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Residue integrals and Waring formulas for algebraic and transcendent systems of equations

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Abstract—We discuss a system of algebraical and transcendental systems of equations of general form. We determine residue integrals over the cycles, connected with the system. We give formulas for their calculation and give multi-dimensional analogs of Waring formulas, i. e., connection between the coefficients of equations with power sums of the roots of system.

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INTRODUCTION

Earlier L.A. Aizenberg, A.P. Yuzhakov and A.K. Tsikhh have obtained formulas for power sums of ruuts of systems of non-linear algebraic equations in $\mathbb{C}^n$ on the base of many-dimensional logarithmic residue; see [1]–[3]. These formulas enable us to find the sums without calculation of the roots themselves, and to build a new method of investigation of systems of algebraic equations in $\mathbb{C}^n$. It is proposed by L.A. Aizenberg [1], and its development is continued in monographs [2], [4]. The main idea of the method is to find power sums of roots of a system in positive degrees, and to use either one-dimensional or many-dimensional Newton recurrent formulas [5]. Unlike the classical exclusion method, this method is less time consuming and does not increase the multiplicity of roots.

The base of the method is a formula [1], which is obtained by means of many-dimensional logarithmic residue for evaluation of sums of meanings of arbitrary polynomial at roots of given system of algebraic equations without calculation of the roots themselves.

As a rule, we cannot obtain formulas for the sums of roots of non-algebraic (transcedent) equations, because the set of the roots can be infinite, and series of their coordinates can be divergent. However, the non-algebraic system of equation arise, for instance, in the problems of chemical kinetics [6], [7]. Thus, the problem of the investigation of that systems is urgent.

The power sums of negative degrees of roots of various transcendent systems are studied in the papers [8]–[16]. These sums are calculated by means of residue integral over skeletons of polydisks with center at the origin. Note that this residue integral in general is not many-dimensional logarithmic residue, or the Grothendieck residue. There are cited formulas of residue integrals for various types of homogeneous systems of lower orders, and established their connections with power sums of roots of the system in negative degree.

More complicated systems are investigated in the works [14], [15]. Here the lower homogeneous parts allow expansion into product of linear factors, and the cycles of integration in the residue integrals, are determined by these factors.

The work [16] deals with the system arising in Zel'dovitch–Semenov model [6], [7] in chemical kinetics.

The subjects of the present paper are algebraic and transcendent systems of equations, where the lower homogeneous parts of functions form non-degenerated system of algebraic equations. We find formulas for the residue integrals, power sums of the roots in negative degree, and many-dimensional analogs of the Waring formula, i. e., the relations between the coefficients of the equations with the residue integrals.

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1. RESIDUE INTEGRALS

Let $f_1(z), \dots, f_n(z)$ be a system of functions, which are holomorphic in a neighborhood of the origin in many-dimensional complex space $\mathbb{C}^n$, $z = (z_1, \dots, z_n)$.

We expand functions $f_1(z), \dots, f_n(z)$ into the Taylor series in a neighborhood of the origin, and consider system of equations

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad j = 1, \dots, n, \tag{1}$$

where P_j is the lower homogeneous part of Taylor expansion of function $f_j(z)$. The degree of all monomials (in totality of variables) in P_j equals to m_j, $j = 1, \dots, n$. In function Q_j the degrees of all monomials are strictly greater than m_j.

The expansions of functions Q_j, P_j, $j = 1, \dots, n$, in a neighborhood of null into the Taylor series, which converge absolutely and uniformly in this neighborhood, have the form

$$Q_j(z) = \sum_{\|\alpha\| > m_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\beta\| = m_j} b^j_\beta z^\beta, \tag{3}$$

$j = 1, \dots, n$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-subscripts, i. e., α_j, β_j are non-negative integer numbers, $j = 1, \dots, n$, $\|\alpha\| = \alpha_1 + \dots + \alpha_n$, $\|\beta\| = \beta_1 + \dots + \beta_n$, monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot \ldots \cdot z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdot \ldots \cdot z_n^{\beta_n}$.

In what follows we assume that the system of polynomials $P_1(z), \dots, P_n(z)$ is *non-degenerated*, i. e., its common zero is the origin only.

We consider the following open set (special analytic polyhedron):

$$D_P(r_1, \dots, r_n) = \{z : |P_j(z)| < r_j,\ j = 1, \dots, n\},$$

where $r_1, \dots, r_n$ are positive values. Its *skeleton* is

$$\Gamma_P(r_1, \dots, r_n) = \Gamma_P(r) = \{z :\ |P_j(z)| = r_j,\ j = 1, \dots, n\}.$$

This set is of importance in the theory of many-dimensional residues (see, for instance, [2]).

Lemma 1 ([2]). *If system of polynomials $P_1, \dots, P_n$ is non-degenerated, then set $D_P(r_1, \dots, r_n)$ is relatively compact, and for almost all (n. e.) $r_1, \dots, r_n$ skeleton $\Gamma_P(r_1, \dots, r_n)$ is smooth compact cycle of dimension n.*

Lemma 2. *Let system of polynomials*

$$P_1(z), \dots, P_n(z) \tag{4}$$

be such that on the coordinate plane $\{z_1 = 0\}$ it contains a non-degenerated subsystem of $n-1$ polynomials depending on variables $z_2, \dots, z_n$. Then the skeleton of polyhedron $\Gamma_P(r_1, \dots, r_n)$ does not intersect coordinate plane $\{z_1 = 0\}$ for a. a. $r_1, \dots, r_n$.

Proof. We consider that non-degenerated subsystem has the form

$$P_2(0, z_2, \dots, z_n), \dots, P_n(0, z_2, \dots, z_n). \tag{5}$$

We use in the proof the concept and properties of resultant, which is introduced by A.K. Tsikh ([3], § 18, item 3) for superdetermined system of functions.

Denote $P' = (P_2, \dots, P_n)$, $z' = (z_2, \dots, z_n)$. By Lemma 1 the polyhedron

$$G'_{P'} = \{z' : |P_j(0, z')| < r_j, \quad j = 2, \dots, n\} \tag{6}$$

is relatively compact set for any $r_2 > 0, \dots, r_n > 0$. As system (5) consists of homogeneous polynomials, then the system defines mapping from $\mathbb{C}^{n-1}$ into $\mathbb{C}^{n-1}$, and it is proper. This means that $P' : G'_{P'} \to B'$ is analytic covering over polydisk $B' = \{|w_2| < r_2, \dots, |w_n| < r_n\}$: for every $w =$

$(w_2,\dots,w_n)\in B'$ pre-image $(P')^{-1}(w')$ consists of the same (taking into account their multiplicities) number of points $(z')^{(\nu)}(w')\in G'_{P'}$, $\nu=1,\dots,p$ (i. e., system of equations $P'(0,z')=w'$ has finite number of roots).

