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Generator of Solutions for 2D Navier-Stokes Equations

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On the paper under consideration the investigation of Navier-Stokes equations for 2D viscous incompressible fluid flow is present. An analysis is based on the first integral of these equations. It is revealed that all ratios are reduced to one governing equation which can be considered as a generator of solutions.

Keywords: viscous incompressible fluid, differential equation, partial derivative, nonlinearity, integral, generator of solutions.

Introduction

The Navier-Stokes equations describe a motion of fluids and gases the presence of viscosity. These equations are used in areas where the effects of viscous friction play a significant role. Hydrology, meteorology, shipbuilding, tribology, oil production, pipeline transport, cardiology, are just some of areas where traditionally used the Navier-Stokes equations [1, 2]. However, despite the great practical significance, many issues associated with the Navier–Stokes equations are studied not enough and need further research [3, 4]. One of the main problems is the lack of a constructive method of solution. How to solve the equations of Navier-Stokes equations with complete preservation of nonlinear terms — this is an unsolved question which is still relevant for today.

In this paper the Navier-Stokes equations for case of $2D$ unsteady motion of a viscous incompressible fluid are under consideration and attempts to develop a comprehensive approach to solution construction. If external forces have the potential initial system of equations in dimensionless variables has the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial (p+\Phi)}{\partial x} + \frac{1}{\mathrm{Re}} \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial (p+\Phi)}{\partial y} + \frac{1}{\mathrm{Re}} \cdot \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

The major unknowns in the equations (1–3) are components of velocity u, v and pressure p; Φ — the potential of external forces, which is a given function; Re is the Reynolds number, representing a positive parameter,

$$Re = \frac{LU_0}{\nu},$$

where L — the scape of length, U_0 — the velocity scape, ν — the coefficient of kinematic viscosity.

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1. First integral

The suggested procedure of solution construction for equations (1–3) is based on the first integrals of these equations. General description and conclusion ratios, representing first integral for case of the $3D$ equations, given in works [5, 6]. In the particular case of $2D$ motion conclusion is presented in the work [7]. For the most simple case like this of the first integral is reduced to five equations as the next

$$p + \Phi + \frac{U^2}{2} + d + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial y}\right) = \alpha_1 + \beta_1, \tag{4}$$

$$u^2 - v^2 + \frac{2}{\mathrm{Re}}\left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial^2 \Psi_3}{\partial x^2} + \frac{\partial^2 \Psi_3}{\partial y^2} + \frac{\partial}{\partial t}\left(\frac{\partial \Psi_1}{\partial x} - \frac{\partial \Psi_2}{\partial y}\right) + 2\left(\alpha_1 - \beta_1\right), \tag{5}$$

$$uv - \frac{1}{\mathrm{Re}}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{\partial^2 \Psi_3}{\partial x \partial y} + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \Psi_2}{\partial x} + \frac{\partial \Psi_1}{\partial y}\right) - \frac{\partial}{\partial t}(\alpha_2 + \beta_2), \tag{6}$$

$$u = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y}\right) + \frac{\partial}{\partial y}\left(\alpha_2 + \delta\right), \tag{7}$$

$$v = -\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y}\right) + \frac{\partial}{\partial x}\left(\beta_2 - \delta\right). \tag{8}$$

In ratios (4–8) together with major unknowns u, v, p appear three new associate unknown Ψ_1, Ψ_2, Ψ_3. These values were not in the original equations (1–3), but they occur as a result of the first integration. Name as stream pseudo-functions were entered for these unknowns [5, 6, 7]. The meaning of other designations of the following: U is the velocity modulus, $U = \sqrt{u^2 + v^2}$, d is dissipative term of the calculated by formula

$$d = -\frac{1}{2}\left(\frac{\partial^2 \Psi_3}{\partial x^2} + \frac{\partial^2 \Psi_3}{\partial y^2}\right). \tag{9}$$

α_j, β_j, $(j = 1, 2)$, δ — an arbitrary functions in two variables arising over integration. Each of these five functions is not dependent on any of the three possible arguments, respectively, x, y, or t

$$\frac{\partial \alpha_j}{\partial x} = \frac{\partial \beta_j}{\partial y} = \frac{\partial \delta_j}{\partial t} = 0. \tag{10}$$

Therefore we face the system of five partial differential equations (4–8) with respect to six unknown u, v, p, Ψ_1, Ψ_2, Ψ_3. These equations have some advantages before the original ones. First advantage is that the order of the derivative on major unknown u, v, p is one less than the same for Navier–Stokes equations (1–3). Another advantage is that the equations (4–8) also allow conversion and consistent simplification.

