partial r}=-\dfrac{Q(t)}{2\pi R_* M_1}\dfrac{R+\zeta-R_*}{R_0+\zeta^0-R_*} \quad \text{on } \ \Gamma_*\equiv\{r=R_*,\ \theta\in[0,2\pi]\},\ t>0,\\ p_2=p_e \quad \text{on } \ \Gamma^*\equiv\{r=R^*,\ \theta\in[0,2\pi]\},\ t>0,\\ \dfrac{\partial\zeta}{\partial t}-b^1_2(\zeta)\dfrac{\partial p_1}{\partial r}-b^1_1(\zeta)\dfrac{\partial p_1}{\partial \theta}-b^2_2(\zeta)\dfrac{\partial p_2}{\partial r}-b^2_1(\zeta)\dfrac{\partial p_2}{\partial \theta}=-\dfrac{Q(t)}{2\pi R},\\ b^1_2(\zeta)\dfrac{\partial p_1}{\partial r}+b^1_1(\zeta)\dfrac{\partial p_1}{\partial \theta}=b^2_2(\zeta)\dfrac{\partial p_2}{\partial r}+b^2_1(\zeta)\dfrac{\partial p_2}{\partial \theta}, \quad p_1=p_2\\ \qquad\qquad \text{on } \Gamma\equiv\{r=R_0+\zeta^0,\theta\in[0,2\pi]\},\ t>0,\\ p_i\Big|_{t=0}=p_i^0 \quad \text{on } \ \Omega_i\ (i=1,2), \quad \zeta\Big|_{t=0}=\zeta^0 \quad \text{on } \ [0,2\pi]. \end{cases} \tag{1.7}$$

Here $\mathcal{L}^i_\zeta\equiv\mathcal{L}^i_\zeta(r,\theta;\partial/\partial r,\partial/\partial\theta)$ is a Laplace operator represented by the composite change of variables of polar coordinates (r,θ) and the mapping from $\Omega_i(t)$ to Ω_i $(i=1,2)$, and

$$b^j_2(\zeta)=\frac{M_j}{2}\left[\left(1+\frac{1}{(R_0+\zeta^0)^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}-\frac{1}{(R_0+\zeta^0)^2}\frac{\partial\zeta}{\partial\theta}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}\right],$$

$$b^j_1(\zeta)=-\frac{M_j}{2}\,\frac{1}{(R_0+\zeta^0)^2}\,\frac{\partial\zeta}{\partial\theta} \quad (j=1,2).$$

In detail, see [9].

We consider problem (1.7) in the standard Hölder spaces, $C^{l+\alpha}(\bar\Omega)$, $C^{l+\alpha,(l+\alpha)/2}_{x,t}(\bar Q_T)$ ($\bar Q_T\equiv\bar\Omega\times[0,T]$; $\Omega\subset\mathbb{R}^n$ $(n\in\mathbb{N})$, a domain; T, any positive number; $l\geqslant0$, an integer; $\alpha\in(0,1)$) with the norms:

$$|u|^{(\alpha)} = |u|^{(0)} + \langle u\rangle^{(\alpha)}, \quad |u|^{(0)} = \sup_{(x,t)\in\bar{Q}_T} |u(x,t)|, \quad \langle u\rangle^{(\alpha)} = \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\alpha/2)},$$

$$\langle u\rangle_x^{(\alpha)} \equiv \sup_{x,y\in\bar{\Omega},\, t\in[0,T]} \frac{|u(x,t)-u(y,t)|}{|x-y|^\alpha}, \quad \langle u\rangle_t^{(\alpha)} \equiv \sup_{x\in\bar{\Omega},\, t,t'\in[0,T]} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha}.$$