Define resultant of function $P_1(0,z')-w_1$ with regard system $P'(0,z')$ ([3], § 18, п. 3) as follows:

$$R(w)=R(w_1,w')=\prod_{\nu=1}^{p}[P_1(0,(z')^{(\nu)}(w'))-w_1]. \tag{7}$$

As shown in ([3], § 18, п. 3) for proper mappings, $R(w)$ is a polynomial. Clearly, $R(0)=0$, and $R(w)$ does not vanish identically. We denote A the set of nulls

$$A=\{w:R(w)=0\}.$$

We see from Definition (7) that $w\notin A$ if and only if the system of equations

$$P_1(0,z')=w_1,\dots,P_n(0,z')=w_n \tag{8}$$

has not roots.

As dimension of set A is $2n-2$, then the study of skeletons $w:|w_1|=r_1,\dots,|w_n|=r_n$, implies that for a. a. $r_1,\dots,r_n$ they do not intersect A. □

Corollary 1. If for every coordinate plane $\{z_j=0\}$ we can find a non-degenerate subsystem of order $(n-1)$ of system (5), then for a. a. $r_1,\dots,r_n$ skeleton $\Gamma_P(r_1,\dots,r_n)$ does not intersect the coordinate planes.

Note that for $n=2$ we do not need any restrictions on the system (besides non-degenerateness).

For sufficiently small r_i the cycles Γ_P are situated inside holomorphy domain of functions f_i. Therefore, the series

$$\sum_{\|\alpha\|>m_i}|a^i_\alpha|r_1^{\alpha_1}\cdots r_n^{\alpha_n}$$

converge for $i=1,2,\dots,n$. Then for sufficiently small $t>0$ we have on cycle $\Gamma_P(tr)=\Gamma_P(tr_1,tr_2,\dots,tr_n)$

$$|P_i(tr)|=\Big|\sum_{\|\beta\|=m_i}b^i_\beta(tr)^\beta\Big|=\sum_{\|\beta\|=m_i}t^{\|\beta\|}|b^i_\beta|r^\beta=t^{m_i}\sum_{\|\beta\|=m_i}|b^i_\beta|r^\beta,\quad i=1,\dots,n,$$

$$|Q_i(tr)|=\Big|\sum_{\|\alpha\|>m_i}a^i_\alpha(tr)^\alpha\Big|\leqslant\sum_{\|\alpha\|>m_i}t^{\|\alpha\|}|a^i_\alpha|r^\alpha=t^{m_i+1}\sum_{\|\alpha\|>m_i}|a^i_\alpha|r^\alpha t^{\|\alpha\|-(m_i+1)}.$$

Hence, for sufficiently small t on the cycle $\Gamma_P(tr)$ there are valid inequalities

$$|P_i(z)|>|Q_i(z)|,\quad i=1,2,\dots,n. \tag{9}$$

Thus,

$$f_i(z)\neq 0\ \text{ на }\ \Gamma_P(tr),\quad i=1,2,\dots,n.$$

In what follows we consider that $t=1$, i. e., inequality (9) holds on cycle $\Gamma_P(r_1,\dots,r_n)$.

We introduce concept of *residue integral* J_γ ([17]). Define

$$J_\gamma=\frac{1}{(2\pi\sqrt{-1})^n}\int_{\Gamma_P}\frac{1}{z^{\gamma+I}}\cdot\frac{df}{f}=$$
$$=\frac{1}{(2\pi\sqrt{-1})^n}\int_{\Gamma_P}\frac{1}{z_1^{\gamma_1+1}\cdot z_2^{\gamma_2+1}\cdot\ldots\cdot z_n^{\gamma_n+1}}\cdot\frac{df_1}{f_1}\wedge\frac{df_2}{f_2}\wedge\ldots\wedge\frac{df_n}{f_n}, \tag{10}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is multisubscript, and $I = (1, \ldots, 1)$. This residue integral is defined if $r_1, \ldots, r_n$ are chosen so that relation (9) holds, and cycle Γ_P does not intersect the coordinate planes (see Corollary 1). Note that this integral is neither many-dimensional logarithmic residue nor Grothendieck residue.

Theorem 1. *If system of polynomials* (4) *is non-degenerated and satisfies assumptions of Corollary* 1*, then for system of form* (1) *we have*

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_P} \frac{1}{z^{\gamma+I}} \cdot \frac{df}{f} = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\|\alpha\| \leqslant \|\gamma\|+n} (-1)^{\|\alpha\|} \int_{\Gamma_P} \left[\frac{\Delta \cdot Q^\alpha \cdot dz}{z^{\gamma+I} \cdot P^{\alpha+I}} \right],$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ *is multisubscript,* Δ *is Jacobian of system* (1), $Q^\alpha = Q_1^{\alpha_1} \cdot \ldots \cdot Q_n^{\alpha_n}$, $P^{\alpha+I} = P_1^{\alpha_1+1} \cdot \ldots \cdot P_n^{\alpha_n+1}$.

Proof. We use the formula of geometric progression and condition (**??**) on Γ_P, and obtain

$$\frac{1}{f_i} = \frac{1}{P_i + Q_i} = \sum_{s=0}^{\infty} (-1)^s \frac{Q_i^s}{P_i^{s+1}}.$$

Then

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\|\alpha\| \geqslant 0} (-1)^{\|\alpha\|} \int_{\Gamma_P} \frac{\Delta}{z_1^{\gamma_1+1} \cdot \ldots \cdot z_n^{\gamma_n+1}} \cdot \frac{Q_1^{\alpha_1} \cdot \ldots \cdot Q_n^{\alpha_n}}{P_1^{\alpha_1+1} \cdot \ldots \cdot P_n^{\alpha_n+1}} dz. \tag{11}$$

The series converges absolutely.

Let us show that the quantity of non-zero terms in this sum is finite. We calculate to this end the degrees (with regard to totality of the variables) of all monomials in the numerator and denominator of the integrand.

The degree (in the totality of the variables) $\deg \Delta$ of monomials, belonging to Δ, is not lesser that $m_1 + \cdots + m_n - n$. For the degree of monomials in Q^α we obtain bound

$$\deg Q_i^{\alpha_i} \geqslant (m_i + 1) \cdot \alpha_i, \quad i = 1, \ldots, n.$$

Therefore, the degree of the numerator is not lesser than

$$m_1 + \cdots + m_n - n + \sum_{s=1}^{n} \alpha_s (m_s + 1).$$

The degree of denominator is $\|\gamma\| + n + m_1(\alpha_1 + 1) + \cdots + m_n(\alpha_n + 1)$.

All terms of sum (11), where degree of numerator exceeds degree of denominator by n, vanish. Indeed, we can change variables in every integral from (11) by formula $z_j \to e^{\theta\sqrt{-1}} z_j$, $j = 1, \ldots, n$, $0 \leqslant \theta \leqslant 2\pi$. This change of variables keeps the integral and the cycles of integration because the polynomials $P_i(z)$ are homogeneous, and the integrand gets factor $e^{\theta\sqrt{-1}}$ with degree equaling to difference between degrees of the numerator (together with dz) and denominator.

