Journal of Siberian Federal University. Mathematics & Physics 2019, 0(0), 000–000

УДК 517.955

# On Carleman-type formulas for solutions to the heat equation

**Ilya A. Kurilenko***
**Alexander A. Shlapunov**†
Institute of Mathematics,
Siberian Federal University,
Svobodny 79, 660041 Krasnoyarsk,
Russia

Received 19.02.2019, received in revised form 10.02.2012, accepted 20.04.2012

*We apply the method of integral representations to study the ill-posed Cauchy problem for the heat equation. More precisely we recover a function, satisfying the heat equation in a cylindrical domain, via its values and the values of its normal derivative on a given part of the lateral surface of the cylinder. We prove that the problem is ill-posed in the natural (anisotropic) spaces (Sobolev and Hölder spaces, etc). Finally, we obtain a uniqueness theorem for the problem and a criterion of its solvability and a Carleman-type formula for its solution.*

*Keywords: The heat equation, ill-posed problems, integral representation method, Carleman formulas.*

The integral representation method is the core of investigations of the ill-posed problems for partial differential equations, see [1, 2, 3]. Lev Aizenberg [4, 5] noted that the Cauchy problem for the Cauchy–Riemann equations is closely related to the problem of analytic continuation even if its entries are not analytic. He found principal ingredients, leading to the construction of integral formulas for its solution (Carleman formulas): a proper integral formula recovering the function via the data on the whole boundary, the uniqueness theorem and an effective tool, providing the analytic continuation from a domain to a larger one. This method was successfully used in the framework of Hilbert space methods to investigate the Cauchy problem for general elliptic systems of partial differential equations, see [6, 7, 8], and even to elliptic complexes of differential operators, see [9]. It provided both a solvability criterion and formulas for exact and approximate solutions. Recently, the scheme of using the concept of analytic continuation was adopted to obtain a solvability criterion for the ill-posed Cauchy problem in Hölder spaces for some parabolic operators as well, see [10, 11].

In this paper we concentrated our efforts in the construction of Carleman-type formulas for solutions to the heat equation. Namely, we recover a function satisfying the heat equation in a cylindrical domain via its values and the values of its normal derivative on a given part of the lateral surface of the cylinder. The principal difference with [10, 11] is that we use anisotropic Hölder and Sobolev spaces and we succeed to involve Hilbert space methods, too.

## 1. Preliminaries

Let $\Omega$ be a bounded domain in $n$-dimensional linear space $\mathbb{R}^n$ with the coordinates $x = (x_1, \dots, x_n)$. As usual we denote by $\overline{\Omega}$ the closure of $\Omega$, and we denote by $\partial\Omega$ its boundary. In the sequel we assume that $\partial\Omega$ is piece-wise smooth. We denote by $\Omega_T$ the bounded open cylinder $\Omega \times (0, T)$ in $\mathbb{R}^{n+1}$ with a positive altitude $T$. Let also $\Gamma \subset \partial\Omega$ be a non empty connected relatively open subset of $\partial\Omega$. Then $\Gamma_T = \Gamma \times (0,\ T)$ and $\overline{\Gamma_T} = \overline{\Gamma} \times [0,\ T]$.

*ilyakurq@gmail.com
†ashlapunov@sfu-kras.ru

We consider functions on subsets in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. As usual, for $s \in \mathbb{Z}_+$ we denote by $C^s(\Omega)$ the space of all $s$ times continuously differentiable functions in $\Omega$. Next, for a (relatively open) set $S \subset \partial\Omega$ denote by $C^s(\Omega \cup S)$ the set of such functions from the space $C^s(\Omega)$ that all their derivatives up to order $s$ can be extended continuously onto $\Omega \cup S$. The standard topology of these metrizable spaces induces the uniform convergence on compact subsets in $\Omega \cup S$ together with all the partial derivatives up to order $s$ (the case $S = \emptyset$ is included). We will also use the so-called Hölder spaces (cf., [12, Ch.1, §1]. Recall that a function $u$, defined on a set $M \in \mathbb{R}^n$, is called *Hölder continuous with a power* $0 < \lambda < 1$ on $M$, if there is such a constant $C > 0$ that

$$< u >_{\lambda,M} = \sup_{x,y \in M,\, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \leq C.$$

Let $C^{s,\lambda}(\Omega \cup S)$ stand for the set of such Hölder continuous functions that all their partial derivatives up to order $s$ are also Hölder continuous functions with a power $\lambda$ over each compact subset of $\Omega \cup S$. Clearly, $C^{s,\lambda}(\Omega \cup \partial\Omega) = C^{s,\lambda}(\overline{\Omega})$ is a Banach space with the norm,

$$\|u\|_{C^{s,\lambda}(\overline{\Omega})} = \sum_{|\alpha| \leq s} \left( \max_{x \in \overline{\Omega}} |\partial^\alpha u(x)| + < \partial^\alpha u >_{\lambda,\overline{\Omega}} \right)$$

see, for instance, [12, 13, 14]. In general, the space $C^{s,\lambda}(\Omega \cup S)$ again can be treated as a metrizable space, generated by a system of seminorms $p^{s,\lambda}_m(u) = \|u\|_{C^{s,\lambda}(\overline{\Omega}_m)}$ with a suitable exhaustion $\{\Omega_m\}_{m\in\mathbb{N}}$ of the set $\Omega \cup S$. By the definition, the space $C^{s,\lambda}(\Omega \cup S)$ is continuously embedded into $C^{s',\lambda'}(\Omega \cup S)$ if $s + \lambda \geq s' + \lambda'$, $\lambda, \lambda' \in [0,1)$.

