Functions with the one-dimensional holomorphic extension property

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This paper presents some results related to the holomorphic extension of functions f, defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$, into this domain. It's about a functions with the one-dimensional holomorphic extension property along the complex lines.

The first result related to our subject was received M.L.Agranovsky and R.E.Valsky in [1], who studied the functions with a one-dimensional holomorphic continuation property into a ball. The proof was based on the automorphism group properties of a sphere.

E.L.Stout in [2] used complex Radon transformation to generalize the Agranovsky and Valsky theorem for an arbitrary bounded domain with a smooth boundary. An alternative proof of the Stout theorem was obtained by A.M.Kytmanov in [3] by applying the Bochner-Martinelli integral. The idea of using the integral representations (Bochner - Martinelli, Cauchy - Fantappiè, logarithmic residue) has been useful in the study the functions with one-dimensional holomorphic continuation property (see review [4]).

The question of finding the different families of complex lines, sufficient for holomorphic extension was put in [5]. As shown in [6], the family of complex lines passing through a finite number of points also, generally speaking, is not sufficient. Thus, a simple analogy theorem of Hartogs should be not expected.

Various other families are given in [7] – [11]. In [12] – [16] it is shown that for holomorphic extension of continuous functions defined on the boundary of the ball, enough $n+1$ points inside the ball, do not lie on a complex hyperplane. By the author and A.Kytmanov this result was generalized on the n-circular [17] and circular domains.

Now we formulate some results about the different sufficient families of complex lines for holomorphic extension.

Let D be a bounded domain in $\mathbb{C}^n$ with a smooth boundary. Consider the complex line of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \zeta = z + bt, t \in \mathbb{C}\} = \{(\zeta_1, \dots \zeta_n) : \zeta_j = z_j + b_j t, j = 1, 2, \dots, n, t \in \mathbb{C}\},$$

where $z \in \mathbb{C}^n$, $b \in \mathbb{CP}^{n-1}$.

We will say that the function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the complex line* $l_{z,b}$, if $\partial D \cap l_{z,b} \neq \varnothing$ and there exists a function $F_{l_{z,b}}$ with the following properties:

1) $F_{l_{z,b}} \in \mathcal{C}(\overline{B} \cap l_{z,b})$,

2) $F_{l_{z,b}} = f$ on the set $\partial D \cap l_{z,b}$,

3) function $F_{l_{z,b}}$ is holomorphic at the interior (with respect to the topology of $l_{z,b}$) points of set $\overline{D} \cap l_{z,b}$.

Let Γ is a set in $\mathbb{C}^n$. Denote by $\mathfrak{L}_\Gamma$ the set of all complex lines $l_{z,b}$ such that $z \in \Gamma$, and $b \in \mathbb{CP}^{n-1}$, i.e., the set of all complex lines passing through $z \in \Gamma$.

We will say that a function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the family* $\mathfrak{L}_\Gamma$, if it has the one-dimensional holomorphic extension property along any complex line $l_{z,b} \in \mathfrak{L}_\Gamma$.

We will say *set* $\mathfrak{L}_\Gamma$ *sufficient for holomorphic extension*, if the function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along all complex lines of the family $\mathfrak{L}_\Gamma$, and then the function f extends holomorphically into D (i.e., f is a CR-function on ∂D).

In what follows we will need the definition of a domain with the Nevanlinna property [18]. Let $G \subset \mathbb{C}$ be a simply connected domain and $t = k(\tau)$ be a conformal mapping of the unit circle $\Delta = \{\tau : |\tau| < 1\}$ on G.

Domain G is *a domain with the Nevanlinna property*, if there are bounded holomorphic functions u and v in G such that almost everywhere on $S = \partial\Delta$, the equality

$$\bar{k}(\tau) = \frac{u(k(\tau))}{v(k(\tau))}$$

holds in terms of the angular boundary values. Essentially this means

$$\bar{t} = \frac{u(t)}{v(t)} \quad \text{on} \quad \partial G.$$

Give a characterization of a domains with the Nevanlinna property (Proposition 3.1 in [18]). Domain G is a domain with the Nevanlinna property if and only if $k(\tau)$ admits a holomorphic pseudocontinuation through S in $\overline{\mathbb{C}} \setminus \overline{\Delta}$, i.e., there are bounded holomorphic functions u_1 and v_1 such that the function $\tilde{k}(\tau) = \dfrac{u_1(\tau)}{v_1(\tau)}$ coincides almost everywhere with the function $k(\tau)$ on S.