Thus, here can be non-zero only the terms such that

$$m_1 + \cdots + m_n - n + \alpha_1(1 + m_1) + \alpha_2(1 + m_2) + \ldots + \alpha_n(1 + m_n) \leqslant \\ \leqslant \|\gamma\| + (\alpha_1 + 1)m_1 + \ldots + (\alpha_n + 1)m_n,$$

$$\|m\| + \|\alpha\| + \sum_{s=1}^{n} \alpha_s m_s \leqslant \|\gamma\| + n + \|m\| + \sum_{s=1}^{n} \alpha_s m_s,$$

i. e. $\|\alpha\| \leqslant \|\gamma\| + n$. □

Our further target is to show that the integrals in formula (11) can be expressed in terms of the Taylor coefficients of functions f_i, and connect them with the power sums of roots of system (1). We need to this end certain restrictions on functions $Q_i(z)$, $i = 1, \ldots, n$.

2. AUXILIARY RESULTS

We call to our mind certain concepts concerning space $\overline{\mathbb{C}}^n$, equaling to product of n Riemann spheres $\mathbb{CP}^1$, i. e., $\overline{\mathbb{C}}^n = \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$.

Let $z_j : w_j$ be homogeneous coordinates in j-th set of space $\overline{\mathbb{C}}^n$, and

$$F_j(z_1, w_1, \ldots, z_n, w_n) = 0, \quad j = 1, \ldots, n, \tag{12}$$

is a system of equations, consisting of polynomials F_j, which are homogeneous with regard to each pair of variables (z_k, w_k), $k = 1, \ldots, n$. We consider only such roots $(z_1, w_1, \ldots, z_n, w_n)$ of system (12) that

$$(z_k, w_k) \in \mathbb{C}^2 \setminus \{(0,0)\}, \quad k = 1, \ldots, n.$$

Any pair of roots of system (12) with proportional coordinates determines a unique root $(z_1 : w_1, \ldots, z_n : w_n)$ в $\overline{\mathbb{C}}^n$.

Let

$$a = (z_1^{(0)} : w_1^{(0)}, \ldots, z_n^{(0)} : w_n^{(0)})$$

be a root of system (12) such that $w_k^{(0)} \neq 0$ for any k. Then point $(z_1, 1, \ldots, z_n, 1)$ is a root of system

$$F_j(z_1, 1, \ldots, z_n, 1) = 0, \quad j = 1, \ldots, n,$$

in $\mathbb{C}^n$. If some coordinates $w_j^{(0)}$ of a root a vanish, this root corresponds to a root at infinity in $\overline{\mathbb{C}}^n$.

In what follows we consider that the systems of form (1) consist of polynomials $f_j(z)$. In order to find roots of that system in the point at infinity $\overline{\mathbb{C}}^n$, we first have to pass to homogeneous coordinates, i.e., to substitute ratios z_k/w_k instead of z_k, reject the obtained denominator, and obtain a system of form (12). We solve it, and find both customary roots and roots in the point at infinity for system (1).

We assume that system $P_1(z), \ldots, P_n(z)$ is non-degenerate and has no infinite roots in $\overline{\mathbb{C}}^n$.

Let us call to our mind that polynomials $Q_i(z)$, $i = 1, \ldots, n$, are of the form (2), i. e.,

$$Q_i(z) = \sum_{\|\alpha\| > m_i} a_\alpha^i z^\alpha.$$

We denote by $\operatorname{ord} Q_i$ the order of polynomial Q_i, i. e., the least of degrees of monomials in Q_i.

Suppose that every i-th equation from (1) satisfies conditions

$$\deg_{z_i} P_i < \operatorname{ord}_{z_i} Q_i, \quad \deg_{z_j} P_i \geqslant \operatorname{ord}_{z_j} Q_i, \quad j \neq i. \tag{13}$$

Here $\deg_{z_i} P(z)$ is degree of polynomial P with regard of variable z_i for fixed other variables, and $\operatorname{ord}_{z_i} Q$ is order of polynomial Q with regard to variable z_i for fixed other variables.

We have $\deg_{z,i} P_i = m_i$. Denote $\operatorname{ord} Q_i = s_i$, $\deg_{z_j} P_i = m_i^j$, $\operatorname{ord}_{z_j} Q_i = s_i^j$. Then $m_i < s_i$, $m_i^i < s_i^i$, $i = 1, \ldots, n$. In addition, $m_i^j \geqslant s_i^j$ for $j \neq i$. We do not exclude cases, where $\sum\limits_{j=1}^{n} m_i^j > m_i$.

We perform in all functions $f_i(z) = P_i(z) + Q_i(z)$, $i = 1, 2, \ldots, n$, change of variables $z_i = \dfrac{1}{w_i}$, $i = 1, \ldots, n$, assuming that all $w_i \neq 0$. As a result we obtain

$$P_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \sum_{\|\beta\| = m_i} b_\beta^i \frac{1}{w_1^{\beta_1}} \cdot \ldots \cdot \frac{1}{w_n^{\beta_n}} = \frac{1}{w_1^{m_i^1}} \cdot \ldots \cdot \frac{1}{w_n^{m_i^n}} \sum_{\|\beta| = m_i} b_\beta^i w_1^{m_i^1 - \beta_1} \cdot \ldots \cdot w_n^{m_i^n - \beta_n},$$

$$Q_i\left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right) = \sum_{\|\alpha\| > m_i} a_\alpha^i \frac{1}{w_1^{\alpha_1}} \cdot \ldots \cdot \frac{1}{w_n^{\alpha_n}} = \frac{1}{w_1^{s_i^1}} \cdot \ldots \cdot \frac{1}{w_n^{s_i^n}} \sum_{\|\alpha\| > m_i} a_\alpha^i w_1^{s_i^1 - \alpha_1} \cdot \ldots \cdot w_n^{s_i^n - \alpha_n}.$$