To investigate the heat equation we need also the anisotropic (parabolic) spaces, see [12, Ch. 1], [14, Ch. 8]. For this purpose, let $C^{2s,s}(\Omega_T)$ stand for the set of all continuous functions $u$ in $\Omega_T$, having in $\Omega_T$ the continuous partial derivatives $\partial_t^j \partial_x^\alpha u$ with all multi-indexes $(\alpha, j) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ satisfying $|\alpha| + 2j \leq 2s$ where, as usual, $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, we denote by $C^{2s+k,s}(\Omega_T)$ the set of continuous functions in $\Omega_T$, such that all partial derivatives $\partial^\beta u$ belong to $C^{2s,s}(\Omega_T)$ if $\beta \in \mathbb{Z}_+^n$ satisfies $|\beta| \leq k$, $k \in \mathbb{Z}_+$. Of course, it is natural to agree that $C^{2s+0,s}(\Omega_T) = C^{2s,s}(\Omega_T)$, $C^{0,0}(\Omega_T) = C(\Omega_T)$ and $C^0(\Omega) = C(\Omega)$. We also denote by $C^{2s+k,s}((\Omega \cup S)_T)$ the set of such functions $u$ from the space $C^{2s+k,s}(\Omega_T)$ that their partial derivatives $\partial_t^j \partial_x^{\alpha+\beta} u$, $2j + |\alpha| \leq 2s$, $|\beta| \leq k$, can be extended continuously onto $(\Omega \cup S)_T$. The standard topology of these metrizable space induces the uniform convergence on compact subsets of $(\Omega \cup S)_T$ together with all partial derivatives used in its definition (the case $S = \emptyset$ is included).

We will also use the so-called anisotropic Hölder spaces (cf., [12, Ch. 1], [14, Ch. 8]). Recall that a function $u(x,t)$, defined on a set $M_T = M \times [0,T] \in \mathbb{R}^{n+1}$, is called *anisotropic Hölder continuous with a power* $0 < \lambda < 1$ on $M_T$, if there is such a constant $C > 0$ that

$$< u >_{\lambda,M_T} = \sup_{t \in [0,T]} < u(\cdot,t) >_{\lambda,M} + \sup_{x \in M} < u(x,\cdot) >_{\lambda/2,[0,T]} \leq C.$$

Let $C^{2s+k,s,\lambda,\lambda/2}((\Omega \cup S)_T)$ stand for the set of anisotropic Hölder continuous functions with a power $\lambda$ over each compact subset of $(\Omega \cup S)_T$ together with all partial derivatives $\partial_x^{\alpha+\beta} \partial_t^j u$ where $|\beta| \leq k$, $|\alpha| + 2j \leq 2s$. Clearly, $C^{2s+k,s,\lambda,\lambda/2}(\overline{\Omega_T})$ is a Banach space with the norm,

$$\|u\|_{C^{2s+k,s,\lambda,\lambda/2}(\overline{\Omega})} = \sum_{|\beta| \leq k} \sum_{|\alpha|+2j \leq 2s} \left( \max_{(x,t) \in \overline{\Omega_T}} |\partial_x^{\alpha+\beta} \partial_t^j u(x,t)| + < \partial^{\alpha+\beta} \partial^j u >_{\lambda,\overline{\Omega_T}} \right)$$

see, for instance, [14, Ch. 8]. In general, the space $C^{2s+k,s,\lambda,\lambda/2}((\Omega \cup S)_T)$ can be treated again as a metrizable space, generated by a system of seminorms $p^{s,\lambda,k}_{m,T}(u) = \|u\|_{C^{2s+k,s,\lambda,\lambda/2}(\overline{\Omega_{m,T}})}$ with a suitable exhaustion $\{\Omega_m\}_{m\in\mathbb{N}}$ of the set $\Omega \cup S$. Obviously, the space $C^{2s+k,s,\lambda,\lambda/2}((\Omega \cup S)_T)$

is continuously embedded into $C^{2s'+k',s',\lambda',\lambda'/2}((\Omega\cup S)_T)$ if $s+\lambda\geq s'+\lambda'$, $\lambda,\lambda'\in[0,1)$ and $k\geq k'$.

In order to invoke the Hilbert space approach, we need anisotropic (parabolic) Sobolev spaces $H^{2s,s}(\Omega_T)$, $s\in\mathbb{Z}_+$, see, [12, 15], i.e. the set of all measurable functions $u$ in $\Omega_T$ such that all generalized partial derivatives $\partial_t^j\partial_x^\alpha u$ with all multi-indexes $(\alpha,j)\in\mathbb{Z}_+^n\times\mathbb{Z}_+$ satisfying $|\alpha|+2j\leq 2s$, belong to the Lebesgue class $L^2(\Omega_T)$. This is a Hilbert space with the inner product

$$(u,v)_{H^{2s,s}(\Omega_T)}=\sum_{|\alpha|+2j\leq 2s}\int_{\Omega_T}\partial_t^j\partial_x^\alpha v(x,t)\,\partial_t^j\partial_x^\alpha u(x,t)dxdt. \tag{1}$$

We also may define $H^{2s,s}(\Omega_T)$ as the completion of the space $C^{2s,s}(\overline{\Omega_T})$ with respect to the norm $\|\cdot\|_{H^{2s,s}(\Omega_T)}$ generated by the inner product (1). For $s=0$ we have $H^{0,0}(\Omega_T)=L^2(\Omega_T)$.

Let now $\Delta_n=\sum\limits_{j=1}^n\partial^2_{x_j,x_j}$ be the Laplace operator in $\mathbb{R}^n$ and let $\mathcal{L}=\partial_t-\mu\Delta_n$ stand for the heat operator in $\mathbb{R}^{n+1}$. This operator plays essential role in the contemporary natural science, see, for instance, [17]. Now let $\partial_\nu=\sum\limits_{j=1}^n\nu_j\partial_{x_j}$ denote the derivative at the direction of the exterior unit normal vector $\nu=(\nu_1,...,\nu_n)$ to the surface $\partial\Omega$. As $\partial\Omega$ is piece-wise smooth, the normal vector $\nu$ is defined almost everywhere on $\partial\Omega$. It is known that the standard initial boundary value problem for the heat operator consists of the recovering of the function $u$ over the cylinder $\Omega_T$ satisfying the heat equation

$$\mathcal{L}u=f \text{ in } \Omega_T, \tag{2}$$

the boundary and initial conditions

$$a(x,t)u(x,t)+b(x,t)\partial_\nu u(x,t)=0 \text{ on } \partial\Omega_T,\ u(x,0)=u_0(x) \text{ on } \overline{\Omega}. \tag{3}$$

for fixed real-valued functions $a$ and $b$ and given data $f$ and $u_0$. Problem (2), (3) is well-posed over the scales of anisotropic Hölder and Sobolev spaces, see, for instance, [12, 13, 14, 15, 16, 17, 18]. Instead, we consider the Cauchy problem in the cylinder $\Omega_T$ in the sense of the Cauchy-Kowalevski Theorem with respect to the space variables, cf. [19].