We also use the semi-norm

$$[u]^{(\alpha,\beta)} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ t,t'\in[0,T]}} \frac{|u(x,t)-u(y,t)-u(x,t')+u(y,t')|}{|x-y|^\alpha\,|t-t'|^\beta} \quad (\alpha,\ \beta\in(0,1)),$$

and introduce the Banach spaces $E^{k+\alpha}(\bar{Q}_T)$ $(k=0,1,2)$ which are the completion of infinitely differential functions in respective norms

$$\|u\|_\alpha = \|u\|_{E^\alpha(\bar{Q}_T)} \equiv E^{\alpha,\alpha/2}[u] = |u|^{(0)} + \langle u\rangle^{(\alpha)} + [u]^{(\alpha,\alpha/2)},$$

$$D^{\alpha,\alpha}[u] = |u|^{(0)} + \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\alpha)} + [u]^{(\alpha,\alpha)},$$

$$\|u\|_{k+\alpha} = \|u\|_{E^{k+\alpha}(\bar{Q}_T)} = E^{\alpha,\alpha/2}[\mathrm{D}_x^k u] + \sum_{j=0}^{k-1} D^{\alpha,\alpha}[\mathrm{D}_x^j u] \quad \left(\mathrm{D}_x^k = \sum_{|j|=k} \frac{\partial^j}{\partial x^j},\ k=1,2\right);$$

$$\hat{E}^{2+\alpha}(\bar{Q}_T) = \left\{u \mid \|u\|_{\hat{E}^{2+\alpha}(\bar{Q}_T)} < \infty\right\}, \quad \|u\|_{\hat{E}^{2+\alpha}(\bar{Q}_T)} = \|u\|_{2+\alpha} + \left\|\frac{\partial u}{\partial t}\right\|_{1+\alpha}.$$

The function spaces on a smooth manifold Γ in $\mathbb{R}^n$ are defined with the help of partition of unity and of local maps.

2. Uniqueness of the solution to problem (1.7)

Our main result for two-phase problem (1.7) is as follows:

Theorem 2.1. *Let $T>0$ and $\alpha\in(0,1)$. Assume that $(p_1^0,p_2^0,\zeta^0)\in C^{3+\alpha}(\bar{\Omega}_1)\times C^{3+\alpha}(\bar{\Omega}_2)\times C^{4+\alpha}([0,2\pi])$ satisfy the compatibility conditions, $\partial p_1^0/\partial r - \partial p_2^0/\partial r > 0$ on Γ, $Q\in C^\alpha([0,T])$ and $p_e\in C_{\theta,t}^{3+\alpha,(3+\alpha)/2}(\Gamma_T^*)$ with $\partial p_e/\partial t|_{t=0}=0$. Then there exists $T_0^*>0$ depending on the data of the problem such that problem (1.7) has a unique solution $(p_1,p_2,\zeta)\in E^{2+\alpha}(\bar{Q}_{1,T_0^*})\times E^{2+\alpha}(\bar{Q}_{2,T_0^*})\times \hat{E}^{2+\alpha}(\Gamma_{T_0^*})$ except for the extension of ζ^0 to $[0,2\pi]\times[0,T]$ satisfying*

$$\|p_1\|_{E^{2+\alpha}(\bar{Q}_{1,T_0^*})} + \|p_2\|_{E^{2+\alpha}(\bar{Q}_{2,T_0^*})} + \|\zeta\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0^*})} \leqslant C. \tag{2.1}$$

In [9] we showed the existence of the solution to problem (1.7) on some time interval $[0,T_0]$ $(0<T_0\leqslant T)$ in the form

$$\begin{cases} p_1 = p_1^* + p_1^0 + \dfrac{r-R_*}{R+\bar{\zeta}-R_*}\dfrac{\partial p_1^0}{\partial r}\zeta^*, \quad p_2 = p_2^* + p_2^0 + \dfrac{r-R^*}{R+\bar{\zeta}-R^*}\dfrac{\partial p_2^0}{\partial r}\zeta^*, \\ \zeta = \zeta^* + \bar{\zeta} \end{cases} \tag{2.2}$$

by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in a parabolic equation. Here $\bar{\zeta}\in C_{\theta,t}^{4+\alpha,(4+\alpha)/2}([0,2\pi]\times[0,T])$ is an extension of ζ^0 such that $(\bar{\zeta},\partial\bar{\zeta}/\partial t,\partial^2\bar{\zeta}/\partial t^2)|_{t=0} = (\zeta^0,\ \partial\zeta/\partial t,\partial^2\zeta/\partial t^2)|_{t=0}$ whose right hand side are obtained from the fourth equation in (1.7) and its derivative in t at $t=0$.