We have

$$f_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=P_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)+Q_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=$$
$$=\frac{1}{w_1^{m_i^1}\cdot\ldots\cdot w_i^{s_i^i}\cdot\ldots\cdot w_n^{m_i^n}}\cdot\left(\widetilde{P}_i(w)+\widetilde{Q}_i(w)\right),\quad (14)$$

where

$$\widetilde{P}_i(w_1,\dots,w_n)=w_1^{m_i^1}\cdot\ldots\cdot w_i^{s_i^i}\cdot\ldots\cdot w_n^{m_i^n}\cdot P_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=$$
$$=w_i^{s_i^i-m_i^i}\sum_{\|\beta\|=m_i}b_\beta^i w_1^{m_i^1-\beta_1}\cdot\ldots\cdot w_n^{m_i^n-\beta_n}=w_i^{s_i^i-m_i^i}\cdot\widetilde{\widetilde{P}}_i$$

and

$$\widetilde{\widetilde{P}}_i=\sum_{\|\beta\|=m_i}b_\beta^i w_1^{m_i^1-\beta_1}\cdot\ldots\cdot w_n^{m_i^n-\beta_n}$$

are homogeneous polynomials. Neither $w_1,\dots,$ nor w_n can be carried over the sum sign in $\widetilde{\widetilde{P}}_i$. Polynomials $\widetilde{Q}_i$ can be written in the form

$$\widetilde{Q}_i(w_1,\dots,w_n)=w_1^{m_i^1}\cdot\ldots\cdot w_i^{s_i^i}\cdot\ldots\cdot w_n^{m_i^n}\cdot Q_i\left(\frac{1}{w},\dots,\frac{1}{w_n}\right)=$$
$$=w_1^{m_i^1}\cdot\ldots\cdot w_i^{s_i^i}\cdot\ldots\cdot w_n^{m_i^n}\cdot\frac{1}{w_1^{s_i^1}}\cdot\ldots\cdot\frac{1}{w_n^{s_i^n}}\sum_{\|\alpha\|>m_i}a_\alpha^i w_1^{s_i^1-\alpha_1}\cdot\ldots\cdot w_n^{s_i^n-\alpha_n}=$$
$$=w_1^{m_i^1-s_i^1}\cdot\ldots\cdot[w_i]\cdot\ldots\cdot w_n^{m_i^n-s_i^n}\cdot\sum_{\|\alpha\|>m_i}a_\alpha^i w_1^{m_i^1-\alpha_1}\cdot\ldots\cdot w_n^{m_i^n-\alpha_n}.$$

Denote by $\widetilde{f}_i$ polynomials

$$\widetilde{f}_i(w)=\widetilde{P}_i(w)+\widetilde{Q}_i(w)=w_i^{s_i^i-m_i^i}\cdot\widetilde{\widetilde{P}}_i+\widetilde{Q}_i(w),\quad i=1,2,\dots,n.\quad (15)$$

We have

$$\deg\widetilde{P}_i>\operatorname{ord}\widetilde{Q}_i,\quad i=1,\dots,n.\quad (16)$$

Lemma 3. *System*

$$\widetilde{\widetilde{P}}_j(w)=0,\quad j=1,\dots,n,\quad (17)$$

have only null solution, i. e., it is non-degenerate.

Proof. We prove ad absurdum that the system has unique zero $w_1=w_2=\ldots=w_n=0$. We apply to this end the fact that before the change of variables system

$$P_j(z)=0,\quad j=1,\dots,n,\quad (18)$$

had unique zero $z_1=z_2=\ldots=z_n=0$.

Let system (17) have a root such that $w_j=0$ for some j. Then this root is root at infinity for system (18), but this is impossible by assumption.

Let system (17) have s solution $w_j=\alpha_j\neq 0,\quad j=1,\dots,n$, Then the inverse change of variables gives $z_j=\frac{1}{\alpha_j},\quad j=1,\dots,n$. and this is a root of system (18), what is impossible, too. □

Lemma 4. *Let us consider system*

$$\widetilde{P}_j(w) = 0, \quad j = 1, \dots, n. \tag{19}$$

If for any family of subscripts $i_1, \dots, i_k$, $i_1 < i_2 < \dots < i_k$, $k = 1, \dots, n$, the systems of equations

$$\widetilde{\widetilde{P}}_{j_1}(w) = 0, \ \dots, \ \widetilde{\widetilde{P}}_{j_{n-k}} = 0$$

for $w_{i_1} = 0, \dots, w_{i_k} = 0$ and for $j_p \neq i_q$ are non-degenerate, then system (19) *also is non-degenerate.*

Proof follows from the form of functions $\widetilde{P}_j(w)$ and Lemma 3.

Note that for $n = 2$ Lemmas 3, 4 are valid without any additional restrictions on $P_1(z_1, z_2)$ and $P_2(z_1, z_2)$.

3. SOME INTEGRAL FORMULAS

We consider system of equations (1) with polynomials $Q_i(z)$ satisfying conditions (13). Let system of functions (19) satisfy assumptions of Lemma 4 and Corollary 1.

Denote by $\Gamma_{\widetilde{P}} = \Gamma_{\widetilde{P}}(\varepsilon)$ cycle

$$\Gamma_{\widetilde{P}} = \{w \in \mathbb{C}^n : |\widetilde{P}_i| = \varepsilon_i, \quad \varepsilon_i > 0, \ i = 1, \dots, n\};$$

due to Corollary 1, it does not intersect coordinate plane for a. a. ε_i, $i = 1, \dots, n$.

We consider residue integral

$$\widetilde{J}_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w^{\gamma+I} \frac{df(1/w)}{f(1/w)},$$

where $w^{\gamma+I} = w_1^{\gamma_1+1} \cdot \ldots \cdot w_n^{\gamma_n+1}$, $f(1/w) = f_1(1/w_1, \dots, 1/w_n) \cdot \ldots \cdot f_n(1/w_1, \dots, 1/w_n)$, $df(1/w) = df_1(1/w_1, \dots, 1/w_n) \wedge \dots \wedge df_n(1/w_1, \dots, 1/w_n)$. As a matter of fact, $\widetilde{J}_\gamma$ is obtained from J_γ (10) by means of change of variables $z_j = 1/w_j$, $j = 1, \dots, n$, in its integrand, and change of the cycle of integration Γ_P by $\Gamma_{\widetilde{P}}$. But we have to prove that these integrals are equal.

Lemma 5. *For any multi-subscript γ we have*

$$\widetilde{J}_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \frac{d\widetilde{f}_1}{\widetilde{f}_1} \wedge \frac{d\widetilde{f}_2}{\widetilde{f}_2} \wedge \dots \wedge \frac{d\widetilde{f}_n}{\widetilde{f}_n}.$$

Proof. By virtue of formula (14)

$$\frac{df_j\left(\frac{1}{w_1}, \frac{1}{w_2}, \dots, \frac{1}{w_n}\right)}{f_j\left(\frac{1}{w_1}, \frac{1}{w_2}, \dots, \frac{1}{w_n}\right)} = \frac{d\widetilde{f}_j(w)}{\widetilde{f}_j(w)} - \sum_{k=1}^n c_k^j \cdot \frac{dw_k}{w_k},$$

where c_k^j are certain constants.

