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L^P-bound for the Fourier Transform of Surface-Carried Measures Supported on Hypersurfaces with D_∞ Type Singularities

Nigina A. Soleeva*
Samarkand State University
Samarkand, Uzbekistan

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Abstract. Estimate for Fourier transform of surface-carried measures supported on non-convex surfaces of three-dimensional Euclidean space is considered in this paper.The exact convergence exponent was found wherein the Fourier transform of measures is integrable in tree-dimensional space. This result gives an answer to the question posed by Erdösh and Salmhofer.

Keywords: Fourier transform, oscillatory integral, surface-carried measure.

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1. Introduction and preliminaries

Let $S \subset \mathbb{R}^3$ be a smooth surface and $\psi \in C_0^\infty(S)$ be a smooth function with compact support on S. Consider the measure $d\mu = \psi d\sigma$, where $d\sigma$ is the surface-carried measure. Fourier transform of the measure is defined by:

$$\hat{\mu}(\xi) := \int_S e^{i(\xi,x)} d\mu.$$

It is well-know that $\hat{\mu}$ is an analytic function.

In this paper the following problem is considered: find $\gamma := \inf\{p : \hat{\mu} \in L^p(\mathbb{R}^3)\}$. This problem has a long history [1,2]. Recently L. Erdös and M. Salmhofer [2] considered the problem for partial class of non-convex surfaces in $\mathbb{R}^3$. The main class of such surfaces was level set of dispersion relation of discrete Schrödinger operator on the lattice $\mathbb{Z}^3$. It should be noted that the phase function of the corresponding oscillatory integrals has singularities of type A_1, A_2, A_3 or D_4. In particular, except the case D_4 one of the principal curvatures does not vanish at every point. The case D_4 type singularities was excluded in [2]. A more general class of hypersurfaces for which the Gaussian curvature has only simple roots was considered [3]. However, it was assumed that only one of the principal curvatures can vanish. The case when both principal curvatures vanish at a point of the surface in $\mathbb{R}^3$ is still one of the open problems.

We consider the problem for hypersurfaces in $\mathbb{R}^3$. More precisely it is assumed that the phase function $(x, \omega)|_S$ (where $\omega \in S^2$ is the unite sphere centred at the origin) is small perturbation of the so-called D_∞ type singularity (see [4] for definitions and basic properties of such singularities).

*niginasol@yahoo.com

It is shown that in this case $\gamma = 3$. It can be shown that for any hypersurface $S \subset \mathbb{R}^3$, $\hat{\mu} \notin L^p(\mathbb{R}^3)$ for $p \leqslant 3$, whenever $\mathrm{Supp}(\mu) \neq \emptyset$.

The main result is the following.

Theorem 1.1. *Let S be an analytic hypersurface in $\mathbb{R}^3$. If S has D_∞ type singularities at the origin then there exists a neighborhood U of the origin such that for any $\psi \in C_0^\infty(U)$ the inclusion $\hat{\mu} \in L^p(\mathbb{R}^3)$ holds for any $p > 3$.*

Moreover, if S is any smooth surface in $\mathbb{R}^3$ and $\psi(0,0) \neq 0$ then $\hat{\mu} \notin L^3(\mathbb{R}^3)$.

The paper is organized as follows. In Section 2 the problem for the model case is considered. In this case the result is obtained with the use of simple methods. The Section 3 is devoted to special function with D_∞ type singularity at the origin.

In Section 4 the general case is considered. Main theorem is proved in Section 5.

2. Model case D_∞

Let us consider a measure supported on hypersurface $x_3 = x_1 x_2^2$. The singularity of that function is called to be D_∞ type singularity at $(0,0)$. The Fourier transform of the measure can be written as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^2} e^{i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_1 x_2^2)} \psi_1(x) dx,$$

where $\psi_1(x_1, x_2) = \psi(x_1, x_2, x_1 x_2^2) / \!/ \sqrt{1 + x_2^4 + 4x_1^2 x_2^2}$.

Following B. Randol [3], we define the following maximal function:

$$M(\omega) = \sup_{r > o} r |\hat{\mu}(r\omega)|,$$

where $r = |\xi|$ and $\omega \in S^2$, S^2 is the unite sphere centred at the origin.