Then (1.7) becomes

$$\begin{cases} \mathcal{L}_*^i p_i^* = \Phi_i \quad \text{in } \Omega_i,\ t>0\ (i=1,2), \\ \dfrac{\partial p_1^*}{\partial r} = \Psi_* \quad \text{on } \Gamma_*,\ t>0, \quad p_2^* = \Psi^* \quad \text{on } \Gamma^*,\ t>0, \\ \dfrac{\partial \zeta^*}{\partial t} - b_2^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial r} - b_1^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial \theta} - b_2^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial r} - b_1^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial \theta} = \Psi_1 + \Psi_2, \\ b_2^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial r} + b_1^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial \theta} - b_2^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial r} - b_1^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial \theta} = -\Psi_1 + \Psi_2, \\ p_1^* - p_2^* + d(\bar\zeta)\zeta^* = \Psi_3 \quad \text{on } \Gamma,\ t>0, \\ p_i^* \big|_{t=0} = 0 \quad \text{on } \Omega_i\ (i=1,2), \quad \zeta^* \big|_{t=0} = 0 \quad \text{on } [0,2\pi]. \end{cases} \tag{2.3}$$

Here $\mathcal{L}_*^i$ is the principal part of $\mathcal{L}_\zeta^i$ with ζ replaced by $\bar\zeta$,

$$\Phi_i = \Phi_i(p_i^*, \zeta^*) = -\mathcal{L}_\zeta^i p_i + \mathcal{L}_*^i p_i^* \quad (i=1,2), \quad \Psi^* = p_e - p_2^0,$$

$$\Psi_* = \Psi_*(\zeta^*) = -\frac{\partial}{\partial r}\left(p_1^0 + \frac{r-R_*}{R+\bar\zeta-R_*}\frac{\partial p_1^0}{\partial r}\right)\zeta^* - \frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}\frac{Q(t)}{2\pi R_* M_1},$$

$$\Psi_j = \Psi_j\left(p_1^*, p_2^*, \zeta^*\right) = b_2^j(\zeta)\frac{\partial p_j}{\partial r} + b_1^j(\zeta)\frac{\partial p_j}{\partial \theta} - b_2^j(\bar\zeta)\frac{\partial p_j^*}{\partial r} - b_1^j(\bar\zeta)\frac{\partial p_j^*}{\partial \theta} - \frac{Q(t)}{4\pi R} - \frac{1}{2}\frac{\partial \bar\zeta}{\partial t}$$

$$(j=1,2),$$

$$\Psi_3 = p_2^0 - p_1^0, \quad d(\bar\zeta) = \frac{R_0+\zeta^0-R_*}{R+\bar\zeta-R_*}\frac{\partial p_1^0}{\partial r} - \frac{R_0+\zeta^0-R^*}{R+\bar\zeta-R^*}\frac{\partial p_2^0}{\partial r}$$

with (p_1, p_2, ζ) replaced by (2.2).

If problem (2.3) admits a unique solution on some time interval $[0, T_0^*]$ $(0 < T_0^* \leqslant T_0)$, then the limit process holds for the full sequence, not the subsequence, on $[0, T_0^*]$, so that the proof of Theorem 2.1 is completed.

In what follows we shall prove the uniqueness of solution to problem (2.3).

Let (p_1^*, p_2^*, ζ^*) and $(p_1^{**}, p_2^{**}, \zeta^{**})$ be two solutions of (2.3) satisfying

$$\|p_1^\dagger\|_{E^{2+\alpha}(\bar Q_{1,T_0})} + \|p_2^\dagger\|_{E^{2+\alpha}(\bar Q_{2,T_0})} + \|\zeta^\dagger\|_{\hat E^{2+\alpha}(\Gamma_{T_0})} \leqslant C_1 \quad (\dagger = *,\ **). \tag{2.4}$$

To the end, as the same way as in [9] it is essential to consider the following four model problems in the whole- and half-spaces:

$$\mathcal{L}u = f \quad \text{in } \mathbb{R}^2,\ t>0, \quad u\big|_{t=0} = 0; \tag{2.5}$$

$$\mathcal{L}u = f \quad (x_1 \in \mathbb{R},\ x_2 > 0,\ t>0), \quad u\big|_{x_2=0} = 0, \quad u\big|_{t=0} = 0; \tag{2.6}$$

$$\mathcal{L}u = f \quad (x_1 \in \mathbb{R},\ x_2 > 0,\ t>0), \quad \frac{\partial u}{\partial x_2}\Big|_{x_2=0} = 0, \quad u\big|_{t=0} = 0; \tag{2.7}$$

