DOI: 10.17516/1997-1397-2021-14-2-150-158
УДК 517.9

# Fixed Points of Set-valued $F$-contraction Operators in Quasi-ordered Metric Spaces with an Application to Integral Equations

**Ehsan Lotfali Ghasab***
**Hamid Majani**†
Department of Mathematics
Shahid Chamran University of Ahvaz
Ahvaz, Iran
**Ghasem Soleimani Rad**‡
Young Researchers and Elite club, West Tehran Branch
Islamic Azad University
Tehran, Iran

Received 01.01.2020, received in revised form 22.09.2020, accepted 20.11.2020

**Abstract.** In this paper, we prove some new fixed point theorems involving set-valued $F$-contractions in the setting of quasi-ordered metric spaces. Our results are significant since we present Banach contraction principle in a different manner from that which is known in the present literature. Some examples and an application to existence of solution of Volterra-type integral equation are given to support the obtained results.

**Keywords:** fixed point, sequentially complete metric spaces, $F$-contraction, ordered-close operator.

**Citation:** E.L. Ghasab, H. Majani, G.S.Rad, Fixed Points of Set-valued $F$-contraction Operators in Quasi-ordered Metric Spaces with an Application to Integral Equations, J. Sib. Fed. Univ. Math. Phys., 2021, 14(2), 150–158. DOI: 10.17516/1997-1397-2021-14-2-150-158.

## 1. Introduction and preliminaries

It is well known that the Banach contraction principle is a very useful and classical tool in nonlinear analysis [3]. After that, the generalization of this principle has been a heavily investigated. For example, in 1969, Nadler [10] extended the Banach contraction principle for set-valued mapping as follows:

**Theorem 1.1.** *Let $(X,d)$ be a complete metric space and $T : X \to CB(X)$ be a set-valued operator. Also, let $H : N(X)^2 \to [0,+\infty]$ be the Hausdorff metric on $N(X)$ which defined by*

$$H(A,B) = \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\right\},$$

*where $D(a,B) = D(B,a) = \inf_{b\in B} d(a,b)$. Assume that there exists $\alpha \in [0,1)$ such that $H(Tx,Ty) \leqslant \alpha d(x,y)$ for all $x,y \in X$. Then $T$ has a fixed point in $X$.*

*e-lotfali@stu.scu.ac.ir https://orcid.org/0000-0002-8418-9351
†Correspondent: h.majani@scu.ac.ir; majani.hamid@gmail.com https://orcid.org/0000-0001-7022-6513
‡gha.soleimani.sci@iauctb.ac.ir; gh.soleimani2008@gmail.com https://orcid.org/0000-0002-0758-2758

Then Ćirić [6] extended Nadler's result as follows:

**Theorem 1.2.** *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ be a set-valued operator. Assume that there exists $\alpha \in [0, 1)$ such that $H(Tx, Ty) \leqslant \alpha M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\right\}.$$

*Then $T$ has a fixed point in $X$.*

In 2011, Amini-Harandi [2] considered some fixed point theorem for set-valued quasi-contraction mappings in metric spaces.

**Theorem 1.3** ([2]). *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ be a $k$-set-valued quasi-contraction with $k \in \left[0, \frac{1}{2}\right)$; that is,*

$$H(Tx, Ty) \leqslant k \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}$$

*for all $x, y \in X$. Then $T$ has a fixed point in $X$.*

On the other hands, Ran and Reurings [12], and Nieto and Rodríguez-López [11] studied the Banach contraction principle distinctly from another point of view. They imposed a partial order to the metric space $(X, d)$ and discussed on the existence and uniqueness of fixed points for contractive conditions and for the comparable elements of $X$ (also, see [1, 4, 6–8, 13, 15]). Moreover, in 2012, Wardowski [14] obtained a new fixed point theorem concerning $F$-contraction for single-valued mapping.

**Theorem 1.4** ([14]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $F$-contraction. Then $T$ has an unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $x^*$.*

In this paper, we obtain several fixed point results for set-valued $F$-contraction mappings in quasi-ordered metric spaces. Also, we prepare some examples and an application to the existence of a solution for Volterra-type integral equation. Throughout this paper, the family of all nonempty closed and bounded subsets of $X$ is denoted by $CB(X)$, and the family of all nonempty subsets of $X$ by $N(X)$.