**Problem 1.** *Let functions $u_1\in C^{1,0}(\overline{\Gamma_T})$, $u_2\in C(\overline{\Gamma_T})$ and $f\in C(\overline{\Omega_T})$ be given. Find a function $u\in C^{2,1}(\Omega_T)\cap C^{1,0}((\Omega\cup\Gamma)_T)$ satisfying the heat equation* (2) *and boundary conditions*

$$u(x,t)=u_1(x,\ t) \text{ on } \overline{\Gamma_T},\quad \partial_\nu u(x,t)=u_2(x,\ t) \text{ on } \overline{\Gamma_T}. \tag{4}$$

The motivation of Problem 1 is rather transparent. It describes the situation where for some reasons at each time $t\in[0,T]$ only part $\overline{\Gamma}$ of the boundary of the "body" $\overline{\Omega}$ is available for measurements.

**Example 1.** If $n>1$ then the famous Hadamard example for the Cauchy problem for the Laplace operator is fit to demonstrate the absence of the continuity of the solution with respect to the data in all reasonable standard spaces (Hölder spaces, Sobolev spaces, etc., see [19]). For instance, denote by $Q_{n,R}$ the $n$-dimensional cube $\{0<x_m<R,\,1\leqslant m\leqslant n\}$ and take the cube $Q_{n,1}$ as the base $\Omega$ of the cylinder $\Omega_T$. Let $\Gamma$ be the face $\{x_n=0\}$ of the cube $Q_{n,1}$ and $\Gamma_T=Q_{n-1,1}\times(0,T)$. Then, for each $a_j\in\mathbb{R}$, $j\in\mathbb{N}$, the function $u_j(x,t)=a_j\cos(jx_1)\cosh(jx_n)$ is a solution to problem (2), (4), with the data

$$f_j=0,\ u_{1,j}=a_j\cos(jx_1),\ u_{2,j}=0.$$

If $\lim_{j\to+\infty}j^{2s+k}a_j=0$ then for each $k\in\mathbb{Z}_+$ and each $\lambda\in[0,1]$ we have

$$f_j\underset{j\to\infty}{\longrightarrow}0 \text{ in } C^\infty(\overline{\Omega_T}),\ u_{1,j}\underset{j\to\infty}{\longrightarrow}0 \text{ in } C^{2s+k,s,\lambda,\lambda/2}(\overline{\Gamma_T}),\ u_{2,j}\underset{j\to\infty}{\longrightarrow}0 \text{ in } C^{2s+k-1,s,\lambda,\lambda/2}(\overline{\Gamma_T}).$$

On the other hand, the sequence $|u_j(x,\ t)|$ is unbounded for each $(x,t) \in \Omega_T$.

If $n = 1$ then we could not directly provide a corresponding example but we reduce the problem to an ill-posed one in Theorem 2 below. However, Problem 1 become ill-posed if we additionally impose rather mild restrictions on the growth of its solutions in $\overline{\Omega}_T$, see [10]. Indeed, consider the sequence of smooth solutions $u_j(x,\ t) = a_j\, e^{\mu j^2 (t-T) + j x_n}$ to problem (2), (4), with the data

$$f_j = 0,\ u_{1,j} = a_j e^{\mu j^2 (t-T)},\ u_{2,j} = j a_j e^{\mu j^2 (t-T)}.$$

If $\lim_{j\to+\infty} j^{2(s+1)} a_j = 0$ then for each $k \in \mathbb{Z}_+$ and each $\lambda \in [0,1]$ we have

$$f_j \underset{j\to\infty}{\longrightarrow} 0 \text{ in } C^\infty(\overline{\Omega_T}),\ u_{1,j} \underset{j\to\infty}{\longrightarrow} 0 \text{ in } C^{2s+k,s,\lambda,\lambda/2}(\overline{\Gamma_T}),\ u_{2,j} \underset{j\to\infty}{\longrightarrow} 0 \text{ in } C^{2s+k-1,s,\lambda,\lambda/2}(\overline{\Gamma_T}).$$

On the other hand, if we choose $a_j = j^{-2(s+1)-1}$ then for any $q \geq 1$ we have

$$\lim_{j\to+\infty} \|u_j\|_{L^q(Q_{n,1}\times[0,T])} = +\infty.$$

## 2. Solvability conditions and Carleman formula

If the surface $\Gamma$ and the data of the problem are real analytic then the Cauchy-Kowalevski Theorem implies that problem (2), (4) has at most one solution in the class of (even formal) power series. However the theorem does not imply the existence of solutions to Problem 1 because it grants the solution in a small neighbourhood of the (analytic) surface $\Gamma_T$ only (but not in a given domain $\Omega_T$!). We begin this section proving that Problem 1 can not have more than one solution in the spaces of differentiable (non-analytic) functions.

To investigate Problem 1, we use an integral representation constructed with the use the fundamental solution to heat operator $\mathcal{L}$, see, for instance, [13, 17]:

$$\Phi(x,t) = \begin{cases} \frac{e^{-\frac{|x|^2}{4\mu\, t}}}{\left(2\sqrt{\pi\mu\, t}\right)^n} & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Consider the cylinder type domain $\Omega_{T_1,T_2} = \Omega_{T_2} \setminus \overline{\Omega_{T_1}}$ with $0 \leq T_1 < T_2$ and a closed measurable set $S \subset \partial\Omega$. For functions $f \in C(\overline{\Omega}_{T_1,T_2})$, $v \in C(S_T)$, $w \in C(S_T)$, $h \in C(\overline{\Omega_{T_1,T_2}})$ we introduce the following potentials:

$$I_{\Omega,T_1}(h)(x,t) = \int\limits_{\Omega} \Phi(x-y,t) h(y,T_1) dy, \quad G_{\Omega,T_1}(f)(x,t) = \int\limits_{T_1}^{t}\int\limits_{\Omega} \Phi(x-y,\ t-\tau) f(y,\tau) dy d\tau,$$

$$V_{S,T_1}(v)(x,t) = \int\limits_{T_1}^{t}\int\limits_{S} \Phi(x-y,t-\tau) v(y,\tau) ds(y) d\tau,$$

$$W_{S,T_1}(w)(x,t) = -\int\limits_{T_1}^{t}\int\limits_{S} \partial_{\nu_y}\Phi(x-y,t-\tau) w(y,\tau) ds(y) d\tau,$$

(see, for instance, [12, Ch. 4, §1], [13, Ch. 1, §3 and Ch. 5, §2], [16, Ch. 3, §10]). The potential $I_{\Omega,T_1}(h)$ is sometimes called *Poisson type integral* and the functions $G_{\Omega,T_1}(f)$, $V_{S,T_1}(v)$, $W_{S,T_1}(w)$ are often referred to as heat potentials or, more precisely, *volume heat potential*, *single layer heat potential* and *double layer heat potential*, respectively. By the construction, all these potentials are (improper) integrals depending on the parameters $(x,t)$.