2. Generator of solutions

Let's analyze the values appearing in equations (4–8) and do some conversion. An unknown p is present only in equation (4) with additively. This equation should be used to determine p at the final stage when the rest of the unknown u, v, Ψ_1, Ψ_2, Ψ_3 have already been found.

To determine the latter have a system of four equations (5–8). In these equations appear three associated unknown Ψ_1, Ψ_2, Ψ_3. Despite the similar signs, these unknown have different connotation. According to (9) unknown Ψ_3 determines the dissipative term d. For this reason, Ψ_3 aptly called the dissipative stream pseudo-function. While the Ψ_1 and Ψ_2, according to (7–8), determine the velocities u, v. These unknown logical to name as velocity stream pseudo-functions.

Unknowns Ψ_1, Ψ_2, Ψ_3 appear in the equations (5–8) differently. Unknown Ψ_3 is present in two equations from four, in only equations (5–6). These equations can be used to determine Ψ_3, when Ψ_1, Ψ_2, u, v have been found. If to unite the left parts of all the terms, not containing Ψ_3, these two equations can be represented as

$$f_1 = -\frac{\partial^2 \Psi_3}{\partial x^2} + \frac{\partial^2 \Psi_3}{\partial y^2}, \tag{11}$$

$$f_2 = -\frac{\partial^2 \Psi_3}{\partial x \partial y}, \tag{12}$$

where f_1, f_2 are some expressions containing Ψ_1, Ψ_2, u, v, α_j, β_j.

Two equations (11–12) can be solved for Ψ_3 only in the case when the left parts meet the the condition of consistency. This condition can be obtained as follows. Calculate the derivative with respect to x from (11) and the derivative with respect to y from (12). Folding the results, we arrive at the equation

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = -\frac{\partial^3 \Psi_3}{\partial x^3}. \tag{13}$$

Next we calculate the derivative with respect to y from (13) and the second derivative with respect to x from (12). Subtracting the results we arrive at the equation

$$\frac{\partial^2 f_1}{\partial x \partial y} = \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2}. \tag{14}$$

Equation (14) is necessary ("almost sufficient") condition of consistency. If this equation is satisfied, then an unknown Ψ_3 can be found. To do this, equation (12) twice in succession to integrate in x and y respectively. The result for the solution it is enough to allow two remaining equation of system (7–8) together with equation (14).

It is vital that the functions f_1 and f_2, appearing in equation (14), are determined only by the unknown u, v, Ψ_1, Ψ_2. Moreover, u, v are expressed through the Ψ_1, Ψ_2, according to (7–8). So, the unknown u, v you can exclude. Enough instead of u, v substitute in (14) right parts of (7–8). Therefore, we come to one equation with respect to Ψ_1, Ψ_2. The result of transform of this equation can be represented as

$$-u\Delta v + v\Delta u + \frac{1}{\mathrm{Re}}\left(-\frac{\partial \Delta u}{\partial y} + \frac{\partial \Delta v}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left(-\frac{\partial \Delta \Psi_1}{\partial y} + \frac{\partial \Delta \Psi_2}{\partial x}\right) = 0. \tag{15}$$

Here instead of u, v means of their expression through the Ψ_1, Ψ_2, according to (7–8). And Δ denotes the Laplace operator in the variables x and y.

So, to determine Ψ_1, Ψ_2 we have one equation (15). At this stage there is a whole range of possibilities. We can Ψ_1 set arbitrarily and by equation (15) allow Ψ_2. Can, on the contrary, set arbitrarily Ψ_2, and the unknown Ψ_1 to find out of (15). We can also impose some conditions on Ψ_1 and Ψ_2, and subject to the conditions both of the unknown to find from equation (15). In any case, both unknown Ψ_1 and Ψ_2 determined as solution of (15).