**Definition 1.1** ([9]). Let $(X, d)$ be a metric space with a quasi-order " $\preceq$ " (pre-order or pseudo-order; that is, a reflexive and transitive relation). We say that $X$ is sequentially complete if every Cauchy sequence whose consecutive terms are comparable in $X$ converges.

**Definition 1.2** ([9]). Let $(X, d)$ be a metric space with a quasi-order " $\preceq$ ". For two subsets $A$, $B$ of $X$, we say that $A \sqsubseteq B$ if each $a \in A$ and each $b \in B$ imply that $a \preceq b$.

**Definition 1.3** ([9]). Let $(X, d)$ be a metric space with a quasi-order " $\preceq$ ".

(i) A subset $D \subset X$ is said to be approximative, if the set-valued mapping $P_D(x) = \{p \in D : d(x, D) = d(p, x)\}$ for all $x \in X$ has nonempty value.

(ii) The set-valued mapping $G : X \longrightarrow N(X)$ is said to be have approximative values (for short, AV), if $Gx$ is approximative for each $x \in X$.

(iii) The set-valued mapping $G : X \longrightarrow N(X)$ is said to be have comparable approximative values (for short, CAV), if $Gx$ has approximative values for each $x \in X$ and for each $z \in X$, there exists $y \in P_{Gz}(x)$ such that $y$ is comparable to $z$.

(iv) The set-valued mapping $G : X \longrightarrow N(X)$ is said to be have upper comparable approximative values (for short, UCAV), if $Gx$ has approximative values and for each $z \in X$, there exists $y \in P_{Gz}(x)$ such that $y \succeq z$.

(v) The set-valued mapping $G : X \longrightarrow N(X)$ is said to be have lower comparable approximative values (for short, LCAV), if $Gx$ has approximative values and for each $z \in X$, there exists $y \in P_{Gz}(x)$ such that $y \preceq z$.

**Definition 1.4** ([9])**.** The set-valued mapping $G$ is said to has a fixed point if there exists $x \in X$ such that $x \in Gx$.

## 2. Main result

From the idea of Wardowski [14], we consider a new type of $F$-contraction for set-valued operator in quasi-ordered metric spaces as follows.

**Definition 2.1.** Let $H : N(X)^2 \to [0, +\infty]$ be the Hausdorff metric on $N(X)$ and $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a mapping satisfying the following conditions:

(F1) $F$ is increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a \leqslant b$, then $F(a) \leqslant F(b)$;

(F2) for each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} a_n = 0$ if and only if $\lim\limits_{n\to\infty} F(a_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $G : X \longrightarrow CB(X)$ is said to be an $F$-contraction if there exists $\tau > 0$ such that

$$H(Gx, Gy) > 0 \Longrightarrow \tau + F(H(Gx, Gy)) \leqslant F(d(x, y)) \tag{1}$$

for all $x, y \in X$.

**Example 2.1.** If $F(a) = \ln a + a$ for all $a > 0$ and $H : N(X)^2 \to [0, +\infty]$ is the Hausdorff metric on $N(X)$, then $F$ satisfies (F1)–(F3) and each mapping $G : X \longrightarrow CB(X)$ is an $F$-contraction such that $H(Gx, Gy)e^{H(Gx,Gy)-d(x,y)} \leqslant e^{-\tau} d(x, y)$ for all $x, y \in X$.

**Example 2.2.** If $F(a) = \ln a$ for all $a > 0$ and $H : N(X)^2 \to [0, +\infty]$ is the Hausdorff metric on $N(X)$, then $F$ satisfies (F1)–(F3) and each mapping $G : X \longrightarrow CB(X)$ is an $F$-contraction such that $H(Gx, Gy) \leqslant e^{-\tau} d(x, y)$ for all $x, y \in X$.

**Definition 2.2.** Ordered-close operator is set-valued operator $G : X \to CB(X)$ if for two monotone sequences $\{x_n\}, \{y_n\} \subset X$ and $x_0, y_0 \in X$; $x_n \to x_0$, $y_n \to y_0$ and $y_n \in G(x_n)$ imply $y_0 \in G(x_0)$.