Let us call to our mind that $\widetilde{f}_i = \widetilde{P}_i + \widetilde{Q}_i = w_i^{s_i^i - m_i^i} \cdot \widetilde{\widetilde{P}}_i + \widetilde{Q}_i$. Then

$$\widetilde{J}_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w^{\gamma+I} \cdot \frac{df}{f} = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n} =$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w^{\gamma+I} \left(\frac{d\widetilde{f}_1(w)}{\widetilde{f}_1(w)} - \sum_{k=1}^n c_k^1 \cdot \frac{dw_k}{w_k}\right) \wedge \dots \wedge \left(\frac{d\widetilde{f}_n(w)}{\widetilde{f}_n(w)} - \sum_{k=1}^n c_k^n \cdot \frac{dw_k}{w_k}\right).$$

Integrals

$$\int_{\Gamma_{\widetilde{P}}} w^{\gamma+I}\frac{dw_1\wedge dw_2\wedge\dots\wedge dw_n}{w_1\cdot\ldots\cdot w_n}$$

vanish by virtue of the same reasons as the integrals in Theorem 1, because the degree of the denominator is n, and, consequently, it is lesser than the degree of the numerator.

We consider now integrals

$$\int_{\Gamma_{\widetilde{P}}} w^{\gamma+I}\frac{d\widetilde{f}_{i_1}(w)}{\widetilde{f}_{i_1}(w)}\wedge\dots\wedge\frac{d\widetilde{f}_{i_l}(w)}{\widetilde{f}_{i_l}(w)}\wedge\frac{dw_{j_1}}{w_{j_1}}\wedge\ldots\wedge\frac{dw_{j_{n-l}}}{w_{j_{n-l}}}, \tag{20}$$

for $0\leqslant l<n$ and sufficiently large ε_j. As

$$\frac{1}{\widetilde{f}_j(w)}=\sum_{p=0}^{\infty}\frac{(-1)^p\widetilde{Q}_j^p(w)}{\widetilde{P}_j^{p+1}},$$

then integrals (20) are absolutely convergent series of integrals

$$\int_{\Gamma_{\widetilde{P}}} w^{\gamma+I}\frac{\widetilde{Q}_1^{p_1}\cdot\widetilde{Q}_2^{p_2}\cdot\ldots\cdot\widetilde{Q}_{i_l}^{p}\cdot h(w)dw_1\wedge dw_2\wedge\dots\wedge dw_n}{\widetilde{P}_1^{p_1+1}\cdot\widetilde{P}_2^{p_2+1}\cdot\ldots\cdot\widetilde{P}_{i_l}^{p_l+1}\cdot w_{j_{i_1}}\cdot\ldots\cdot w_{j_{n-l}}},$$

where $h(w)$ is holomorphic function of w. All they vanish. Indeed, the denominator does not contain factors w_i, and some of $\widetilde{P}_j$. Therefore, cycle $\Gamma_{\widetilde{P}}$ is boundary of chain $S_j=\{|\widetilde{P}_1(w)|=\varepsilon_1,\dots,|\widetilde{P}_{j-1}(w)|=\varepsilon_{j-1},|\widetilde{P}_j(w)|<\varepsilon_j,|\widetilde{P}_{j+1}(w)|= \ =\varepsilon_{j+1},\dots,|\widetilde{P}_n(w)|=\varepsilon_n\}$, which is situated in the domain of holomorphy of the integrand. We conclude the proof by means of the Stokes formula. □

As the functions in system (15) satisfy inequalities (16), and system of functions $\widetilde{P}_1(w),\dots,\widetilde{P}_n(w)$ is non-degenerate, then by virtue of the well known Besout theorem the system of equations

$$\widetilde{f}_j(w)=0,\quad j=1,\dots,n, \tag{21}$$

has finite number of roots (counting any root as many times as its multiplicity), and this number equals to the product of degrees of polynomials $\widetilde{P}_j(w)$.

Lemma 6. *Let $w_{(1)},\dots,w_{(s)}$ be roots of system* (21) (*taking into account their multiplicities*), *where $w_{(j)}=(w_{j1},w_{j2},\dots,w_{jn})$, $j=1,2,\dots,s$. Then*

$$\widetilde{J}_\gamma=\sum_{j=1}^{s}w_{j1}^{\gamma_1+1}\cdot w_{j2}^{\gamma_2+1}\cdot\ldots\cdot w_{jn}^{\gamma_n+1}. \tag{22}$$

The assertion of Lemma follows from the formula of many-dimensional logarithmic residue and the Yuzhakov theorem on displaced frame ([2], § 4).

If some of $w_{(j)}$ have null coordinates, formula (22) does not contain them. If not all coordinates of root $w_{(j)}$ vanish, the point with coordinates $z_{jm}=\frac{1}{w_{jm}}$, $m=1,2,\dots,n$, is a root of system (1). The sum of multiplicities of that roots equals to $p\leqslant s$. They does not belong to coordinate planes.

Theorem 2. *There is valid equality*

$$\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdot\ldots\cdot z_{jn}^{\gamma_n+1}}=$$
$$=\sum_{\|\alpha\|\leqslant\|\gamma\|+n}(-1)^{\|\alpha\|}\int_{\Gamma_{\widetilde{P}}}\left[\widetilde{\Delta}\cdot w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot\ldots\cdot w_n^{\gamma_n+1}\cdot\frac{\widetilde{Q}_1^{\alpha_1}\cdot\widetilde{Q}_2^{\alpha_2}\cdot\ldots\cdot\widetilde{Q}_n^{\alpha_n}}{\widetilde{P}_1^{\alpha_1+1}\cdot\widetilde{P}_2^{\alpha_2+1}\cdot\ldots\cdot\widetilde{P}_n^{\alpha_n+1}}\right]dw,$$

where $\widetilde{\Delta}$ *is Jacobian of system* (15).

Proof follows from Lemma 6 and Theorem 1.

4. SEPARATING CYCLES

Consider cycle $\Gamma_P = \Gamma_P(\varepsilon_1, \dots, \varepsilon_n)$. We pass to new variables $z_j = \dfrac{1}{w_j}$, $j = 1, \dots, n$, and obtain

$$\widetilde{\widetilde{\Gamma}}_P = \{w : |\widetilde{\widetilde{P}}_j(w)| = |w_1^{m_j^1} w_2^{m_j^2} \cdot \ldots \cdot w_n^{m_j^n}| \cdot \varepsilon_j, \quad j = 1, \dots, n\}.$$

As system $\widetilde{\widetilde{P}}_1(w), \dots, \widetilde{\widetilde{P}}_n(w)$ is non-degenerate, then this cycle belongs to set

$$X = \{\mathbb{C}^n \setminus \{\{w_1 = 0\} \cup \{w_2 = 0\} \cup \dots \cup \{w_n = 0\}\}\},$$

which is domain of holomorphy (Stein variety in $\mathbb{C}^n$).

Define divisors $F_j = \{w : \widetilde{\widetilde{P}}_j(w) = 0\}$, and denote

$$F = F_1 \cup F_2 \cup \dots \cup F_n.$$

Lemma 7. *There is valid inclusion* $\widetilde{\widetilde{\Gamma}}_P \subset X \setminus F$.