Next, we need the so-called Green formula for the heat operator.

**Lemma 1.** *For all $0 \le T_1 < T_2$ and all $u \in C^{2,1}(\Omega_{T_1,T_2}) \cap C^{1,0}(\overline{\Omega_{T_1,T_2}})$ with $\mathcal{L}u \in C(\overline{\Omega_{T_1,T_2}})$ the following formula holds:*

$$\left.\begin{array}{r} u(x,t),\ (x,t) \in \Omega_{T_1,T_2} \\ 0,\ (x,t) \notin \overline{\Omega_{T_1,T_2}} \end{array}\right\} = \Big( I_{\Omega,T_1}(u) + G_{\Omega,T_1}(\mathcal{L}u) + V_{\partial\Omega,T_1}(\partial_\nu u) + W_{\partial\Omega,T_1}(u) \Big)(x,t). \tag{5}$$

*Proof.* See, for instance, [10], [20, Ch. 6, §12] (cf. also [21, Theorem 2.4.8] for more general linear operators). □

As is known, the heat equation is hypoelliptic. Moreover, any $C^{2,1}(\Omega_{T_1,T_2})$-solution $P$ to the heat equation $\mathcal{L}P = 0$ in the cylinder domain $\Omega_{T_1,T_2}$ belongs to $C^\infty(\Omega_{T_1,T_2})$ and, actually $P(x,t)$ is real analytic with respect to the space variable $x \in \Omega$ for each $t \in (T_1, T_2)$, see, for instance, [17, Ch. VI, §1, Theorem 1]. Then Green formula (5) and the information on the kernel $\Phi$ provide us with a uniqueness theorem for Problem 1.

**Theorem 1** (A uniqueness theorem)**.** *If $\Gamma$ has at least one interior point in the relative topology of $\partial\Omega$ then Problem 1 has no more than one solution.*

*Proof.* See, for instance, [10, Theorem 1, Corollary 1]. □

Now we are ready to formulate a solvability criterion for Problem 1. As before, we assume that $\Gamma$ is a relatively open connected set of $\partial\Omega$. Then we may find a set $\Omega^+ \subset \mathbb{R}^n$ in such a way that the set $D = \Omega \cup \Gamma \cup \Omega^+$ would be a bounded domain with piece-wise smooth boundary. It is convenient to set $\Omega^- = \Omega$. For a function $v$ on $D_T$ we denote by $v^+$ its restriction to $\Omega^+_T$ and, similarly, we denote by $v^-$ its restriction to $\Omega_T$. It is natural to denote limit values of $v^\pm$ on $\Gamma_T$, when they are defined, by $v^\pm_{|\Gamma_T}$. Actually, a solvability criterion for Problem 1 was obtained in [10]. In this section we would like to improve these results in order to invoke Hilbert space methods. More precisely, the following theorem is a modification of [10, Theorem 2], related to the use of anisotropic Hölder spaces. Though the proofs of the theorems are very similar, we obtain additional essential information about the solution to Problem 1 allowing us to use anisotropic Sobolev spaces and so called bases with double orthogonality property.

**Theorem 2** (Solvability criterion)**.** *Let $\lambda \in (0,1)$, $\partial\Omega$ belong to $C^{1+\lambda}$ and let $\Gamma$ be a relatively open connected subset of $\partial\Omega$. If $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$, $u_1 \in C^{1,0,\lambda,\lambda/2}(\overline{\Gamma_T})$, $u_2 \in C^{0,0,\lambda,\lambda/2}(\overline{\Gamma_T})$ then Problem (2), (4) is solvable in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T)$ if and only if there is a function $F \in C^\infty(D_T)$ satisfying the following two conditions: 1) $\mathcal{L}F = 0$ in $D_T$, 2) $F = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$ in $\Omega^+_T$.*

*Proof. Necessity.* Let a function $u(x,t) \in C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T)$ satisfy (2), (4). Clearly, the function $u(x,t)$ belongs to the space $C^{2,1,\lambda,\lambda/2}(\overline{\Omega'_T})$ for each cylindrical domain $\Omega'_T$ with such a base $\Omega'$ that $\Omega' \subset \Omega$ and $\overline{\Omega'} \cap \partial\Omega \subset \Gamma$. Besides, $\mathcal{L}u = f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega'_T})$. Without loss of the generality we may assume that the interior part $\Gamma'$ of the set $\overline{\Omega'} \cap \partial\Omega$ is non-empty. Consider in the domain $D_T$ the functions

$$\mathcal{F} = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) \text{ and } F = \mathcal{F} - \chi_{\Omega_T} u, \tag{6}$$

where $\chi_M$ is the characteristic function of a set $M \subset \mathbb{R}^{n+1}$. By the very construction condition 2) is fulfilled for it. Note that $\chi_{\Omega_T} u = \chi_{\Omega'_T} u$ in $D'_T$, where $D' = \Omega' \cup \Gamma' \cup \Omega^+$. Then Lemma 1 yields

$$F = G_{\Omega\setminus\overline{\Omega'},0}(f) + V_{\overline{\Gamma}\setminus\Gamma',0}(u_2) + W_{\overline{\Gamma}\setminus\Gamma',0}(u_1) - I_{\Omega',0}(u) \text{ in } D'_T. \tag{7}$$

Arguing as in the proof of Theorem 1 we conclude that each of the integrals on the right-hand side of (7) is smooth outside the corresponding integration set and each satisfies homogeneous heat equation there. In particular, we see that $F \in C^\infty(D'_T)$ and $\mathcal{L}F = 0$ in $D'_T$ because of [17, Ch. VI, §1, Theorem 1]. Obviously, for any point $(x,t) \in D_T$ there is a domain $D'_T$ containing $(x,\ t)$. That is why $\mathcal{L}F = 0$ in $D_T$, and hence $F$ belongs to the space $C^\infty(D_T)$. Thus, this function satisfies condition 1), too.