**Theorem 2.1.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Also, let the mapping $G : X \longrightarrow CB(X)$ be an ordered-close set-valued $F$-contraction and has UCAV. Then $G$ has a fixed point $x^* \in X$.*

*Proof.* Let $x_0 \in X$. If $x_0 \in Gx_0$, then our proof is complete. Otherwise, since $G$ has UCAV, there exists $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \preceq x_1$ such that $d(x_0, x_1) = \inf\limits_{x\in Gx_0} d(x_0, x) = D(x_0, Gx_0)$. Continue this procedure, we obtain a non-decreasing sequence $\{x_n\}$, where $x_n \in Gx_{n-1}$ with $x_{n-1} \preceq x_n$ and $x_{n-1} \neq x_n$ such that $d(x_n, x_{n+1}) = \inf\limits_{x\in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. On the other hand,

$$D(x_n, Gx_n) \leqslant \sup_{x\in Gx_{n-1}} D(x, Gx_n) \leqslant H(Gx_n, Gx_{n-1}).$$

Therefore, $d(x_n, x_{n+1}) \leqslant H(Gx_n, Gx_{n-1})$. From (F1), we have $F(d(x_n, x_{n+1})) \leqslant F(H(Gx_n, Gx_{n-1}))$. In addition, $G$ is $F$-contraction. Thus,

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leqslant F(H(Gx_n, Gx_{n-1})) \\ &\leqslant F(d(x_n, x_{n-1})) - \tau \\ &\leqslant F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\leqslant \\ &\;\vdots \\ &\leqslant F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{2}$$

We obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Denote $\gamma_n = d(x_n, x_{n+1})$. By (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{4}$$

By (2), we have

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leqslant \gamma_n^k (F(\gamma_0) - n\tau) - \gamma_n^k F(\gamma_0) = -\gamma_n^k n\tau \leqslant 0 \tag{5}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (5), and applying (3) and (4), we obtain $\lim_{n\to\infty} n\gamma_n^k = 0$. Hence, there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leqslant 1$ for each $n \geqslant n_1$. Consequently, we have

$$\gamma_n \leqslant \frac{1}{\sqrt[k]{n}} < 1 \tag{6}$$

for all $n \geqslant n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence, let $m, n \in \mathbb{N}$ with $m > n \geqslant n_1$. From the definition of the metric and (6), we obtain

$$d(x_n, x_m) \leqslant \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n < \sum_{i=n}^{\infty} \gamma_i \leqslant \sum_{i=n}^{\infty} \frac{1}{\sqrt[k]{i}}. \tag{7}$$

From (7) and the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{\sqrt[k]{i}}$, we conclude that $\{x_n\}$ is Cauchy sequence. From the completeness of $X$, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n \to x^*$. Since $G$ is ordered-close operator, $\{x_n\}$ is monotone and $x_{n+1} \in G(x_n)$, we deduce $x^* \in G(x^*)$ and $x^*$ is a fixed point of $G$. □

**Theorem 2.2.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Also, let the mapping $G : X \to CB(X)$ be an ordered-close set-valued $F$-contraction and has LCAV. Then $G$ has a fixed point $x^* \in X$.*

*Proof.* The proof is similar to Theorem 2.1. □

**Example 2.3.** Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ by $S_1 = 1$ and $S_n = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{N}$. Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Also, we define the relation " $\preceq$ " on $X$ by $x \preceq y \Leftrightarrow S_p \leqslant S_q$ for all $x = S_p$, $y = S_q \in X$. Then $(X, d, \preceq)$ is a sequentially complete metric space. Also, let the mapping $G : X \to CB(X)$ be a ordered-close set-valued mapping and has LCAV defined by $G(S_1) = \{S_1\}$ and $G(S_n) = [1, S_{n-1}]$ for all $n > 1$. Then $G$ is an $F$-contraction with $F$ as in Example 2.1 and $\tau = 1$. To see this, let us consider the following calculations:

For each $m, n \in \mathbb{N}$ with $m > 2$ and $n = 1$, we have

$$H(G(S_m), G(S_1)) = \max\left\{ \sup_{a \in G(S_m)} D(a, G(S_1)), \sup_{b \in G(S_1)} D(b, G(S_m)) \right\} = d(S_{m-1}, S_1)$$

and

$$\begin{aligned} \frac{H(G(S_m), G(S_1))}{d(S_m, S_1)} e^{H(G(S_m),G(S_1)) - d(S_m,S_1)} &= \frac{d(S_{m-1}, S_1)}{d(S_m, S_1)} e^{d(S_{m-1},S_1) - d(S_m,S_1)} = \\ &= \frac{S_{m-1} - 1}{S_m - 1} e^{S_{m-1} - S_m} = \\ &= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned}$$

Now, for each $m, n \in \mathbb{N}$ with $m > n > 1$, we have

$$H(G(S_m), G(S_n)) = \max\left\{ \sup_{a \in G(S_m)} D(a, G(S_n)), \sup_{b \in G(S_n)} D(b, G(S_m)) \right\} = d(S_{m-1}, S_{n-1})$$

and

$$\begin{aligned} \frac{H(G(S_m), G(S_n))}{d(S_m, S_n)} e^{H(G(S_m),G(S_n)) - d(S_m,S_n)} &= \frac{d(S_{m-1}, S_{n-1})}{d(S_m, S_n)} e^{d(S_{m-1},S_{n-1}) - d(S_m,S_n)} = \\ &= \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{S_n - S_{n-1} + S_{m-1} - S_m} = \\ &= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} < e^{-1}. \end{aligned}$$

Therefore, by Theorem 2.2, $S_1$ is a fixed point of $G$.

**Theorem 2.3.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Suppose that the mapping $G : X \to CB(X)$ is an ordered-close set-valued $F$-contraction and has $AV$. If there exists $x_0 \in X$ such that $\{x_0\} \sqsubseteq Gx_0$, then $G$ has a fixed point $x^* \in X$.*

*Proof.* If $x_0 \in Gx_0$, then the proof is finished. Otherwise, by Definition 1.2, we have $x \succeq x_0$ for any $x \in Gx_0$. Since $G$ has approximative values, there exists $x_1 \in Gx_0$ with $x_1 \succeq x_0$ and $x_0 \neq x_1$ such that $d(x_0, x_1) = D(x_0, Gx_0)$. Continue this procedure, we have a non-decreasing sequence $\{x_n\}$ with $x_{n-1} \preceq x_n$, where $x_n \in Gx_{n-1}$ and $x_n \neq x_{n-1}$ such that $d(x_n, x_{n+1}) = = \inf_{x \in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. The rest of this proof is the same as that of Theorem 2.1. □

**Theorem 2.4.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Suppose that the mapping $G : X \to CB(X)$ be an ordered-close set-valued $F$-contraction and has AV. If there exists $x_0 \in X$ such that $Gx_0 \sqsubseteq \{x_0\}$, then $G$ has a fixed point $x^* \in X$.*

*Proof.* The proof is similar to Theorem 2.2. □

**Theorem 2.5.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Also, let the mapping $G : X \longrightarrow CB(X)$ be an ordered-close set-valued and has UCAV. If we have*

$$F(H(Gx, Gy)) \leqslant F(M(x, y)) - \tau \tag{8}$$

*for all $x, y \in X$, where*

$$M(x,y) = \max\left\{d(x,y), D(x,Gx), D(y,Gy), \frac{1}{2}[D(x,Gy) + D(y,Gx)]\right\},$$

*then $G$ has a fixed point $x^* \in X$.*

*Proof.* Let $x_0 \in X$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, Since $G$ has UCAV, there exists $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_0 \preceq x_1$ such that $d(x_0, x_1) = \inf\limits_{x \in Gx_0} d(x_0, x) = D(x_0, Gx_0)$. Continue this procedure, we obtain a non-decreasing sequence $\{x_n\}$ with $x_{n-1} \preceq x_n$, where $x_n \in Gx_{n-1}$ and $x_n \neq x_{n-1}$ such that $d(x_n, x_{n+1}) = \inf\limits_{x \in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. On the other hand,

$$D(x_n, Gx_n) \leqslant \sup_{x \in Gx_{n-1}} D(x, Gx_n) \leqslant H(Gx_n, Gx_{n-1}).$$