$$\begin{cases} \mathcal{L}u^+ = 0 \ (x_1 \in \mathbb{R}, x_2 > 0, t > 0), \quad \mathcal{L}u^- = 0 \ (x_1 \in \mathbb{R}, x_2 < 0, t > 0), \\ \dfrac{\partial \varrho}{\partial t} - b^+\dfrac{\partial u^+}{\partial x_2} - b^-\dfrac{\partial u^-}{\partial x_2}\Big|_{x_2=0} = g_1, \quad -b^+\dfrac{\partial u^+}{\partial x_2} + b^-\dfrac{\partial u^-}{\partial x_2}\Big|_{x_2=0} = g_2, \\ -u^+ + u^- + d\varrho\big|_{x_2=0} = g_3, \quad (u^+, u^-, \varrho)\big|_{t=0} = 0. \end{cases} \tag{2.8}$$

Moreover, it suffices to assume that $\mathcal{L} = \Delta$, and $b^\pm$ and d are positive constants, and set $b \equiv (2db^+b^-)/(b^+ + b^-)$ in (2.5)–(2.8).

In estimating the difference $(p_1^*, p_2^*, \zeta^*) - (p_1^{**}, p_2^{**}, \zeta^{**})$, we trace a proof in [9] with the help of a fundamental solution and Green functions of (2.5)–(2.8) instead of Γ_ε, G_ε, N_ε and Z_ε in [9]. It is clear that the solutions to problems (2.5)–(2.7) for $\mathcal{L} = \Delta$ are given by

$$u(x,t) = \int_{\mathbb{R}^2} \Gamma_0(x-y) f(y,t)\,\mathrm{d}y, \quad u(x,t) = \int_{\mathbb{R}^2} G_0(x-y) f(y,t)\,\mathrm{d}y,$$
$$u(x,t) = \int_{\mathbb{R}^2} N_0(x-y) f(y,t)\,\mathrm{d}y,$$

respectively, where $\Gamma_0(x) = -\log|x|/(2\pi)$, $G_0(x_1,x_2) = \Gamma_0(x_1,x_2) - \Gamma_0(x_1,-x_2)$, and $N_0(x_1,x_2) = \Gamma_0(x_1,x_2) + \Gamma_0(x_1,-x_2)$. Whereas, the solution $(u^+, u^-, \varrho) = (\mathcal{FL})^{-1}[(\tilde{u}^+, \tilde{u}^-, \tilde{\varrho})]$ of problem (2.8) is represented by virtue of Green function

$$Z_0(x_1,t) = (\mathcal{FL})^{-1}[\tilde{Z}_0] = (\mathcal{FL})^{-1}\left[\frac{1}{s+b|\xi|}\right]:$$
$$\tilde{u}^+ = (\mathcal{FL})[u^+] = \tilde{v}^+(\xi,s)\,\mathrm{e}^{-|\xi|x_2} \quad (x_2>0), \quad \tilde{u}^- = (\mathcal{FL})[u^-] = \tilde{v}^-(\xi,s)\,\mathrm{e}^{|\xi|x_2} \quad (x_2<0),$$
$$\tilde{v}^+ = \frac{1}{(b^++b^-)\,|\xi|}\,\tilde{g}_2 - \frac{b^-}{b^++b^-}\,\tilde{g}_3 + \tilde{Z}_0\left(\frac{db^-}{b^++b^-}\,\tilde{g}_1 - \frac{b-db^-}{b^++b^-}\,\tilde{g}_2 + \frac{bb^-}{b^++b^-}\,|\xi|\tilde{g}_3\right),$$
$$\tilde{v}^- = \frac{1}{(b^++b^-)\,|\xi|}\,\tilde{g}_2 + \frac{b^+}{b^++b^-}\,\tilde{g}_3 + \tilde{Z}_0\left(-\frac{db^+}{b^++b^-}\,\tilde{g}_1 - \frac{b-db^+}{b^++b^-}\,\tilde{g}_2 - \frac{bb^+}{b^++b^-}\,|\xi|\tilde{g}_3\right),$$
$$\tilde{\varrho} = \tilde{Z}_0\left(\tilde{g}_1 - \frac{b^+-b^-}{b^++b^-}\,\tilde{g}_2 + \frac{2b^+b^-}{b^++b^-}\,|\xi|\tilde{g}_3\right).$$

Here $\tilde{u} = (\mathcal{FL})[u]$ is the Fourier transformation in x_1 and Laplace transformation in t of u, and $(\mathcal{FL})^{-1}[\tilde{u}]$ is its inverse transformation.