Let us note that $\hat{\mu}(\xi) = O(|\xi|^{-N})$ (as $|\xi| \to \infty$) provided $|\xi_3| \leqslant \max\{|\xi_1|, |\xi_2|\}$ and ψ is a smooth function concentrated in a sufficiently small neighbourhood of the origin [5]. It is also assumed that $|\xi_3| \geqslant \max\{|\xi_1|, |\xi_2|\}$. Let us consider the associated oscillatory integral

$$J(\lambda, s) = \int_{\mathbb{R}^2} e^{i\lambda \Phi(x,s)} \psi_1(x) dx,$$

where $\Phi(x, s) = x_1 x_2^2 + s_1 x_1 + s_2 x_2$, $\lambda = \xi_3$, $s_j = \dfrac{\xi_j}{\lambda}$, $j = 1, 2$.

One can define the Randol type maximal function [3] associated with the oscillatory integral $J(\lambda, s)$ as

$$M(s) = \sup_{\lambda \neq 0} |\lambda| |J(\lambda, s)|.$$

Now, the following statement is proved.

Theorem 2.1. *The inclusion $M \in L^{3-0}_{loc}(\mathbb{R}^2)$ holds true.*

Taking into account that ψ has a compact support and using integration by parts, the integral

$$J_1(\lambda, s_1, x_2) = \int_{\mathbb{R}} e^{i\lambda x_1 (x_2^2 + s_1)} \psi(x_1, x_2) dx_1$$

can be estimated by

$$|J_1(\lambda, s_1, x_2)| \leqslant \frac{c \|\psi\|_{c^2}}{1 + |\lambda|^2 |x_2^2 + s_1|^2}.$$

Consider the following integral

$$J^1(\lambda, s_1) = \int_{\mathbb{R}} \frac{dx_2}{1 + |\lambda|^2 |x_2^2 + s_1|^2}.$$

First, we prove the auxiliary statement.

Lemma 2.1. *The following estimate holds true:*

$$|J^1(\lambda, s_1)| \leqslant \frac{c}{|\lambda||s_1|^{\frac{1}{2}}}.$$

Proof. First consider the case $\lambda|s_1| \leqslant 1$. If $s_1 = 0$ then there is nothing to prove. Let us assume that $s_1 \neq 0$. In this case we use change of variables $x_2 = |s_1|^{\frac{1}{2}} y_2$ and obtain

$$J^1(\lambda, s_1) = |s_1|^{\frac{1}{2}} \int_{\mathbb{R}} \frac{dy_2}{1 + |\lambda s_1|^2 |y_2^2 + \mathrm{sgn}(s_1)|^2}.$$

For the sake of definiteness we assume that $\mathrm{sgn}(s_1) = -1$, e.g. $s_1 < 0$. Actually the case $\mathrm{sgn}(s_1) = 1$ or equivalently $s_1 > 0$ is much more easy to prove. Thus, we have

$$J(\lambda, s_1) = |s_1|^{\frac{1}{2}} \int_{\mathbb{R}} \frac{dy_2}{1 + |\lambda s_1|^2 |y_2^2 - 1|^2}.$$

It is easy to see that the following estimate

$$\int\limits_{|\lambda s_1||y_2^2 - 1| > 1} \frac{dy_2}{|y_2^2 - 1|} \leqslant C |\lambda s_1|^{\frac{1}{2}}$$

holds. Indeed

$$\int\limits_{|y_2^2 - 1| > \frac{1}{\lambda s_1}} \frac{dy_2}{|y_2^2 - 1|} = 2 \int\limits_{y_2 > \sqrt{1 + \frac{1}{|\lambda s_1|}}} \frac{dy_2}{y_2^2 - 1} =$$

$$= \int\limits_{\sqrt{1 + \frac{1}{|\lambda s_1|}}}^{\infty} \left(\frac{1}{y_2 - 1} - \frac{1}{y_2 + 1} \right) dy_2 = \ln \frac{y_2 - 1}{y_2 + 1} \Bigg|_{\sqrt{1 + \frac{1}{|\lambda s_1|}}}^{\infty} =$$