The above definition and statement will be applied to bounded domains G with a boundary of class $\mathcal{C}^2$, therefor (due to the principle of correspondence of boundaries) the function $k(\tau)$ extends to $\overline{\Delta}$ as a function of class $\mathcal{C}^1(\overline{\Delta})$ and $\tilde{k}(\tau)$ extends to $\mathbb{C} \setminus \overline{\Delta}$ as a function of class $\mathcal{C}(\mathbb{C} \setminus \overline{\Delta})$. Therefor in whole the function $\bar{t} = \dfrac{u(\tau)}{v(\tau)}$ is a meromorphic function in $\mathbb{C}$. Various example of domains with the Nevanlinna property are given in [18]. For example, if ∂G is a real-analytic, then $k(\tau)$ is a rational function with no poles on the closure Δ.

In our further consideration we will need the domain G to possess *the strengthened Nevanlinna property*, that is the function $u_1(\tau) \neq 0$ in $\mathbb{C} \setminus \Delta$ and $\tilde{k}$ has at infinity zero of no more than first order. If $G = \Delta$ then $\bar{\tau} = \dfrac{1}{\tau}$ on $\partial\Delta$. Therefor meromorphic function $\dfrac{1}{\tau}$ has a zero of the first order at ∞.

For example, such domain will include domains for which $k(\tau)$ is a rational function with no poles on $\overline{\Delta}$ and no zeros in $\mathbb{C} \setminus \Delta$.

We will to say that the domain $D \subset \mathbb{C}^n$ possess *the strengthened Nevanlinna property in the point* $z \in D$ if the section $D \cap l_{z,b}$ possess the strengthened Nevanlinna property for any $b \in \mathbb{CP}^{n-1}$.

We formulate some results about the different families of complex lines sufficient for holomorphic extension.

Now we consider families of complex lines passing through a generic manifold. The real dimension of such a manifold is at least n. Recall that a smooth manifold Γ of class $\mathcal{C}^\infty$ is said to be *generic* if the complex linear span of the tangent space $T_z(\Gamma)$ coincides with $\mathbb{C}^n$ for each point $z \in \Gamma$. We denote the family of all complex lines intersecting Γ by $\mathfrak{L}_\Gamma$.

Theorem 1. *Let Γ be a germ of a generic manifold in $\mathbb{C}^n \setminus \overline{D}$ and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_\Gamma$, then the function f extends holomorphically into the domain D.*

Here we consider a generic manifold Γ lying in the domain D.

Theorem 2. *Let Γ be a germ of a generic manifold in D and a function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_\Gamma$ and the connected components of the intersection $D \cap l$ be domains with the strengthened Nevanlinna property, then the function f extends holomorphically into the domain D.*

Let Γ be the germ of a complex manifold of dimension $(n-1)$ in $\mathbb{C}^n$, which lies outside $\overline{D}$. Having done the shift and unitary transformation, we can assume that $0 \in \Gamma$, $0 \notin \overline{D}$ and that the complex hypersurface Γ in some neighborhood U of 0 has the form

$$\Gamma = \{z \in U : \ z_n = \varphi(z'), \ z' = (z_1, \dots, z_{n-1})\},$$

where φ is the holomorphic function in a neighborhood of zero in $\mathbb{C}^{n-1}$ and $\varphi(0) = 0$, $\dfrac{\partial \varphi}{\partial z_k}(0) = 0$, $k = 1, \dots, n-1$.

We will assume that there is a direction $b^0 \neq 0$ such that

$$\langle b^0, \bar{\zeta} \rangle \neq 0 \quad \text{for all} \quad \zeta \in \overline{D}. \tag{1}$$

Theorem 3. *Let D be a simply connected bounded domain and condition (1) be fulfilled and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_\Gamma$, then the function f extends holomorphically into the domain D.*

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be a unit ball in $\mathbb{C}^n$ centered at the origin and let $S = \partial B$ be the boundary of the ball.

We denote by $\mathfrak{A}$ the set of points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \dots, n+1$, which do not lie on the complex hyperplane in $\mathbb{C}^n$.

Theorem 4. *Let a function $f \in \mathcal{C}(S)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function f extends holomorphically into the ball B.*

This theorem was proved for circular domains with the strengthened Nevanlinna property.

Theorem 5. *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$ and possess the strengthened Nevanlinna property in the points from the set $\mathfrak{A}$ and a function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function f extends holomorphically into the domain D.*

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