Therefore, $d(x_n, x_{n+1}) \leqslant H(Gx_n, Gx_{n-1})$. Now, from $(F1)$ and (8) we have

$$F(d(x_n, x_{n+1})) \leqslant F(H(Gx_n, Gx_{n-1})) \leqslant F(M(x_n, x_{n-1})) - \tau$$

for all $n \in \mathbb{N}$, where

$$M(x_n, x_{n-1}) = \\ = \max\left\{d(x_n, x_{n-1}), D(x_n, Gx_n), D(x_{n-1}, Gx_{n-1}), \frac{1}{2}[D(x_n, Gx_{n-1}) + D(x_{n-1}, Gx_n)]\right\}.$$

Once more, note that $x_{n+1} \in Gx_n$ and $D(x_n, Gx_n) = d(x_n, x_{n+1})$. Hence, we have

$$\begin{aligned} M(x_n, x_{n-1}) &\leqslant \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{1}{2}[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]\right\} \leqslant \\ &\leqslant \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right\} \leqslant \\ &\leqslant \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned}$$

If $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then $F(d(x_n, x_{n+1})) \leqslant F(d(x_n, x_{n+1})) - \tau$, which contradicts with $\tau > 0$. Thus, we have $F(d(x_n, x_{n+1})) \leqslant F(d(x_n, x_{n-1})) - \tau$. The rest of the proof is in the similar manner given in Theorem 2.1. □

**Theorem 2.6.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Assume that the mapping $G : X \longrightarrow CB(X)$ is an ordered-close set-valued and has LCAV, and $F(H(Gx, Gy)) \leqslant F(M(x, y)) - \tau$ for all $x, y \in X$, where*

$$M(x,y) = \max\left\{d(x,y), D(x,Gx), D(y,Gy), \frac{1}{2}[D(x,Gy) + D(y,Gx)]\right\}.$$

*Then $G$ has a fixed point $x^* \in X$.*

*Proof.* Let $x_0 \in X$. If $x_0 \in Gx_0$, then the proof is complete. Otherwise, Since $G$ has LCAV, there exists $x_1 \in Gx_0$ with $x_0 \neq x_1$ and $x_1 \preceq x_0$ such that $d(x_0, x_1) = \inf\limits_{x \in Gx_0} d(x_0, x) = D(x_0, Gx_0)$. Continue this procedure, we obtain a non-increasing sequence $\{x_n\}$ with $x_n \preceq x_{n-1}$, where $x_n \in Gx_{n-1}$ and $x_n \neq x_{n-1}$ such that $d(x_n, x_{n+1}) = \inf\limits_{x \in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. The rest of this proof is the same as that of Theorem 2.5. □

**Theorem 2.7.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Assume that the mapping $G : X \to CB(X)$ is an ordered-close set-valued and has AV, and $F(H(Gx, Gy)) \leqslant F(M(x, y))-\tau$ for all $x, y \in X$, where*

$$M(x,y) = \max\left\{d(x,y), D(x,Gx), D(y,Gy), \frac{1}{2}[D(x,Gy)+D(y,Gx)]\right\}.$$

*If there exists $x_0 \in X$ such that $\{x_0\} \sqsubseteq Gx_0$, then $G$ has a fixed point $x^* \in X$.*

*Proof.* If $x_0 \in Gx_0$, then the proof is finished. Otherwise, by Definition 1.2, we have $x \succeq x_0$ for any $x \in Gx_0$. Since $G$ has approximative values, there exists $x_1 \in Gx_0$ with $x_1 \succeq x_0$ and $x_0 \neq x_1$ such that $d(x_0, x_1) = D(x_0, Gx_0)$. Continue this procedure, we have a non-decreasing sequence $\{x_n\}$ with $x_{n-1} \preceq x_n$, where $x_n \in Gx_{n-1}$ and $x_n \neq x_{n-1}$ such that $d(x_n, x_{n+1}) = = \inf_{x\in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. The rest of this proof is the same as that of Theorem 2.5. □