$$= \ln \left(\frac{\sqrt{1 + \frac{1}{|\lambda s_1|}} + 1}{\sqrt{1 + \frac{1}{|\lambda s_1|}} - 1} \right) = \ln \left(|\lambda s_1| \left(2 + \frac{1}{|\lambda s_1|} + 2\sqrt{1 + \frac{1}{|\lambda s_1|}} \right) \right) =$$

$$= \ln \left(1 + 2|\lambda s_1| + 2\sqrt{|\lambda s_1|^2 + |\lambda s_1|} \right) \leqslant 2|\lambda s_1| + 2\sqrt{|\lambda s_1|^2 + |\lambda s_1|} =$$

$$= \sqrt{|\lambda s_1|} (2\sqrt{|\lambda s_1|} + 2\sqrt{1 + |\lambda s_1|}) \leqslant \sqrt{|\lambda s_1|}(2 + 2\sqrt{2}) = c\sqrt{|\lambda s_1|}$$

for $|\lambda s_1| \leqslant 1$. An analogical estimate holds true for $|\lambda s_1| \leqslant 2$.

Also $\mu\{y_2 : |\lambda s_1||y_2^2 - 1| \leqslant 1\} \leqslant \dfrac{c}{(\lambda|s_1|)^{\frac{1}{2}}}$. Hence the inequality

$$|J(\lambda, s_1)| \leqslant \frac{c}{\lambda |s_1|^{\frac{1}{2}}}$$

holds true provided $\lambda|s_1| \leqslant 2$.

Now, we consider the case $|\lambda s_1| \geqslant 2$. In this case, we have

$$\int\limits_{|y_2^2-1|\geqslant 1} \frac{dy_2}{|\lambda s_1|^2|y_2^2-1|^2} = \frac{c}{|\lambda s_1|^2}.$$

It is easy to see that the following estimate

$$\int\limits_{1\geqslant|y_2^2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2^2-1|^2} \leqslant c|\lambda s_1|$$

holds. Indeed, using symmetry of arguments, the last integral can be estimated as

$$\int\limits_{1\geqslant|y_2^2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2^2-1|^2} \leqslant 2 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2-1|^2|y_2+1|^2} \leqslant$$

$$\leqslant 2 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{|y_2-1|^2} \leqslant 4 \int\limits_{|y_2-1|>|\lambda s_1|^{-1}} \frac{dy_2}{(y_2-1)^2} = 4|\lambda s_1|.$$

On the other hand the inequality $\mu\{y_2 : |y_2^2-1| < |\lambda s_1|^{-1}\} \leqslant c|\lambda s_1|^{-1}$ holds true for the measure of the set $\{y_2 : |y_2^2-1| < |\lambda s_1|^{-1}\}$. Hence we obtain

$$|J^1(\lambda, s)| \leqslant \frac{c}{\lambda |s_1|^{\frac{1}{2}}}.$$

Lemma is proved. □

It is easy to see that the oscillatory integral $J(\lambda, s)$ can be estimated as follows:

$$|J(\lambda, s)| \leqslant \int_{-N}^{N} |J_1(\lambda, s, x_2)| dx_2,$$

where the number N is

$$N = \max\{|x_2| : \text{there exist } x_1, \text{ such that } (x_1, x_2) \in \text{ Supp}\,\psi\}. \tag{1}$$

Hence

$$|J(\lambda, s)| \leqslant c\|\psi\|_{c^2} |J^1(\lambda, s)|.$$

Consequently, it follows from the Lemma that

$$|J(\lambda, s)| \leqslant \frac{c\|\psi\|_{c^1}}{|\lambda||s_1|^{\frac{1}{2}}}$$

because ψ has a compact support. If $|s| > m$, where m is a big positive number depending on the support of ψ, then the phase function has no critical point. Hence we can use integration by parts and obtain

$$J(\lambda, s) \leqslant \frac{c}{\lambda |s|}.$$

Therefore we have

$$\chi_{\{|s|>m\}}(s) M(s) \leqslant \frac{c}{|s|} \in L^\infty(\mathbb{R}^2 \backslash B(0, m)), \tag{2}$$

where $B(0,m)$ is the ball of radius m centred at the origin, and $\chi_{\{|s|>m\}}$ is the indicator function of the set $\{|s|>m\}$. Let us denote the indicator function of the set A by χ_A, e.g., $\chi_A(x)=1$ for $x\in A$ otherwise $\chi_A(x)=0$.