Lemma 2.2. *When $b=1$, we have the following estimates of Z_0:*

$$|Z_0(x_1,t)| \leqslant C_2\,\frac{1}{\sqrt{x_1^2+t^2}}, \quad \left|\frac{\partial}{\partial t}\,Z_0(x_1,t)\right| + \left|\frac{\partial}{\partial x_1}\,Z_0(x_1,t)\right| \leqslant C_2\,\frac{1}{x_1^2+t^2},$$
$$\left|\frac{\partial^2}{\partial t\partial x_1}\,Z_0(x_1,t)\right| + \left|\frac{\partial^2}{\partial x_1^2}\,Z_0(x_1,t)\right| \leqslant C_2\,\frac{1}{(x_1^2+t^2)^{3/2}}.$$

Using Lemma 2.2, we estimate these solutions of (2.5)–(2.8) in the same way as in [9] (cf. [1,2]). For a general domain by using the regularizer method for elliptic system ([6–8]), we finally obtain the estimate of $(p_1^*, p_2^*, \zeta^*) - (p_1^{**}, p_2^{**}, \zeta^{**})$ with the help of Young's and interpolation inequalities and (2.4):

$$\|p_1^* - p_1^{**}\|_{E^{2+\alpha}(\bar{Q}_{1,t})} + \|p_2^* - p_2^{**}\|_{E^{2+\alpha}(\bar{Q}_{2,t})} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)} \leqslant \tag{2.9}$$
$$\leqslant C_3\left(\sum_{i=1}^{2}\Big(\|\Phi_i(p^*,\zeta^*) - \Phi_i(p^{**},\zeta^{**})\|_{E^{\alpha}(\bar{Q}_{i,t})} + \|\Psi_i(p^*,\zeta^*) - \Psi_i(p^{**},\zeta^{**})\|_{E^{1+\alpha}(\bar{\Gamma}_t)}\Big) + \right.$$
$$\left. + \|\Psi_*(\zeta^*) - \Psi_*(\zeta^{**})\|_{E^{1+\alpha}(\bar{\Gamma}_{*t})}\right) \leqslant$$
$$\leqslant C_3\,(\beta + C_\beta t^\chi F(4C_1))\Big(\|p_1^* - p_1^{**}\|_{E^{2+\alpha}(\bar{Q}_{1,t})} + \|p_2^* - p_2^{**}\|_{E^{2+\alpha}(\bar{Q}_{2,t})} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)}\Big)$$

for any $t \in (0, T_0)$ and any $\beta > 0$, where C_β is a positive constant depending on β non-increasingly, χ is a constant depending on α, $F(\cdot)$ is a polynomial in its argument.

Now choosing first $\beta = 1/(4C_3)$, and then

$$T_0^* = \min\left\{T_0, \left(\frac{1}{4C_3C_\beta F(4C_1)}\right)^{1/\chi}\right\},$$

we conclude from (2.9) that the solution to problem (2.3) is unique on $[0, T_0^*]$.

3. Uniqueness of the solution to the one-phase problem

Our next main result for one-phase problem is as follows:

Theorem 3.1. *Let $T > 0$ and $\alpha \in (0, 1)$. Assume that $(p^0, \zeta^0) \in C^{3+\alpha}(\bar{\Omega}) \times C^{4+\alpha}([0, 2\pi])$ satisfy the compatibility conditions, $\partial p^0/\partial r < 0$ on Γ, $Q \in C^\alpha([0, T])$ and $p_e \in C_{\theta,t}^{3+\alpha,(3+\alpha)/2}(\Gamma_T^*)$ with $\partial p_e/\partial t|_{t=0} = 0$. Then there exists $T_0^* > 0$ depending on the data of the problem such that one-phase problem has a unique solution $(p, \zeta) \in E^{2+\alpha}(\bar{Q}_{T_0^*}) \times \hat{E}^{2+\alpha}(\Gamma_{T_0^*})$ except for the extension of ζ^0 to $[0, 2\pi] \times [0, T]$ satisfying*

$$\|p\|_{E^{2+\alpha}(\bar{Q}_{T_0^*})} + \|\zeta\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0^*})} \leqslant C'. \tag{3.1}$$

Like the two-phase problem we transform the one-phase problem into just the same equations as (2.3). In [10] the existence of the solution (p^*, ζ^*) (cf. (2.2)) to one-phase problem on some time interval $[0, T_0]$ $(0 < T_0 < T)$ was shown.