*Sufficiency.* Let there be a function $F \in C^\infty(D_T)$, satisfying conditions 1) and 2) of the theorem. Consider on the set $D_T$ the function

$$U = \mathcal{F} - F. \tag{8}$$

As $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$ then the results of [12, Ch. 4, §§11-14], [13, Ch. 1, §3] imply

$$G_{\Omega,0}(f) \in C^{2,1,\lambda,\lambda/2}(\overline{\Omega^\pm_T}) \cap C^{1,0,\lambda,\lambda/2}(D_T) \tag{9}$$

and, moreover,

$$\mathcal{L}G^-_{\Omega,0}(f) = f \text{ in } \Omega_T, \quad \mathcal{L}G^+_{\Omega,0}(f) = 0 \text{ in } \Omega^+_T. \tag{10}$$

Since $u_2 \in C^{0,0,\lambda,\lambda/2}(\overline{\Gamma_T})$ then the results of [12, Ch. 4, §§11-14], [13, Ch. 5, §2] yield

$$V_{\overline{\Gamma},0}(u_2) \in C^\infty(\Omega^\pm_T) \cap C^{1,0,\lambda,\lambda/2}((\Omega^\pm \cup \Gamma)_T), \quad \mathcal{L}V_{\overline{\Gamma},0}(u_2) = 0 \text{ in } \Omega_T \cup \Omega^+_T. \tag{11}$$

On the other hand, the behaviour of the double layer potential $W_{\overline{\Gamma},0}(u_1)$ is similar to the behaviour of the normal derivative of single layer potential $V_{\overline{\Gamma},0}(u_1)$. Hence

$$W_{\overline{\Gamma},0}(u_1) \in C^\infty(\Omega^\pm_T) \cap C^{0,0,\lambda,\lambda/2}((\Omega^\pm \cup \Gamma)_T), \quad \mathcal{L}W_{\overline{\Gamma},0}(u_1) = 0 \text{ in } \Omega_T \cup \Omega^+_T. \tag{12}$$

**Lemma 2.** *Let $S \subset \overline{\Gamma} \in C^{1+\lambda}$. If $u_1 \in C^{1+\lambda,\lambda}(\overline{\Gamma_T})$, then the potential $W^-_{\overline{\Gamma},0}(u_1)$ belongs to the space $C^{1,0,\lambda,\lambda/2}(\Omega_T \cup S_T)$ if and only if $W^+_{\overline{\Gamma},0}(u_2) \in C^{1,0,\lambda,\lambda/2}(\Omega^+_T \cup S_T)$.*

*Proof.* It is similar to the proof of the analogous lemma for Newton double layer potential (see, for instance, [6, lemma 1.1]). Actually, one needs to use Lemma 1 instead of the standard Green formula for the Laplace operator, see [10, Lemma 2] for a different function class. □

Since $F \in C^\infty(D_T) \subset C^{1,0,\lambda,\lambda/2}((\Omega^+ \cup \Gamma)_T)$ then it follows from the discussion above that $W^+_{\overline{\Gamma},0}(u_2) \in C^{1,0,\lambda,\lambda/2}((\Omega^+ \cup \Gamma)_T)$. Thus, formulas (8)–(12) and Lemma 2 imply that $U$ belongs $C^{2,1,\lambda,\lambda/2}(\Omega^\pm_T) \cap C^{1,0,\lambda,\lambda/2}((\Omega^\pm \cup \Gamma)_T)$ and

$$\mathcal{L}U = \chi_{D_T} f \text{ in } \Omega_T \cup \Omega^+_T.$$

In particular, (2) is fulfilled for $U^-$. Let us show that the function $U^-$ satisfies (4). Since $F \in C^\infty(D_T)$ we see that $\partial^\alpha F^- = \partial^\alpha F^+$ on $\Gamma_T$ for $\alpha \in \mathbb{Z}_+$ with $|\alpha| \leqslant 1$ and

$$\partial^\alpha F^+_{|\Gamma_T} = \left(\partial^\alpha G^+_{\Omega,0}(f) + \partial^\alpha V^+_{\overline{\Gamma},0}(u_2) + \partial^\alpha W^+_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T}.$$

It follows from formulas (9) and (11) that the volume potential $G^+_{\Omega,0}(f)$ and the single layer potential $V^+_{\overline{\Gamma},0}(u_2)$ are continuous if the point $(x,t)$ passes over the surface $\Gamma_T$. Then

$$U^-_{|\Gamma_T} = W^-_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} - W^+_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} = u_1.$$

because of the theorem on jump behaviour of the double layer potential $W_{\overline{\Gamma},0}(u_1)$ (see, for instance, [13, Ch. 5, §2, theorem 1] for the corresponding heat potential), i.e. equality the first equation in (4) is valid for $U^-$.

Formula (9) means that that the normal derivative of the volume potential $G^+_{\Omega,0}(f)$ is continuous if the point $(x,t)$ passes over the surface $\Gamma_T$. Therefore

$$\partial_\nu U_{|\Gamma_T} = \partial_\nu V^-_{\overline{\Gamma},0}(u_2)_{|\Gamma_T} - \partial_\nu V^+_{\overline{\Gamma},0}(u_2)_{|\Gamma_T} + \partial_\nu W^-_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} - \partial_\nu W^+_{\overline{\Gamma},0}(u_1)_{|\Gamma_T}. \tag{13}$$

By theorem on jump behaviour of the normal derivative of the single layer potential (see, for instance, [16, Ch. 3, §10, theorem 10.1] for the corresponding heat potential),

$$\partial_\nu V^-_{\overline{\Gamma},0}(u_2)_{|\Gamma_T} - \partial_\nu V^+_{\overline{\Gamma},0}(u_2)_{|\Gamma_T} = u_2. \tag{14}$$

Finally, we need the following lemma.