The relation (2) suggests that it is sufficiently to consider the oscillatory integral and the associated maximal function on the set $\{|s|\leqslant m\}$. Let us assume that $x=x^0\in \mathrm{Supp}\,(\psi)$ is a critical point, and $s=s^0\in\overline{B(0,m)}$ is a fixed point. If x_0 is not a critical point of the phase function $\Phi(x,s^0)$ then one can use integration by parts and obtain better estimate than needed. Equations for critical points are

$$(x_2^0)^2+s_1^0=0,\quad 2x_1^0x_2^0+s_2^0=0.$$

Let us assume that $s_2^0\neq 0$. Then $x_1^0x_2^0\neq 0$. Hence $x_1^0\neq 0$ and also $x_2^0\neq 0$, $s_1^0\neq 0$. Let us consider the integral

$$J^{\chi}(\lambda,s):=\int_{\mathbb{R}^2} e^{i\lambda\Phi(x,s)}\psi(x)\chi(x)dx,$$

where χ is a smooth cut-off function defined in a sufficiently small neighbourhood of x^0 and s is close to s^0. One can use stationary phase method in two variables because

$$Hess\Phi(x^0,s^0)=-4(x_2^0)^2\neq 0.$$

Therefore for $|s-s^0|<\varepsilon$ we have the estimate

$$|J^{\chi}(\lambda,s)|\leqslant\frac{c}{\lambda}$$

provided χ is a smooth function defined in a sufficiently small neighbourhood of x^0. If x^0 is not a critical point then one can use integration by parts and obtain the same type of estimate (even better estimate than needed). Hence $M(s)$ is a bounded function in $V(s^0)$, where $V(s^0)$ is a sufficiently small neighbourhood of $s^0\neq 0$. Let us consider the case when $s^0=0$, e.g., when s belongs to a sufficiently small neighbourhood of the origin. This case will be considered in the next section.

3. Case $\{|s_1|^{\frac{1}{2}}\geqslant|s_2|\}$

Then trivial estimate for $J(\lambda,s)$ is

$$|J(\lambda,s)|\leqslant\frac{c}{|\lambda||s_1|^{\frac{1}{2}}}\leqslant\frac{c}{|\lambda||s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}}$$

and the estimate is obtained because $\dfrac{1}{|s_1|^{\frac{1}{3}}|s_2|^{\frac{1}{3}}}\in L^{3-0}(V)$, where V is a bounded neighbourhood of the origin.

Let us assume that $|s_2|\geqslant|s_1|^{\frac{1}{2}}$.

Let us consider the one-dimensional integral

$$J_2(\lambda,s_2,x_1)=\int_{\mathbb{R}} e^{i\lambda(x_1x_2^2+s_2x_2)}\psi(x_1,x_2)dx_2.$$

If $|\lambda x_1|\leqslant 1$ then we have the trivial estimate

$$\int_{[0,\lambda^{-1}]}|J_2(\lambda,s_2,x_1)|dx_1\leqslant c|\lambda|^{-1}.$$

Hence we may assume $|\lambda x_1| > 1$. If $|\lambda x_1| > 1$ and $|x_1| \leqslant |s_2|$ then the phase function has no critical point on the support of ψ provided $N < \frac{1}{2}$, where N is defined by relation (1). Then one can use double integration by parts and obtain

$$|J_2(\lambda, s_2, x_1)| \leqslant \frac{c\|\psi\|c_2}{|\lambda x_1|^2}.$$

Therefore

$$\int\limits_{[0,|s_2|]} |J_2(\lambda, s_2, x_1)| dx_1 \leqslant \frac{c\|\psi\|_{c_2}}{|\lambda|}.$$