Proof. We have

$$P_j\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right) = \frac{1}{w_1^{m_1^j}} \frac{1}{w_2^{m_1^j}} \cdot \ldots \cdot \frac{1}{w_n^{m_1^j}} \widetilde{\widetilde{P}}_j, \quad j = 1, \dots, n$$

(see section 1). Therefore, if и $\widetilde{\widetilde{P}}_j(w) = 0$, then at least of w_k vanishes. □

Lemma 8. *Cycle* $\widetilde{\widetilde{\Gamma}}_P$ *is separating with respect to set* $X \setminus F$.

Proof. By definition [2], [3] a cycle $\widetilde{\widetilde{\Gamma}}_P$ is *separating* with respect to set $X \setminus F$, if it lies in $X \setminus F$ and is homological to zero on set

$$X \setminus (F_1 \cup F_2 \cup \dots \cup F_{j-1} \cup F_{j+1} \cup \dots \cup F_n)$$

for any $j = 1, \dots, n$.

According Lemma 7 we have $\widetilde{\widetilde{\Gamma}}_P \subset X \setminus F$.

Let us show, for instance, that $\widetilde{\Gamma}_P$ is homological to zero on set $X \setminus (F_2 \cup \dots \cup F_n)$. It suffices to show to this end, that cycle $\widetilde{\Gamma}_P$ is boundary of a chain from $X \setminus (F_2 \cup \dots \cup F_n)$.

We consider chain

$$S = \{w : |\widetilde{\widetilde{P}}_1| < \varepsilon_1 |w_1^{m_1^1} \cdot \ldots \cdot w_n^{m_1^n}|,\ |\widetilde{\widetilde{P}}_2| = \varepsilon_2 |w_1^{m_2^1} \cdot \ldots \cdot w_2^{m_2^n}|, \dots, |\widetilde{\widetilde{P}}_n| = \varepsilon_n |w_1^{m_n^1} \cdot \ldots \cdot w_n^{m_n^n}|\}.$$

Clearly, S lies on set $X \setminus (F_2 \cup \dots \cup F_n)$, and its boundary coincides with $\widetilde{\Gamma}_P$.

We return to variables z, and obtain

$$S = \{z : |P_1(z)| < \varepsilon_1,\ |P_2(z)| = \varepsilon_2, \dots, |P_n(z)| = \varepsilon_n\}.$$

This set is relatively compact by virtue of Lemma 1. □

Hence, by the Tsikh theorem [2], [3] cycle $\widetilde{\widetilde{\Gamma}}_P$ lies in the group of homologies generated by cycle $\Gamma_{\widetilde{\widetilde{P}}} = \{w : |\widetilde{\widetilde{P}}_1| = \varepsilon_1, \dots, |\widetilde{\widetilde{P}}_n| = \varepsilon_n\}$. As cycle $\widetilde{\widetilde{\Gamma}}_P$ also generates this group, then there is valid

Theorem 3. *Cycles* $\widetilde{\widetilde{\Gamma}}_P$ *and* $\Gamma_{\widetilde{\widetilde{P}}}$ *are homological in* X.

As mapping $z_j = 1/w_j$, $j = 1, \dots, n$, is diffeomorphism of X, then there is valid

Corollary 2. Cycles Γ_P and $\Gamma_{\widetilde{P}}$ are homological.

Theorem 4. *Cycle*

$$\Gamma_P = \{z \in \mathbb{C}^n : |P_i| = r_i, \quad r_i > 0,\ i = 1, 2, \dots, n\}$$

is homological to cycle

$$\Gamma_{\widetilde{P}} = \{w \in \mathbb{C}^n : |\widetilde{P}_i| = \varepsilon_i, \quad \varepsilon_i > 0,\ i = 1, 2, \dots, n\}.$$

Proof. Let us consider cycle Γ_P. We perform there the change of variables $z_1 = 1/w_1, z_2 = 1/w_2, \dots, z_n = 1/w_n$, and obtain cycle $\widetilde{\widetilde{\Gamma}}_P$, i. e.,

$$\Big\{w : |\widetilde{\widetilde{P}}_1| = |w_1^{m_1^1} \cdot w_2^{m_1^2} \cdot \ldots \cdot w_n^{m_1^n}| \cdot r_1, |\widetilde{\widetilde{P}}_2| = |w_1^{m_2^1} \cdot w_2^{m_2^2} \cdot \ldots \cdot w_n^{m_2^n}| \cdot r_2, \dots, |\widetilde{\widetilde{P}}_n| = \\ = |w_1^{m_n^1} \cdot w_2^{m_n^2} \cdot \ldots \cdot w_n^{m_n^n}| \cdot r_n\Big\}.$$

Then we multiply each of these equations by $w_i^{s_i^i - m_i^i}$, $i = 1, 2, \dots, n$, and obtain

$$|w_1^{s_1^1 - m_1^1} \cdot \widetilde{\widetilde{P}}_1| = |w_1^{s_1^1} \cdot w_2^{m_1^2} \cdot \ldots \cdot w_n^{m_1^n}| \cdot r_1, |w_2^{s_2^2 - m_2^2} \cdot \widetilde{\widetilde{P}}_2| = |w_1^{m_2^1} \cdot w_2^{s_2^2} \cdot \ldots \cdot w_n^{m_2^n}| \cdot r_2, \dots,$$

$$|w_n^{s_n^n - m_n^n} \cdot \widetilde{\widetilde{P}}_n| = |w_1^{m_n^1} \cdot w_2^{s_n^2} \cdot \ldots \cdot w_n^{m_n^n}| \cdot r_n.$$

The left-hand side here is $\widetilde{P}_i$, $i = 1, 2, \dots, n$. Thus, we have equality

$$\widetilde{\widetilde{\Gamma}}_P = \{w : |\widetilde{P}_1| = |w_1^{s_1^1} \cdot w_2^{m_1^2} \cdot \ldots \cdot w_n^{m_1^n}| \cdot r_1, |\widetilde{P}_2| = |w_1^{m_2^1} \cdot w_2^{s_2^2} \cdot \ldots \cdot w_n^{m_2^n}| \cdot r_2, \dots, |\widetilde{P}_n| = \\ = |w_1^{m_n^1} \cdot w_2^{m_n^2} \cdot \ldots \cdot w_2^{s_n^n}| \cdot r_n\}.$$

The furthest proof of homology of cycles $\Gamma_{\widetilde{P}}$ и Γ_P repeats the previous considerations. □

5. ALGEBRAIC SYSTEMS OF EQUATIONS

We assume here that system (18) is non-degenerate, has not roots in the points at infinity in $\overline{\mathbb{C}}^n$ and satisfies assumptions of Corollary 1 and Lemma 4. For $n=2$ the furthest assertions hold only under the assumption that system (18) is non-degenerate.