After that, we can find all the other interesting values. On the equations (7–8), there exist u, v. Then using (6) is defined Ψ_3. And finally, equation (4) with account of (9) is p.

As a result of one equation with two unknowns generates solutions of the $2D$ Navier-Stokes equations. This equation is (15) and it can be called a generator of solutions. Note that with this approach, all of the nonlinear terms of the equations is completely stored and we obtain the exact solution of the Navier-Stokes equations.

3. Implementation

We give a concrete example of implementation and consider how the work described above generating ratio.

We construct a class of solutions of the $2D$ Navier–Stokes equations, which holds equalities as the next $\Psi_1 = A(t)\exp(k_1x + l_1y)$, and $\Psi_2 = B(t)\exp(k_2x + l_2y)$, where $A(t)$, $B(t)$ are some functions in time, k_j, l_j — some constants.

Suppose, for simplicity, $\alpha_j = 0$, $\beta_j = 0$, $(j = 1, 2)$, $\delta = 0$. Calculating velocities, according to (7–8), we arrive at the expressions

$$u = \frac{1}{2}\left(-A(t)l_1^2 \exp(k_1x + l_1y) + B(t)k_2l_2 \exp(k_2x + l_2y)\right), \tag{16}$$

$$v = -\frac{1}{2}\left(-A(t)l_1k_1 \exp(k_1x + l_1y) + B(t)k_2^2 \exp(k_2x + l_2y)\right). \tag{17}$$

Further changes are the following. Substitute the expressions for Ψ_1, Ψ_2, u, v in (15). We obtain the equality to zero of a linear combination of five different functions: $\exp(2k_1x + 2l_1y)$, $\exp(2k_2x + 2l_2y)$, $\exp((k_1 + k_2)x + (l_1 + l_2)y)$, $\exp(k_1x + l_1y)$, $\exp(k_2x + l_2y)$.

To ensure the feasibility of equations (15), to the coefficients on each of these functions put equal to zero. The result is five ratios. First two of them are identical. Three other ones are as follows

$$k_2l_1(k_2l_1 - k_1l_2)(k_2^2 + l_2^2 - k_1^2 - l_1^2) = 0, \tag{18}$$

$$\frac{dA}{dt} = A \cdot \frac{k_1^2 + l_1^2}{\mathrm{Re}}, \tag{19}$$

$$\frac{dB}{dt} = B \cdot \frac{k_2^2 + l_2^2}{\mathrm{Re}}. \tag{20}$$

Expressions (19–20) are ordinal differential equations with respect to $A(t)$ and $B(t)$. Solutions for these equations are defined by expressions

$$A(t) = A(0)\exp(c_1t), \tag{21}$$

$$B(t) = B(0)\exp(c_2t), \tag{22}$$

where $A(0)$, $B(0)$ are some nonzero constants, while decay constants over time defined as

$$c_1 = \frac{k_1^2 + l_1^2}{\mathrm{Re}}, \tag{23}$$

$$c_2 = \frac{k_2^2 + l_2^2}{\mathrm{Re}}. \tag{24}$$

The ratio of (18) is the characteristic equation for the exponent indicators. Its solution leads to two main non-trivial possibilities

$$\frac{l_1}{k_1} = \frac{l_2}{k_2}, \tag{25}$$

$$l_1^2 + k_1^2 = l_2^2 + k_2^2. \tag{26}$$

For each specific task selection of roots is carried out differently. Let us consider in more detail the second possibility. We assume that k_j, l_j are non-zero and satisfy (26). Then decay constants are the same $c_1 = c_2 = c$. Let's find for this case all the remaining unknown. For velocities receive expression

$$u = \frac{1}{2}\left(-A(0)l_1^2 \exp(k_1x + l_1y + c \cdot t) + B(0)k_2l_2 \exp(k_2x + l_2y + ct)\right). \tag{27}$$

$$v = -\frac{1}{2}\left(-A(0)k_1 l_1 \exp(k_1 x + l_1 y + c\cdot t) + B(0)k_2^2 \exp(k_2 x + l_2 y + ct)\right). \quad (28)$$