Finally, let us suppose that $|x_1| > |s_2|$. Then we use stationary phase method in x_2 and obtain

$$J_2(\lambda, s_2, x_1) = \frac{c}{|\lambda x_1|^{\frac{1}{2}}} e^{-\frac{s_2^2}{4x_1}\lambda} \psi\Big(x_1, -\frac{s_2}{2x_1}\Big) + R(\lambda, x_1, s_2).$$

For the remainder term $R(\lambda, x_1, s)$ we have $|R(\lambda, x_1, s_2)| \leqslant \frac{c}{1 + |\lambda x_1|^{\frac{3}{2}}}$. Then $\int |R(\lambda, x_1)| dx_1 \leqslant \frac{c}{|\lambda|}$. Thus, it is sufficiently to consider the integral

$$J_1(\lambda, s) = \int_{\mathbb{R}} \frac{e^{i\lambda s_2^2(-\frac{1}{4x_1} + \frac{s_1}{s_2^2} x_1)}}{|x_1|^{\frac{1}{2}}} \psi\Big(x_1, -\frac{s_2}{2x_1}\Big) dx_1.$$

If $|\lambda s_2^2| < 1$ then we have $|J_1| \leqslant \frac{c}{|\lambda|^{\frac{1}{2}} |s_2|}$. Hence we assume $|\lambda s_2^2| > 1$. Let us estimate the integral

$$J_1^+(\lambda, s) = \int_{\mathbb{R}_+} e^{i\lambda s_2^2(-\frac{1}{4x_1} + \frac{s_1}{s_2^2} x_1)} \frac{\psi\big(x_1, -\frac{s_2}{2x_1}\big)}{x_1^{\frac{1}{2}}} dx_1.$$

Using the change of variables $x_1 = y_1^2$, we obtain

$$J_1^+(\lambda, s) = 2 \int_{\mathbb{R}_+} e^{i\lambda s_2^2(-\frac{1}{4y_1^2} + \sigma_1 y_1^2)} \psi\Big(y_1^2, -\frac{s_2}{2y_1^2}\Big) dy_1,$$

where $\sigma_1 := \frac{s_1}{s_2^2}$.

The phase function has no critical points provided ψ is a smooth function defined in a sufficiently small neighbourhood of the origin so one can use integration by parts.

Thus, we obtain

$$|J_1(\lambda, s)| \leqslant \frac{c\|\psi\|_{c^1}}{|s_2||\lambda|^{\frac{1}{2}}}.$$

Let us show that

$$\frac{\chi_{\{|s_1| \leqslant s_2^2\}}}{s_2} \in L^{3-0}(V).$$

Indeed for $p < 3$ we have

$$\int_0^1 \frac{ds_2}{|s_2|^p} \int_0^{s_2} ds_1 = \int_0^1 \frac{ds_2}{|s_2|^{p-2}} < +\infty.$$

Combining the obtained estimates for the Rendol maximal function for oscillatory integral, we obtain

$$M(S) \leqslant c\bigg(\frac{\chi_{\{|s_1| \geqslant s_2^2\}}(s)}{|s_1|^{\frac{1}{2}}} + \frac{\chi_{\{s_2^2 \geqslant |s_1|\}}(s)}{|s_2|}\bigg).$$

Since $M \in L^{3-0}_{loc}(\mathbb{R}^2)$ our consideration is completed.

4. The general case

The following proposition holds true.

Proposition. Let us assume that $\Phi(x_1, x_2)$ has D_∞ type singularity at the origin

$$\Phi(x_1, x_2) = x_1 x_2^2 + R(x_1, x_2),$$

where $R(x_1, x_2) = O(|x|^4)$.

Then there exist analytic functions φ, ψ and b such that function Φ can be written as

$$\Phi(x_1, x_2) = b(x_1, x_2)(x_1 - \varphi(x_2))(x_2 - \psi(x_1))^2,$$

where $\varphi(0) = \varphi'(0) = 0$, $\psi(0) = \psi'(0) = 0$, $b(0,0) \neq 1$ (see [2] and [6]).