**Theorem 2.8.** *Let $(X, d, \preceq)$ be a sequentially complete metric space. Assume that the mapping $G : X \to CB(X)$ is an ordered-close set-valued and has AV, and $F(H(Gx, Gy)) \leqslant F(M(x, y))-\tau$ for all $x, y \in X$, where*

$$M(x,y) = \max\left\{d(x,y), D(x,Gx), D(y,Gy), \frac{1}{2}[D(x,Gy)+D(y,Gx)]\right\}.$$

*If there exists $x_0 \in X$ such that $Gx_0 \sqsubseteq \{x_0\}$, then $G$ has a fixed point $x^* \in X$.*

*Proof.* If $x_0 \in Gx_0$, then the proof is finished. Otherwise, by Definition 1.2, we have $x_0 \succeq x$ for any $x \in Gx_0$. Since $G$ has approximative values, there exists $x_1 \in Gx_0$ with $x_0 \succeq x_1$ and $x_0 \neq x_1$ such that $d(x_0, x_1) = D(x_0, Gx_0)$. Continue this procedure, we have a non-increasing sequence $\{x_n\}$ with $x_n \preceq x_{n-1}$, where $x_n \in Gx_{n-1}$ and $x_n \neq x_{n-1}$ such that $d(x_n, x_{n+1}) = = \inf_{x\in Gx_n} d(x_n, x) = D(x_n, Gx_n)$. The rest of this proof is the same as that of Theorem 2.5. □

# 3. Application to integral equation

As an application of our results, we will consider the following Volterra integral equation:

$$x(t) = \int_0^t K(t, s, x(s))ds + g(t), \tag{9}$$

where $I = [0, 1]$, $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $g \in C(I, \mathbb{R})$ for all $t \in I$.

Let $C(I, \mathbb{R})$ be the Banach space of all real continuous functions defined on $I$ with the sup norm $||x||_\infty = \max_{t\in I} |x(t)|$ for all $x \in C(I, \mathbb{R})$ and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ be the space of all continuous functions defined on $I \times I \times C(I, \mathbb{R})$. Alternatively, the Banach space $C(I, \mathbb{R})$ can be endowed with Bielecki norm $||x||_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}$ for all $x \in C(I, \mathbb{R})$ and $\tau > 0$, and the induced metric $d_B(x, y) = ||x-y||_B$ for all $x, y \in C(I, \mathbb{R})$ (see [5]). Also, let $f : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by $fx(t) = \int_0^t K(t, s, x(s))ds + g(t)$ and $g \in C(I, \mathbb{R})$. Moreover, we define the relation " $\preceq$ " on $C(I, \mathbb{R})$ by $x \preceq y \Leftrightarrow ||x||_\infty \leqslant ||y||_\infty$ for all $x, y \in C(I, \mathbb{R})$. Clearly the relation " $\preceq$ " is a quasi-order relation.

**Theorem 3.1.** *Let $(C(I, \mathbb{R}), d_B, \preceq)$ be a sequentially complete metric space. Suppose that $G : C(I, \mathbb{R}) \to CB(C(I, \mathbb{R}))$ is a set-valued operator such that $G(x) = \{fx(t)\}$ and has UCAV. Let $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ be an operator satisfying the following conditions:*

*(i)* *$K$ is continuous;*

*(ii)* $\int_0^t K(t,s,.)$ *for all* $t,s \in I$ *is increasing;*

*(iii)* *there exists* $\tau > 0$ *such that* $|K(t,s,x(s)) - K(t,s,y(s))| \leqslant e^{-\tau}|x(s) - y(s)|$ *for all* $x,y \in C(I,\mathbb{R})$ *and all* $t,s \in I$.

*Then, the Volterra-type integral equation* (9) *has a solution in* $C(I,\mathbb{R})$.