For unknown Ψ_3 we have equation (6), which in light of the previous equation is transformed to

$$\frac{\partial^2 \Psi_3}{\partial x \partial y} = \frac{1}{4}(B(0)^2 k_2^3 l_2 \exp(2k_2 x + 2l_2 y + 2c\cdot t) + A(0)^2 l_1^3 k_1 \exp(2k_1 x + 2l_1 y + 2c\cdot t) -$$
$$- A(0)B(0)k_2 l_1 (k_1 l_2 + k_2 l_1)\exp((k_1+k_2)x + (l_1+l_2)y + 2c\cdot t)) +$$
$$+ \frac{1}{\mathrm{Re}}\left(A(0)l_1 k_1^2 \exp(k_1 x + l_1 y + ct) + B(0)k_2 l_2^2 \exp(k_2 x + l_2 y + ct)\right). \quad (29)$$

The consistent integration of this equation in x and y with zero additive functions leads to result

$$\Psi_3 = \frac{1}{4}(B(0)^2 \frac{k_2^2}{4} \exp(2k_2 x + 2l_2 y + 2c\cdot t) + A(0)^2 \frac{l_1^2}{4} \exp(2k_1 x + 2l_1 y + 2c\cdot t) -$$
$$- A(0)B(0)\frac{k_2 l_1 (k_1 l_2 + k_2 l_1)}{(k_1+k_2)(l_1+l_2)} \exp((k_1+k_2)x + (l_1+l_2)y + 2c\cdot t)) +$$
$$+ \frac{1}{\mathrm{Re}}(A(0)k_1 \exp(k_1 x + l_1 y + ct) + B(0)l_2 \exp(k_2 x + l_2 y + ct)). \quad (30)$$

Note that (30) satisfies equation (5). Direct substituting (30) into (5) leads to relation

$$2(k_1+k_2)(l_1+l_2)(l_1 l_2 - k_1 k_2) = (k_1 l_2 + l_1 k_2)((l_1+l_2)^2 - (k_1+k_2)^2), \quad (31)$$

which is obviously valid, due to (26).

Therefore, unknown Ψ_3 is defined. Now you should contact to (4). From this equation, taking into account (9), we find $p + \Phi$. Calculations lead to a result

$$p + \Phi = \frac{A(0)B(0)}{4}\frac{k_1 - k_2}{l_1 + l_2} k_2 l_1 \cdot (k_2 l_1 - k_1 l_2)\exp((k_1+k_2)x + (l_1+l_2)y + 2ct). \quad (32)$$

So, the major unknowns are defined. Constructed solution correspond to the wave motion of a viscous fluid. The most interesting case is when the solution fade over time. For this case decay constant c should be negative. According to formulas (23–24), this can realize if

$$k_j^2 + l_j^2 < 0. \quad (33)$$

This inequality can only be performed on complex values k_j, l_j.

Let's consider, for example, the simplest case, when k_j, l_j purely imaginary. Suppose $k_j = in_j$, $l_j = im_j$, where i is the imaginary unit, n_j, m_j are some real numbers. Then $k_j^2 + l_j^2 = -n_j^2 - m_j^2$ and inequality (33) is obviously valid. Substituting $k_j = in_j$, $l_j = im_j$ on (27–28), (32) and separating real and imaginary parts, we arrive at expressions as the next

$$u = \frac{1}{2}\left(A(0)m_1^2 \cos(n_1 x + m_1 y) - B(0)n_2 m_2 \cos(n_2 x + m_2 y)\right)\exp\left(\frac{-n_1^2 - m_1^2}{\mathrm{Re}}\cdot t\right), \quad (34)$$

$$v = -\frac{1}{2}\left(A(0)n_1 m_1 \cos(n_1 x + m_1 y) - B(0)n_2^2 \cos(n_2 x + m_2 y)\right)\exp\left(\frac{-n_2^2 - m_2^2}{\mathrm{Re}}\cdot t\right), \quad (35)$$

$$p + \Phi = \frac{A(0)B(0)}{4}\frac{(n_1 - n_2)}{(m_1 + m_2)} \cdot n_2 m_1 (n_2 m_1 - n_1 m_2) \times$$
$$\times \cos((n_1+n_2)x + (m_1+m_2)y) \cdot \exp\left(\frac{-2n_1^2 - 2m_1^2}{\mathrm{Re}}\cdot t\right). \quad (36)$$

Solution of the $2D$ Navier-Stokes equations constructed above corresponds to standing waves in deep water [2].