Let us assume that $\psi(x_1) = x_1^{m_1}\tilde{\psi}(x_1)$, $\tilde{\psi}(0) \neq 0$ and $\varphi(x_2) = x_2^{m_2}\tilde{\varphi}(x_2)$, $\tilde{\varphi}(0) \neq 0$. Then

$$\Phi(x, s) = b(x_1, x_2)(x_1 - \varphi(x_2))(x_2 - \psi(x_1))^2 + s_1 x_1 + s_2 x_2.$$

Using the change of variables

$$x_1 - \varphi(x_2) \longrightarrow x_1, \quad x_2 - \psi(x_1) \longrightarrow x_2,$$

we obtain

$$\Phi(x, s) = \tilde{b}(x_1, x_2)x_1 x_2^2 + s_1(x_1 + \varphi(x_2)) + s_2(x_2 + \psi(x_1)).$$

Let D be the annulus $D = \{\frac{1}{2} \leqslant |x| \leqslant 2\}$ and $\operatorname{Supp}\chi \subset \mathrm{D}$ with $\chi \in C^\infty(D)$ satisfying

$$\sum_{\kappa=\kappa_0}^{\infty} \chi(2^{\frac{\kappa}{3}}x) = 1 \ \text{ for } \ x \neq 0, \quad |x| << 1.$$

Then we have

$$J(\lambda, s) = \int a(x_1, x_2)e^{i\lambda\Psi_1(x,s)}dx = \sum_{\kappa=\kappa_0}^{\infty} \int a(x_1, x_2)\chi(2^{\frac{\kappa}{3}}x)e^{i\lambda\Psi_1(x,s)}dx.$$

Let

$$J_\kappa = \int a(x_1, x_2)\chi(2^{\frac{\kappa}{3}}x)e^{i\lambda\Psi_1(x,s)}dx.$$

Let us use scaling $2^{\frac{\kappa}{3}}x \longrightarrow x$ and obtain

$$J_\kappa = 2^{-\frac{2\kappa}{3}} \int a(2^{-\frac{\kappa}{3}}x)\chi(x)e^{i\lambda 2^\kappa \Psi_1(x,s)}dx,$$

$$\Psi(x, s) = \tilde{b}(2^{-\frac{\kappa}{3}}x)x_1 x_2^2 + 2^{\frac{2\kappa}{3}}s_1(x_1 + x_2^{m_1}2^{-\frac{\kappa}{3}(m_2-1)}\tilde{\varphi}(2^{-\frac{\kappa}{3}}x_2)) + \\ + 2^{\frac{2\kappa}{3}}s_2(x_2 + x_1^{m_1}2^{-\frac{\kappa}{3}(m_1-1)}\tilde{\psi}(2^{-\frac{\kappa}{3}}x_1)).$$

Note that $x \in D$. If $|2^{\frac{2\kappa}{3}}s_1| >> 1$ or $|2^{\frac{2\kappa}{3}}s_2| >> 1$ then using integration by parts, we obtain

$$|J_\kappa| \leqslant c\frac{2^{-\frac{2\kappa}{3}}}{|\lambda 2^{-\kappa}|(|2^{\frac{2\kappa}{3}}s_1| + 2^{\frac{2\kappa}{3}}|s_2|)}.$$

Let us take the integral

$$\int\limits_{2^{\frac{2\kappa}{3}}|s|>1} \frac{2^{\frac{\kappa}{3}p}ds}{(|2^{\frac{2\kappa}{3}}s_1| + 2^{\frac{2\kappa}{3}}|s_2|)^p}.$$

After the change variable $2^{\frac{2\kappa}{3}}s=\sigma$ we have

$$\int\limits_{|\sigma|>1} 2^{\frac{\kappa p}{3}-\frac{4\kappa}{3}}\frac{d\sigma}{|\sigma|^p}=2^{\frac{\kappa}{3}(p-4)}\int\limits_{|\sigma|>1}\frac{d\sigma}{|\sigma|^p}=2^{\frac{\kappa}{3}(p-4)}c_p.$$

Thus, if $p<4$ then the series $\sum\limits_{\kappa=\kappa_0}^{\infty}\dfrac{2^{\frac{\kappa}{3}}\chi|2^{\frac{2\kappa}{3}}s|}{|2^{\frac{2\kappa}{3}}s_1|+2^{\frac{2\kappa}{3}}|s_2|}$ converges in L^p. Let $2^{\frac{2\kappa}{3}}s=\sigma$ and $|\sigma|\leqslant 1$.