**Lemma 3.** *Let $\Gamma \in C^{1+\lambda}$ and $u_2 \in C^{0,0,\lambda,\lambda/2}(\overline{\Gamma_T})$. If $W^-_{\overline{\Gamma},0}(u_1) \in C^{1,0,\lambda,\lambda/2}((\Omega \cup \Gamma)_T)$ or $W^+_{\overline{\Gamma},0}(u_1) \in C^{1,0,\lambda,\lambda/2}((\Omega^+ \cup \Gamma)_T)$ then*

$$\partial_\nu W^-_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} - \partial_\nu W^+_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} = 0. \tag{15}$$

*Proof.* It is similar to the proof of the analogous lemma for the the heat double layer potential (see, for instance, [10, lemma 3] for a different function class). □

Using lemma 3 and formulas (13), (14), we conclude that $\partial_\nu U^-_{|\Gamma_T} = u_2$, i.e. the second equation in (4) is fulfilled for $U^-$. Thus, function $u(x,t) = U^-(x,t)$ satisfies conditions (2), (4). The proof is complete. □

We note that both [10, Theorem 2] and Theorem 2 are analogues of Theorem by Aizenberg and Kytmanov [5] describing solvability conditions of the Cauchy problem for the Cauchy–Riemann system (cf. also [6] in the Cauchy Problem for Laplace Equation or [8] in the Cauchy problem for general elliptic systems). Formula (8), obtained in the proof of Theorem 2, gives the unique solution to Problem 1. Clearly, if we will be able to write the extension $F$ of the sum of potentials $G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$ from $\Omega^+_T$ onto $D_T$ as a series with respect to special functions or a limit of parameter depending integrals then we will get Carleman-type formula for solutions to Problem 1 (cf. [5]). The simplest formulas of this type for the Cauchy-Riemann system was constructed by Goluzin and Krylov, see [22] or [4], for the special domains (the so-called "lunes", i.e. parts of a disc $\Omega$ on the complex planes, separated from the origin by a smooth curve $\Gamma \subset \Omega$). Aizenberg and Kytmanov [5] supplement the Goluzin–Krylov formula with a simple solvability criterion for the Cauchy problem for holomorphic functions in the lunes based on the Cauchy–Hadamard formula for power series. Let us extend this approach for the case $n = 1$.

With this purpose we introduce the following Carleman kernels:

$$\mathfrak{C}_N(x,y,t-\tau) = \Phi(x-y,t-\tau) - \Phi(y,t-\tau)\sum_{k=0}^{N} \frac{x^k}{(t-\tau)^k}\left(\sum_{\substack{m+j=k\\0\le j\le m}} \frac{(2y)^{m-j}(-1)^j(t-\tau)^j}{(4\mu)^m(m-j)!j!}\right).$$

**Corollary 1.** *Let $n=1$, $a>0$, $\Omega=(a,1)$ and $D=(-1,1)$. Under the hypotheses of Theorem 2, Problem 1 is solvable in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T)\cap C^{1,0,\lambda,\lambda/2}(\Omega_T\cup\Gamma_T)$ if and only if*

$$\limsup_{k\to\infty}\Big(|c_k(t)|\Big)^{1/k} \le 1 \textit{ for each } t\in(0,T)$$

*where $c_k(t)$ are the Taylor coefficients of the function $F(x,t)$ with respect to the variable $x$ at the origin for fixed value $t\in(0,T)$ given by (19) below. Besides, the solution, if exists, equals to*

$$u(x,t) = \lim_{N\to+\infty}\left(\int_0^t\int_\Omega \mathfrak{C}_N(x,y,t-\tau)f(y,\tau)dyd\tau + \int_0^t\int_\Gamma \mathfrak{C}_N(x,y,t-\tau)u_2(y,\tau)ds(y)d\tau + \right. \tag{16}$$

$$\left.\int_0^t\int_\Gamma \partial_\nu(y)\mathfrak{C}_N(x,y,t-\tau)u_1(y,\tau)ds(y)d\tau\right).$$

*Proof.* The first part of the corollary follows immediately from Theorem 2, formula (8) for the solution to Problem 1 and the Cauchy-Hadamard formula for the radius of the convergence of a power series. In particular,

$$U(x,t) = \lim_{N\to\infty} \Big(\mathcal{F}(x,t) - \sum_{k=0}^{N} c_k(t) x^k\Big), \quad (x,t) \in \Omega_T. \tag{17}$$

On the other hand, we may write easily the Taylor series with respect to the variable $x$ near the origin for the kernel $\Phi(x-y,t-\tau)$ if $t>\tau$:

$$\Phi(x-y,t-\tau) = \Phi(y,t-\tau) e^{\frac{2xy-x^2}{4\mu(t-\tau)}} = \sum_{k=0}^{\infty} \Phi_k(y,t-\tau) x^k \tag{18}$$

where the series converges uniformly on the compacts in the set $\{|x|<a, y\in\overline{\Omega}, t>\tau\geq 0\}$ and

$$\Phi_k(y,t-\tau) = \frac{\Phi(y,t-\tau)}{(t-\tau)^k} \left( \sum_{\substack{m+j=k\\ 0\leq j\leq m}} \frac{(2y)^{m-j}(-1)^j (t-\tau)^j}{(4\mu)^m (m-j)! j!} \right).$$

As $y\in\Omega$ then $y\neq 0$ and the kernels $\Phi_k(y,t-\tau)$ and $\partial_\nu \Phi_k(y,t-\tau)$ are integrable over subsets of $\overline{\Omega}_T$. Thus, using (6), after termwise integration of the Taylor series (18), we identify the Taylor coefficients $c_k(t)$ as follows:

$$c_k(t) = \int_0^t \int_\Omega \Phi_k(y,t-\tau) f(y,\tau) dy d\tau + \int_0^t \int_S \Big(\Phi_k(y,t-\tau) u_2(y,\tau) + \partial_\nu \Phi_k(y,t-\tau) u_1(y,\tau)\Big) ds(y) d\tau. \tag{19}$$

Finally, combining formulas (6), (17), (19), we arrive at Carleman-type formula (16). □

For $n>1$ the situation differs drastically because of the nature of the multi-dimensional Cauchy-Hadamard formula for multiple power series. Instead, for the multidimensional Cauchy-Riemann system Aizenberg and Kytmanov [5] suggested to use Hilbert space theory and the so-called bases with the double orthogonality property (cf. [6], [8], [7] for elliptic systems).