If in expressions (27–28),(32) values k_j, l_j believe common species complex numbers, $k_j = r_j + is_j$, $l_j = \xi_j + i\zeta_j$, then we obtain the solution corresponding to progressive waves in deep water. For this case decay constant is a complex number with negative real part $c = -\lambda^2 + i\omega$. On addition, should be performed ratios as the next

$$r_1^2 - s_1^2 + \xi_1^2 - \zeta_1^2 = r_2^2 - s_2^2 + \xi_2^2 - \zeta_2^2, \tag{37}$$

$$r_1 s_1 + \xi_1\zeta_1 = r_2 s_2 + \xi_2\zeta_2, \tag{38}$$

$$\lambda^2 = \frac{s_1^2 - r_1^2 + \zeta_1^2 - \xi_1^2}{\mathrm{Re}}, \tag{39}$$

$$\omega = \frac{2(r_1 s_1 + \xi_1\zeta_1)}{\mathrm{Re}}. \tag{40}$$

As a result of velocities are obtained in the form

$$\begin{aligned} u = \frac{1}{2}B(0)\,((r_2\xi_2 - s_2\xi_2)\cos(s_2x + \zeta_2y + \omega t) -&\\ -(r_2\zeta_2 + s_2\xi_2)\sin(s_2x + \zeta_2y + \omega t))\exp(r_2x + \xi_2y - \lambda^2 t) -&\\ -\frac{1}{2}A(0)\left((\xi_1^2 - \zeta_1^2)\cos(s_1x + \zeta_1y + \omega t) +\right.&\\ \left. +2\xi_1\zeta_1\sin(s_1x + \zeta_1y + \omega t)\exp(r_1x + \xi_1y - \lambda^2 t)\right),& \end{aligned} \tag{41}$$

$$\begin{aligned} v = -\frac{1}{2}B(0)\left((r_2^2 - s_2^2)\cos(s_2x + \zeta_2y + \omega t) -\right.&\\ -2r_2s_2\sin(s_2x + \zeta_2y + \omega t))\exp(r_2x + \zeta_2y - \lambda^2 t) +&\\ +\frac{1}{2}A(0)\,((r_1\xi_1 - s_1\zeta_1)\cos(s_1x + \zeta_1y + \omega t) -&\\ -(r_1\zeta_1 + s_1\xi_1)\sin(s_1x + \zeta_1y + \omega t))\exp(r_1x + \xi_1y - \lambda^2 t).& \end{aligned} \tag{42}$$

Here r_j, s_j, ξ_j, ζ_j are some real numbers satisfying two equations (37–38) and inequality $r_1^2 - s_1^2 + \xi_1^2 - \zeta_1^2 < 0$. An exact solutions of $2D$ Navier–Stokes equations constructed above are new.

Conclusion

Thus, for $2D$ Navier-Stokes equations the generator of solutions is constructed. It is equation (15). It represent the equation of the fifth order with respect to two unknowns. This equation opens way to construction of exact solutions. Allowing this equation we can consistently find all unknown.

This procedure leads to some conclusions about structure of solutions for the $2D$ Navier–Stokes equations. For anyone solution u, v, p all the relationships representing the first integral and all subsequent ratio discussed above are hold. So, the velocities must be of the type described by equations (7–8). Similarly, pressure p must be defined by equation (4). That is, without consideration of additive functions, p must be represented as the sum of four different in nature components.

Thus, each of the major unknown u, v, p must be presented with sum of components of a specific type and it general structure is clear.

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Генератор решений 2D-уравнений Навье-Стокса

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Рассматриваются уравнения Навье-Стокса для 2D-движений вязкой несжимаемой жидкости. Исследование основывается на первом интеграле этих уравнений. Показано, что существует одно определяющее соотношение, которое можно рассматривать как генератор решений.

Ключевые слова: вязкая несжимаемая жидкость, дифференциальное уравнение, частная производная, нелинейность, интеграл, генератор решений.