Now, we use compactness arguments.

Let us assume that $\sigma=\sigma^0\neq 0$ and (x_1^0,x_2^0) is a critical point of the phase function.

Then $\Phi_\kappa(x,\sigma)$ can be considered as a small perturbation of the function

$$\Phi=\tilde{b}(0,0)x_1x_2^2+\sigma_1^0x_1+\sigma_2^0x_2,$$

where $(x_1,x_2)\in D$. If $(\sigma_1^0,\sigma_2^0)\neq(0,0)$ then $x_2^0\neq 0$. Hence

$$Hess\Phi=\begin{vmatrix} 0 & 2x_2^0\\ 2x_2^0 & 2x_1^0\end{vmatrix}b^2(0,0)=-4(x_2^0)^2b^2(0,0)\neq 0.$$

Then we can use stationary phase method in two variables and obtain

$$|J^\chi|\leqslant\frac{c}{|\lambda|}$$

in a neighbourhood of σ^0.

Finally, let us consider the case when $(\sigma_1^0,\sigma_0^0)=(0,0)$. Since $(x_1,x_2)\in D$, then $x_2^0=0$ and $x_1^0\neq 0$. Thus $x_1^0\sim 1$.

$$\tilde{b}(2^{\frac{\kappa}{3}}x)x_1x_2^2+\sigma_1(x_2^{m_1}2^{-\frac{\kappa}{3}(m_2-1)}\tilde{\varphi}(2^{-\frac{\kappa}{3}}x_2))+\sigma_2x_2.$$

$$x_2=-\frac{\sigma_2}{2x_1}g(2^{-\frac{\kappa}{3}}x_1,2^{-\frac{\kappa}{3}(m_2-1)}\sigma_1)$$

Using stationary phase method in x_2, we obtain oscillatory integral with phase $g(0,0)\neq 0$.

$$\Phi_\kappa(\sigma,x_1):=\frac{\sigma_2^2}{4x_1}G\big(2^{-\frac{\kappa}{3}}x_1,2^{-\frac{\kappa}{3}(m_2-1)}\sigma_1\big)+\sigma_1x_1+\sigma_2x_1^{m_1}2^{-\frac{\kappa}{3}(m_2-1)}\tilde{\psi}(2^{-\frac{\kappa}{3}}x_1)$$

$x_1\sim 1$, $\sigma_2^2\sim\sigma_2 2^{-\frac{\kappa}{3}(m_2-1)}2^{\frac{\kappa}{3}(m_2-1)}\sigma_2\sim 1$.

Let us consider the following one-dimensional oscillatory integral

$$J_\kappa(\lambda,\sigma)=\frac{2^{\frac{\kappa}{6}}}{\lambda^{\frac{1}{2}}}\int\limits_{\mathbb{R}}e^{i\lambda 2^{-\kappa}\Phi_\kappa(\sigma,x_1)}a(x_1)dx_1$$

where $|\lambda 2^{-\kappa}|>1$.

We prove the following Lemma.

Lemma 4.1. *Let $x_1^0\neq 0$ be a fixed point. Then there exist a cut-off function χ supported in a neighborhood of x_1^0,k_0,c_0,c such that for any $\kappa>\kappa_0$ the following estimate holds true:*

$$|J_\kappa^\chi|\leqslant\frac{2^{\frac{\kappa}{3}}c}{\lambda^{\frac{1}{2}}}\left(\frac{1}{|\sigma|^{\frac{1}{3}}|\sigma_2|^{\frac{1}{3}}}+\frac{\chi_{|\sigma_1|\leqslant c\sigma_2^2}(\sigma_1,\sigma_2)}{|\sigma_2|^{\frac{1}{2}}|\sigma_1-c_0\sigma_2^2|^{\frac{1}{4}}}\right).$$