Following this idea, we assume that the surface $\partial\Omega$ and that the data $u_j$ are smoother than in Theorem 2. Namely, we need the following lemma.

**Lemma 4.** *Let $\partial\Omega \in C^{3+\lambda}$ and let $\Gamma$ be a relatively open subset of $\partial\Omega$ with boundary $\partial\Gamma \in C^{2+\lambda}$. If $u_1 \in C^{2,1,\lambda,\lambda/2}(\overline{\Gamma_T})$, $u_2 \in C^{2,1,\lambda,\lambda/2}(\overline{\Gamma_T})$ then there exist functions $\tilde u_j \in C^{2,1,\lambda,\lambda/2}(\partial\Omega_T)$ such that $\tilde u_j = u_j$ on $\overline{\Gamma_T}$, $j=1,2$ and a function $\tilde u \in C^{2,1,\lambda,\lambda/2}(\overline{\Omega_T})$ such that $\tilde u = \tilde u_1$ on $(\partial\Omega)_T$ and $\partial_\nu \tilde u = \tilde u_2$ on $(\partial\Omega)_T$.*

*Proof.* We may adopt the standard arguments from [23, Lemma 6.37] related to isotropic spaces. Indeed, according to it, under our assumptions, for every $v_0 \in C^{2,\lambda}(\overline{\Gamma})$ there is $v \in C^{2,\lambda}(\partial\Omega)$ such that $v = v_0$ on $\overline{\Gamma}$. The construction of the extension involves the rectifying diffeomorphism of $\partial\Gamma$ and a suitable partition of unity of a neighbourhood of $\partial\Gamma$, only. Thus, we conclude there are functions $\tilde u_j \in C^{2,1,\lambda,\lambda/2}(\partial\Omega_T)$ such that $\tilde u_j = u_j$ on $\overline{\Gamma_T}$, $j=1,2$.

Next, we use the existence of the Poisson kernel $P_{\Delta^2,\Omega}(x,y)$ for the Dirichlet problem related to the operator $\Delta_n^2$, see [24]. It is known that the problem is well-posed over the scale of Hölder spaces in $\Omega$. Namely, as $\partial\Omega \in C^{3,\lambda}$, for each $1\leq s\leq 3$ and $v_1 \in C^{s,\lambda}(\partial\Omega)$. $v_2 \in C^{s-1,\lambda}(\partial\Omega)$ the integral

$$\mathcal{P}_{\Delta^2,\Omega}(v_1,v_2)(x) = \int_{\partial\Omega} \Big( (\partial_{\nu(y)} P_{\Delta^2,\Omega})(x,y) v_1(y) + P_{\Delta^2,\Omega}(x,y) v_2(y) \Big) ds(y)$$

belongs to $C^{s,\lambda}(\overline{\Omega})$ and satisfies $\Delta_n^2 v = 0$ in $\Omega$ and $v = v_1$, $\partial_\nu v = v_2$ on $\partial\Omega$. Now, we may set

$$\tilde{u}(x,t) = \mathcal{P}_{\Delta^2,\Omega}(\tilde{u}_1(\cdot,t),0)(x) + \mathcal{P}_{\Delta^2,\Omega}(0,\tilde{u}_2(\cdot,t))(x).$$

Then the first integral belongs to $C^{2,\lambda}(\overline{\Omega})$ and the second belongs to $C^{3,\lambda}(\overline{\Omega})$ with respect to the variable $x$ for each $t \in [0,T]$. By the construction, $\tilde{u}(\cdot,t) = \tilde{u}_1(\cdot,t)$ on $\partial\Omega$ and $\partial_\nu \tilde{u}(\cdot,t) = \tilde{u}_2(\cdot,t)$ on $\partial\Omega$ for each $t \in [0,T]$. It remains to check that $\tilde{u}$ belongs to $C^{2,1,\lambda,\lambda/2}(\overline{\Omega_T})$ as the sum of integrals depending on the parameter $(x,t) \in \overline{\Omega_T}$. □

The statements of such type actually reflect the fact that the system of boundary operator, related to the investigated problem, is compatible. In the theory of elliptic operators this means that the system $\{I, \partial_\nu\}$ is a Dirichlet system on $\Gamma$, see, for instance, [7, 8]. For example, under the assumptions of Lemma 4, for each pair $v_1 \in C^{2,\lambda}(\overline{\Gamma})$ and $v_2 \in C^{1,\lambda}(\overline{\Gamma})$ there is $v \in C^{2,\lambda}(\overline{\Omega})$ such that $v = v_1$ on $\overline{\Gamma}$ and $\partial_\nu v = v_2$ on $\overline{\Gamma}$. In Lemma 4 we use smoother data $u_2$ on $\overline{\Gamma_T}$ because the heat operator under the consideration is not elliptic and the behaviour with respect to the variable $t$ can not be improved by actions with respect to the variable $x$.

Under the assumptions of Lemma 4, we set

$$\tilde{\mathcal{F}} = G_{\Omega,0}(f) + V_{\partial\Omega,0}(\tilde{u}_2) + W_{\partial\Omega,0}(\tilde{u}_1) + I_{\Omega,0}(\tilde{u}). \tag{20}$$

**Corollary 2.** *Let $\lambda \in (0,1)$, $\partial\Omega$ belong to $C^{3+\lambda}$ and let $\Gamma$ be a relatively open connected subset of $\partial\Omega$ with boundary $\partial\Gamma \in C^{2+\lambda}$. If $f \in C^{0,0,\lambda,\lambda/2}(\overline{\Omega_T})$, $u_1 \in C^{2,1,\lambda,\lambda/2}(\overline{\Gamma_T})$, $u_2 \in C^{2,1,\lambda,\lambda/2}(\overline{\Gamma_T})$ then Problem (2), (4) is solvable in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2,1}(\Omega_T)$ if and only if there is a function $\tilde{F} \in C^\infty(D_T) \cap H^{2,1}(D_T)$ satisfying the following two conditions: 1') $\mathcal{L}\tilde{F} = 0$ in $D_T$, 2') $\tilde{F} = \tilde{\mathcal{F}}$ in $\Omega_T^+$.*