Lemma 9. *There is valid equality*

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdot \ldots \cdot z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n} = \\ = \frac{(-1)^n}{(2\pi\sqrt{-1})^n} \int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \frac{d\widetilde{f}_1}{\widetilde{f}_1} \wedge \frac{d\widetilde{f}_2}{\widetilde{f}_2} \wedge \ldots \wedge \frac{d\widetilde{f}_n}{\widetilde{f}_n} = (-1)^n \widetilde{J}_\gamma.$$

Proof. We obtain the desired equality by means of change of variables $z_j = 1/w_j$, $j = 1, \dots, n$ in J_γ, and application of Theorem 4 and Lemma 5. The sign changes because this transformation changes orientation of space $\mathbb{C}^n$. □

In what follows we need generalized formula for transformation of the Grothendieck residue (see [18] and [4], ch. 2).

Theorem 5 ([18]). *Let $h(w)$ be holomorphic function, and polynomials $f_k(w)$ and $g_j(w)$, $j,k = 1,\dots,n$, satisfy relations*

$$g_j = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1,2,\dots,n,$$

where matrix $A = \|a_{jk}\|_{j,k=1}^{n}$ consists of polynomials. Let us consider cycles

$$\Gamma_f = \{w:\ |f_j(w)| = r_j,\ j = 1,\dots,n\}, \quad \Gamma_g = \{w:\ |g_j(z)| = r_j,\ j = 1,\dots,n\},$$

where all r_j are positive.

Then there is valid equality

$$\int_{\Gamma_f} h(w)\frac{dw}{f^\alpha} = \sum_{K,\ \sum\limits_{j=1}^{n} k_{sj} = \beta_s} \frac{\beta!}{\prod\limits_{s,j=1}^{n}(k_{sj})!} \int_{\Gamma_g} h(w) \frac{\det A \prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}\, dw}{g^\beta}, \tag{23}$$

where $\beta! = \beta_1!\beta_2!\dots\beta_n!$, $\beta = (\beta_1,\beta_2,\dots,\beta_n)$, and the summation is performed integer non-negative matrices $K = \|k_{sj}\|_{s,j=1}^{n}$ such that $\sum\limits_{s=1}^{n} k_{sj} = \alpha_j$, and $\beta_s = \sum\limits_{j=1}^{n} k_{js}$. Here $f^\alpha = f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^\beta = g_1^{\beta_1}\cdots g_n^{\beta_n}$.

Theorem 6. *There are valid formulas*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdot\ldots\cdot z_{jn}^{\gamma_n+1}} = \frac{(-1)^n}{(2\pi\sqrt{-1})^n}\int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot\ldots\cdot w_n^{\gamma_n+1}\cdot\frac{d\widetilde{f}_1}{\widetilde{f}_1}\wedge\frac{d\widetilde{f}_2}{\widetilde{f}_2}\wedge\ldots\wedge\frac{d\widetilde{f}_n}{\widetilde{f}_n} =$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n} \frac{(-1)^{n+\|\alpha\|}}{(2\pi\sqrt{-1})^n}\int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot\ldots\cdot w_n^{\gamma_n+1}\cdot\frac{\widetilde{\Delta}\cdot\widetilde{Q}_1^{\alpha_1}\cdot\widetilde{Q}_2^{\alpha_2}\cdot\ldots\cdot\widetilde{Q}_n^{\alpha_n}dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\widetilde{P}_1^{\alpha_1+1}\cdot\widetilde{P}_2^{\alpha_2+1}\cdot\ldots\cdot\widetilde{P}_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\|\leqslant\|\gamma\|+n} \frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n} k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\widetilde{\Delta}\cdot\det A\cdot Q^\alpha\prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n} w_j^{\beta_j N_j+\beta_j+N_j}}\right], \tag{24}$$

where $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$, and functional $\mathfrak{M}$ maps Laurent polynomials onto their free members.

Proof. As we have proved earlier (see Theorem 2),

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n}\int_{\Gamma_P}\frac{1}{z_1^{\gamma_1+1}\cdot z_2^{\gamma_2+1}\cdot\ldots\cdot z_n^{\gamma_n+1}}\cdot\frac{df_1}{f_1}\wedge\frac{df_2}{f_2}\wedge\ldots\wedge\frac{df_n}{f_n} =$$

$$= \frac{(-1)^n}{(2\pi\sqrt{-1})^n}\int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot\ldots\cdot w_n^{\gamma_n+1}\cdot\frac{d\widetilde{f}_1}{\widetilde{f}_1}\wedge\frac{d\widetilde{f}_2}{\widetilde{f}_2}\wedge\cdots\wedge\frac{d\widetilde{f}_n}{\widetilde{f}_n} =$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n} \frac{(-1)^{||\alpha||+n}}{(2\pi\sqrt{-1})^n}\int_{\Gamma_{\widetilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdot\ldots\cdot w_n^{\gamma_n+1}\cdot\frac{\widetilde{\Delta}\cdot\widetilde{Q}_1^{\alpha_1}\cdot\widetilde{Q}_2^{\alpha_2}\cdot\ldots\cdot\widetilde{Q}_n^{\alpha_n}dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\widetilde{P}_1^{\alpha_1+1}\cdot\widetilde{P}_2^{\alpha_2+1}\cdot\ldots\cdot\widetilde{P}_n^{\alpha_n+1}}.$$

The system of homogeneous polynomials $\widetilde{P}_1,\dots,\widetilde{P}_n$ has only one common zero at the origin. Hence, by the Hilbert theorem on zeros (see, for instance, [19]) there exist positive integer numbers

$N_1, \dots, N_n$, such that

$$w_j^{N_j+1} = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1, 2, \dots, n,$$

i. e., we can use the monomials $w_j^{N_j+1}$ as functions $g_j(w)$. By the Macaulay theore (see [20], and [3]), one can select these numbers N_j such that $N_j \leqslant k_1 + \dots + k_n - n$.

We use formula (23), the concept of functional $\mathfrak{M}$, and substitute instead of g_j the monomials $w_j^{N_j+1}$ in the last integrals. As a result, we obtain the last equality of the theorem. □

Note that Theorem 6 for $n = 2$ holds without any additive restrictions on system of polynomials $P_1,\ P_2$, besides its non-degeneracy.

Formula (24) is many-dimensional analogy of the Waring formula for algebraic systems of equations.

Note that paper [21] contains consideration of general algebraic systems of equations, and expansions of their solutions into hyper-geometric series. In addition, there are proved there are proved analogies of the Waring formulas for systems

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)} \cup \{0\}} x_\lambda^{(j)} y^\lambda = 0, \quad \lambda_1 + \dots + \lambda_n < m_j, \quad j = 1, \dots, n,$$

i. e., the higher homogeneous parts are monomials. We consider here other (more general) systems of equations with functions of the form (15).