Proof of the Lemma follows from the results presented in [7]. It is easy to see that for any $p < 3$ $\Psi_0 \in L^p_{loc}(\mathbb{R}^2)$, where

$$\Psi_0(\sigma_1, \sigma_2) = \frac{1}{|\sigma|^{\frac{1}{3}}|\sigma_2|^{\frac{1}{3}}} + \frac{\chi_{|\sigma_1| \leqslant c\sigma_2^2}(\sigma_1, \sigma_2)}{|\sigma_2|^{\frac{1}{2}}|\sigma_1 - c_0\sigma_2^2|^{\frac{1}{4}}}.$$

Corollary. There exists κ_0 such that for any $\kappa > \kappa_0$ the following estimate holds true:

$$|J_\kappa(\lambda, \sigma)| \leqslant \frac{\Psi(\sigma_1, \sigma_2)2^{\frac{\kappa}{3}}}{|\lambda|^{\frac{1}{2}}},$$

where $\Psi \in L^{3-0}_{loc}(\mathbb{R})$. The following theorem holds true.

Theorem 4.1. *Let s be an analytic hypersurface such that it has D_∞ type of singularity at the origin. Then there exists a neighbourhood $U \subset \mathbb{R}^3$ such that for any $\Psi \in C_0^\infty(U)$, $M \in L^{3-0}(S^2)$.*

5. Summation of the Fourier transform of measures

Let S be an analytic hypersurface and

$$d\mu = \psi(x)dS.$$

We prove the following Theorem.

Theorem 5.1. *Let S be an analytic hypersurface. If S has D_∞ type of singularity at the origin then there exists a neighbourhood U of the origin such that for any $\Psi \in C_0^\infty(U)$ the inclusion $\hat{\mu} \in L^p(\mathbb{R}^3)$ holds for any $p > 3$.*

Proof. It is well known that there exists a neighbourhood U of the origin such that for any $\Psi \in C_0^\infty(U)$ the following estimate holds true (see [8])

$$|\mu(\hat{\xi})| \leqslant \frac{c}{(1 + |\xi|)^{\frac{1}{2}}}. \tag{3}$$

According to Theorem 4.1, there exists a function $\Psi(\omega) \in L^{3-0}(s^2)$ such that

$$|\hat{\mu}(r\omega)| \leqslant \frac{\Psi(\omega)}{(1 + r)}. \tag{4}$$

Let $p > 3$ be a fixed number. Let us take $q < 3$. We interpolate estimates (3) and (4) and obtain

$$|d\hat{\mu}(r\omega)| \leqslant \frac{c}{(1 + r)^{\frac{\alpha}{2}+\beta}}\Psi(\omega)^\beta.$$

If $p > 3$ one can choose α and β such that $p\Big(\dfrac{\alpha}{2} + \beta\Big) > 3$ and $p\beta < 3$.

For instance, we take a sufficiently small positive number $\delta > 0$ and set $\beta = \dfrac{3 - \delta}{p}$ and $\alpha = \dfrac{p - 3 + \delta}{p}$. Then it is easy to see that

$$\int_{\mathbb{R}^3} |d\hat{\mu(\xi)}|^p \xi \leqslant c \int_0^\infty \frac{r^2 dr}{(1 + z)^{(\frac{\alpha}{2}+\beta)p}} \int_{S^2} (\Psi(\omega))^{p\beta} d\omega < +\infty.$$

Theorem 5.1 is proved. □

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L^p-оценки преобразования Фурье поверхностных мер, сосредоточенных на гиперповерхностях с особенностью типа D_∞

Нигина А. Солеева
Самаркандский государственный университет
Самарканд, Узбекистан

Аннотация. В этой статье рассматриваются оценки преобразования Фурье мер, сосредоточенных на невыпуклых поверхностях трехмерного евклидова пространства. Мы найдем точный покозатель, для которого преобразование Фурье мер с этой степенью интегрируемо по трехмерному пространству. Этот результат дает ответ на вопрос, поставленный Эрдошем и Салмхофером.

Ключевые слова: преобразование Фурье, осцилляторный интеграл, поверхностная мера.