Let (p^*, ζ^*) and (p^{**}, ζ^{**}) be two solutions satisfying

$$\|p^\dagger\|_{E^{2+\alpha}(\bar{Q}_{T_0})} + \|\zeta^\dagger\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0})} \leqslant C_1' \quad (\dagger = *,\ **). \tag{3.2}$$

For one-phase case the essential model problems are the same as (2.5), (2.7), (2.8) with (u^+, u^-, ϱ) replaced by $(u, 0, \varrho)$.

$$\mathcal{L}u = f \quad \text{in } \mathbb{R}^2,\ t > 0, \quad u\,\big|_{t=0} = 0; \tag{3.3}$$

$$\mathcal{L}u = f \quad (x_1 \in \mathbb{R},\ x_2 > 0,\ t > 0)\,, \quad \frac{\partial u}{\partial x_2}\,\Big|_{x_2=0} = 0, \quad u\,\big|_{t=0} = 0; \tag{3.4}$$

$$\begin{cases} \mathcal{L}u = 0 \ \ (x_1 \in \mathbb{R}, x_2 > 0, t > 0)\,, \\ \dfrac{\partial \varrho}{\partial t} - b\dfrac{\partial u}{\partial x_2}\,\Big|_{x_2=0} = g_1, \quad u - d\varrho\,\big|_{x_2=0} = g_2, \quad (u, \varrho)\,\big|_{t=0} = 0. \end{cases} \tag{3.5}$$

The solution $(u, \varrho) = (\mathcal{F}\mathcal{L})^{-1}[(\tilde{u}, \tilde{\varrho})]$ of problem (3.5) is represented by virtue of Green function Z_0 (cf. in Sec. 2):

$$\tilde{u} = \tilde{v}(\xi, s)\,\mathrm{e}^{-|\xi|x_2} \ \ (x_2 > 0), \quad \tilde{v} = d\,\tilde{\varrho} + g_2, \quad \tilde{\varrho} = \tilde{Z}_0\,(\tilde{g}_1 - b\,d\,|\xi|\,\tilde{g}_2)\,.$$

Just in the same way as the two-phase problem, we can estimate the solutions of (3.3)–(3.5) with the help of Lemma 2.2, and for a general domain the regularizer method for elliptic system leads to the estimate of $(p^*, \zeta^*) - (p^{**}, \zeta^{**})$:

$$\begin{aligned} &\|p^* - p^{**}\|_{E^{2+\alpha}(\bar{Q}_t)} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)} \leqslant \\ &\leqslant C_3' \left(\beta 7 + C_{\beta'} t^{\chi'} F'(4C_1')\right) \left(\|p^* - p^{**}\|_{E^{2+\alpha}(\bar{Q}_t)} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)}\right) \end{aligned} \tag{3.6}$$

for any $t \in (0, T_0)$ and any $\beta' > 0$, where $C_{\beta'}$ is a positive constant depending on β' non-increasingly, χ' is a constant depending on α, $F'(\cdot)$ is a polynomial of its argument.

Now choosing first $\beta' = 1/(4C_3')$, and then

$$T_0^* = \min\left\{T_0, \left(\frac{1}{4C_3' C_{\beta'} F'(4C_1')}\right)^{1/\chi}\right\},$$

we conclude from (3.6) that the solution to one-phase problem is unique on $[0, T_0^*]$.

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О единственности классических решений задач радиальной вязкой пальцеобразной структуры в ячейке Хеле-Шоу

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Аннотация. В [9, 10] мы установили существование классических решений двухфазной и однофазной задач радиальной вязкой аппликатуры соответственно в ячейке Хеле-Шоу параболической регуляризацией и обращением в нуль коэффициента производной по времени в параболическом уравнении. В этой статье мы показываем единственность таких решений соответствующих задач.

Ключевые слова: классическое решение, уникальное наличие, радиальная вязкая пальцеобразная структура.