6. TRANSCENDENT SYSTEMS OF EQUATIONS

Consider more general situation. Let functions f_j be meromorphic, and

$$f_j(z) = \frac{f_j^{(1)}(z)}{f_j^{(2)}(z)}, \quad j = 1, 2, \dots, n, \tag{25}$$

where $f_j^{(1)}(z)$ and $f_j^{(2)}(z)$ are entire functions in $\mathbb{C}^n$ expandable into uniformly convergent in $\mathbb{C}^n$ infinite products

$$f_j^{(1)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(1)}(z), \quad f_j^{(2)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(2)}(z),$$

$f_j^{(2)}(0) \neq 0$, and each factor is representable as $P_{j,s}(z) + Q_{j,s}(z)$, where functions $Q_{j,s}(z)$ satisfy conditions (13), $s = 1, 2, \dots$.

For any collection of subscripts $j_1, \dots, j_n$, where $j_1, \dots, j_n \in \mathbb{N}$, and any family of numbers $i_1, \dots, i_n$, where $i_1, \dots, i_n$ are equal either to 1 or to 2, systems of non-linear equations

$$f_{1,j_1}^{(i_1)}(z) = 0, \quad f_{2,j_2}^{(i_2)}(z) = 0, \ \dots, \ f_{n,j_n}^{(i_n)}(z) = 0 \tag{26}$$

have by virtue of Lemma 6 and Theorem 2 only finite number of roots outside the coordinate planes.

The set of roots of all that systems (situated outside the coordinate planes) is no more than countable. Therefore, we can enumerate them (taking into account their multiplicities):

$$z_{(1)},\ z_{(2)}, \dots, z_{(l)}, \dots .$$

Denote

$$\sigma_{\beta+I} = \sum_{l=1}^{\infty} \frac{\varepsilon_l}{z_{1(l)}^{\beta_1+1} \cdot z_{2(l)}^{\beta_2+1} \cdot \dots \cdot z_{n(l)}^{\beta_n+1}}. \tag{27}$$

Here $\beta_1, \ldots, \beta_n$ are, as above, non-negative integer numbers, and sign ε_l equals $+1$, if $z_{(l)}$ is root of system (26) containing even number of functions $f_{j_s}^{(2)}$, and -1 otherwise. The points $z_{(l)}$ are roots or singular points (poles) for system (26) consisting of functions of form (25). All functions f_j are holomorphic in a neighborhood of origin, and integrals J_β are defined for them, because these functions are of form (1).

There exists a relation between growth of null set of holomorphic function of finite order and order itself (see [22], ch. 3), similar to analogous connection for functions of single variable. But, generally speaking, in the case of several variables we have not any connection between orders of entire functions and growth of their common zeros.

Theorem 7. *The series* (27) *absolutely converges for system of equations with meromorphic functions* (25)*, and there are valid formulas*

$$J_\beta = (-1)^n \sigma_{\beta+I}.$$

Proof. As

$$d\frac{f_j^{(1)}(z)}{f_j^{(2)}(z)} = \frac{d\, f_j^{(1)}(z)}{f_j^{(1)}(z)} - \frac{d\, f_j^{(2)}(z)}{f_j^{(2)}(z)},$$

then

$$d\frac{f_1^{(1)}(z)}{f_1^{(2)}(z)} \wedge d\frac{f_2^{(1)}(z)}{f_2^{(2)}(z)} \wedge \ldots \wedge d\frac{f_n^{(1)}(z)}{f_n^{(2)}(z)} = \left(\frac{d\, f_1^{(1)}(z)}{f_1^{(1)}(z)} - \frac{d\, f_1^{(2)}(z)}{f_1^{(2)}(z)}\right) \wedge \left(\frac{d\, f_2^{(1)}(z)}{f_2^{(1)}(z)} - \frac{d\, f_2^{(2)}(z)}{f_2^{(2)}(z)}\right) \wedge$$
$$\wedge \ldots \wedge \left(\frac{d\, f_n^{(1)}(z)}{f_n^{(1)}(z)} - \frac{d\, f_n^{(2)}(z)}{f_n^{(2)}(z)}\right) = \sum (-1)^s \frac{d\, f_1^{(i_1)}(z)}{f_1^{(i_1)}(z)} \wedge \frac{d\, f_2^{(i_2)}(z)}{f_2^{(i_2)}(z)} \wedge \ldots \wedge \frac{d\, f_n^{(i_n)}(z)}{f_n^{(i_n)}(z)}, \quad (28)$$

where s is number of factors with $i_l = 2$, and the sum is taken over all possible collections of numbers $i_1, i_2, \ldots, i_n$ equaling to either 1 or 2.

The relations (28) imply that it suffices to prove the theorem for entire functions $f_j(z)$.

In this case

$$\frac{d\, f_j(z)}{f_j(z)} = \frac{d \prod\limits_{s=1}^{\infty} f_{js}(z)}{\prod\limits_{s=1}^{\infty} f_{js}(z)} = \sum_{s=1}^{\infty} \frac{d\, f_{js}(z)}{f_{js}(z)}.$$

The series under consideration uniformly converges on γ_r. Indeed, one can verify easily, that if a sequence of continuous functions f_m uniformly converges on a compact set K to a function f such that $f \neq 0$ on K, then beginning from certain number we have $f_m \neq 0$ on K, and sequence $1/f_m$ uniformly converges to $1/f$ on K. In just the same way one can verify, that term-by-term multiplication of uniformly convergent on a compact set sequences keeps the uniform convergence.

By assumption all products $\prod\limits_{s=1}^{\infty} f_{js}(z)$ uniformly converge to a non-zero on $\Gamma_f(r)$ function. Hence, the series

$$\sum_{s=1}^{\infty} \frac{d\, f_{js}(z)}{f_{js}(z)} = \frac{d \prod\limits_{s=1}^{\infty} f_{js}(z)}{\prod\limits_{s=1}^{\infty} f_{js}(z)} = \lim_{m\to\infty} \frac{d \prod\limits_{s=1}^{m} f_{js}}{\prod\limits_{s=1}^{m} f_{js}}$$

is uniformly convergent on $\Gamma_f(r)$. Thus, the integral J_β is determined, and equals to convergent series of integrals

$$\frac{1}{(2\pi i)^n} \int_{\Gamma_f(r)} \frac{1}{z^{\beta+I}} \cdot \frac{d\, f_{1s_1}(z)}{f_{1s_1}(z)} \wedge \frac{d\, f_{2s_2}(z)}{f_{2s_2}(z)} \wedge \ldots \wedge \frac{d\, f_{ns_n}(z)}{f_{ns_n}(z)},$$

where summation is performed with regard to cubes. Therefore, the series from $\sigma_{\beta+I}$ converges. As sum of this series does not depend on permutation of its terms, then it converges absolutely.

The desired formula for each of these integrals is proved (see Theorem 6). □

Theorem 7 is analog of the Waring formula for transcendent systems of equations.

The question on representation of functions in the form of product of entire functions is well studied on the complex plane. Its answer is given by classical Hadamard theorem. Analogs of the Hadamard theorem for functions of several variables are known (see [22], [23]), but, generally speaking, these analogs do not give representations of functions in the form of infinite products. A sufficient condition for expandability into infinite product is obtained in [24].

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