*Proof.* By definition of $G$, we have $H(Gx,Gy) = d_B(f(x),f(y))$ for all $x,y \in C(I,\mathbb{R})$. Thus,

$$\begin{aligned} H(Gx,Gy) = d_B(f(x),f(y)) &= \sup_{t\in I}\left\{\Big|\int_0^t K(t,s,x(s))ds - \int_0^t K(t,s,y(s))ds\Big|e^{-\tau t}\right\} \\ &\leqslant \sup_{t\in I}\left\{\int_0^t |K(t,s,x(s)) - K(t,s,y(s))|e^{-\tau t}ds\right\} \\ &\leqslant \sup_{t\in I}\left\{\int_0^t e^{-\tau}|x(s) - y(s)|e^{-\tau t}ds\right\} \\ &\leqslant ||x-y||_B \sup_{t\in I}\left\{\int_0^t e^{-\tau}ds\right\} \\ &= e^{-\tau}d_B(x,y). \end{aligned}$$

Taking logarithms, we have $\ln(H(Gx,Gy)) \leqslant \ln(e^{-\tau}d_B(x,y))$, which implies that $(\tau + \ln(H(Gx,Gy))) \leqslant \ln(d_B(x,y))$. Now, consider the function $F(t) = \ln(t)$ for all $t \in C(I,\mathbb{R})$ and $\tau > 0$. Then, all conditions of Theorem 2.1 are satisfied. Consequently, Theorem 2.1 ensures the existence of fixed point of $G$ that this fixed point is the solution of the integral equation. □

*We are grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (Grant number: SCU.MM99.25894).*

## References

[1] R.P.Agarwal, M.A.El-Gebeily, D.O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87**(2008), 1–8.

[2] A.Amini-Harandi, Fixed point theory for set-valued quasi-contraction maps in metric spaces, *Appl. Math. Lett.*, **24**(2011), 1791–1794. DOI: 10.1016/j.aml.2011.04.033

[3] S.Banach, Surles operations dans les ensembles abstraits et leurs application auxequations integrales, *Fund. Math.*, **3**(1922), 133–181.

[4] T.G.Bhaskar, V.Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*, **65**(2006), 1379–1393.

[5] A.Bielecki, Une remarque sur la methode de Banach-Cacciopoli-Tikhonov dans la theorie des equations differentielles ordinaires, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, **4**(1956), 261–264.

[6] Lj.Ćirić, Fixed points for generalized multi-valued contractions, *Math. Ves.*, **24**(1972), 265–272.

[7] E.L.Ghasab, H.Majani, E.Karapinar, G.Soleimani Rad, New fixed point results in $F$-quasi-metric spaces and an application, *Adv. Math. Phys.*, **2020**, 2020:9452350. DOI: 10.1155/2020/9452350

[8] Z.Kadelburg, S.Radenović, Notes on some recent papers concerning $F$-contractions in $b$-metric spaces, *Construct. Math. Anal.*, **1**(2018), 108–112.

[9] H.P.Masiha, F.Sabetghadam, Fixed point results for multi-valued operators in quasi-ordered metric spaces, *Appl. Math. Lett.*, **25**(2012), 1856–1861.

[10] S.B.Nadler, Multi-valued contraction mappings, *Pacific J. Math.*, **30**(1969), 475–488.

[11] J.J.Nieto, R.Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order.*, **22**(2005), 223–239.

[12] A.C.M.Ran, M.C.B.Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132**(2004), 1435–1443.

[13] R.Saadati, S.M.Vaezpour, Monotone generalized weak contractions in partially ordered metric spaces, *Fixed Point Theory.*, **11**(2010), 375–382.

[14] D.Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**, 2012:94.

[15] D.Wardowski, Solving existence problems via $F$-contractions, *Proc. Amer. Math. Soc.*, **146**(2018), 1585–1598.

## Неподвижные точки многозначных операторов $F$-сжатия в квазиупорядоченных метрических пространствах с приложением к интегральным уравнениям

**Эхсан Л. Гасаб**
**Хамид Маджани**
Университет Шахида Чамрана в Ахвазе
Ахваз, Иран
**Гасем С. Рад**
Исламский университет Азад
Тегеран, Иран

**Аннотация.** В этой статье мы докажем некоторые новые теоремы о неподвижных точках, включающие многозначные $F$-сжатия в условиях квазиупорядоченных метрических пространств. Наши результаты важны, поскольку мы представляем принцип банахового сжатия иначе, чем тот, который известен в настоящей литературе. Для подтверждения полученных результатов приведены некоторые примеры и приложение к существованию решения интегрального уравнения типа Вольтерра.

**Ключевые слова:** неподвижная точка, $F$-сжатие, секвенциально полные метрические пространства, оператор упорядоченного замыкания.