*Proof.* First of all, we note that, by Green formula (5), we have $\tilde{\mathcal{F}} = G_{\Omega,0}(f - \mathcal{L}\tilde{u})$ and then $\tilde{\mathcal{F}} \in C^{2,1,\lambda,\lambda/2}(\overline{\Omega_T^\pm})$ because of (9). On the other hand,

$$\tilde{\mathcal{F}} - \mathcal{F} = V_{\partial\Omega\setminus\Gamma,0}(\tilde{u}_2) + W_{\partial\Omega\setminus\Gamma,0}(\tilde{u}_1) + I_{\Omega,0}(\tilde{u}). \tag{21}$$

This means that the function $\tilde{\mathcal{F}} - \mathcal{F}$ satisfies the $\mathcal{L}(\tilde{\mathcal{F}} - \mathcal{F}) = 0$ in $D_T$ and hence the function $\mathcal{F}$ extends to $D_T$ as a solution of the heat equation if and only if function $\tilde{\mathcal{F}}$ extends to $D_T$ as a solution of the heat equation, too.

Let Problem (2), (4) be solvable in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2,1}(\Omega_T)$. Then formulas (6) and (21) imply

$$\tilde{F} = \tilde{\mathcal{F}} - \chi_{\Omega_T} u \in H^{2,1}(\Omega_T^\pm) \text{ and } \mathcal{L}\tilde{F} = 0 \text{ in } D_T.$$

Now, as $\tilde{F} \in H^{2,1}(\Omega_T^\pm) \cap C^\infty(D_T)$ (see [17, Ch. VI, §1, Theorem 1]) we conclude that $\tilde{F} \in H^{2,1}(D_T)$, i.e. conditions 1'), 2') of the corollary are fulfilled.

If conditions 1'), 2') of the corollary hold true then conditions 1), 2) of Theorem 2 are fulfilled, too. Moreover, formulas (8) and (21) imply that in $D_T$ we have

$$U = \mathcal{F} - F = \tilde{\mathcal{F}} - \tilde{F} \in H^{2,1}(\Omega_T^\pm) \tag{22}$$

and the $U^-$ is the solution to Problem 1 in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) H^{2,1}(\Omega_T^\pm)$ by Theorem 2. □

Let us describe the way for the construction of the bases with the double orthogonality property related to Problem 1. For $s \in \mathbb{Z}_+$, denote by $H^{2s,s}_{\mathcal{L}}(\Omega_T)$ the space of weak (i.e. in the sense of distributions) solutions to the heat equation (2) belonging to $H^{2s,s}(\Omega_T)$. Obviously, it is a closed subspace of $H^{2s,s}(\Omega_T)$.

**Lemma 5.** *Let $\omega$ be a subset of $D$ with piece-wise smooth boundary. There exists an orthonormal basis $\{b_\nu\}$ in the space $H^{2,1}_{\mathcal{L}}(D_T)$ such that the system $\{b_{\nu|\omega_T}\}$ is an orthogonal basis in $L^2_{\mathcal{L}}(\omega_T)$.*

*Proof.* Indeed, by the definition the space $H^{2,1}(D_T)$ is continuously embedded to $L^2(\omega_T)$. As any weak solution to the heat equation in $D_T$ is a weak solution the heat equation in $\omega_T$, too, we conclude that $H^{2,1}_{\mathcal{L}}(D_T)$ is continuously embedded to $L^2_{\mathcal{L}}(\omega_T)$. Let us denote by $R_{D,\omega}$ the corresponding embedding operator, $R_{D,\omega} : H^{2,1}_{\mathcal{L}}(D_T) \to L^2_{\mathcal{L}}(\omega_T)$. By the construction, the range of the operator $R$ is dense in $L^2_{\mathcal{L}}(\omega_T)$.

Moreover, by Fubini Theorem the anisotropic Sobolev space $H^{2,1}_{\mathcal{L}}(D_T)$ can be continuously embedded to the Bochner type space $\mathcal{B}((0,T,H^2(D),L^2(D))$ consisting of maps $v : [0,T] \to H^2(D)$ such that $\partial_t v \in L^2(D)$, see [25, Ch 1, §5]. According to the Kondrashov Theorem, the embedding $H^2(D) \to L^2(D)$ is compact. Applying famous compactness theorem for Bochner type spaces (see, for instance, [25, Ch 1, §5, Theorem 5.1] we see that the space $\mathcal{B}((0,T,H^2(D),L^2(D))$ is embedded compactly to $L^2((0,T),D) = L^2(D_T)$. Thus, the space $H^{2,1}_{\mathcal{L}}(D_T)$ is embedded compactly to $L^2_{\mathcal{L}}(D_T)$ and then to $L^2_{\mathcal{L}}(\omega_T)$, i.e. the operator $R_{D,\omega}$ is compact.

Let $R^*_{D,\omega}$ be the Hilbert space adjoint operator for $R_{D,\omega}$, i.e. $R^*_{D,\omega} : L^2_{\mathcal{L}}(\omega_T) \to H^{2,1}_{\mathcal{L}}(D_T)$. Then the Hilbert-Schmidt spectral theorem grants the existence of an orthonormal basis $\{b_\nu\}$ in the space $H^{2,1}_{\mathcal{L}}(D_T)$, consisting of the eigen-vectors of the compact self-adjoint operator $R^*_{D,\omega}R_{D,\omega} : H^{2,1}_{\mathcal{L}}(D_T) \to H^{2,1}_{\mathcal{L}}(D_T)$. Finally, applying we see that the system $\{b_\nu\}$ is the basis with the double orthogonality property, we are looking for, see [7, Example 1.9]. □

Let $\omega$ be a relatively compact subset of $\Omega^+ \subset D$ with piece-wise smooth boundary and let $\{b_\nu\}$ be the basis with the double orthogonality property granted by Lemma 5. We introduce now the following Carleman kernels:

$$\mathfrak{C}^{(\omega)}_N(x,y,t,\tau) = \Phi(x-y,t,\tau) - \sum_{\nu=0}^{N} \Big( b_\nu(x,t) \int_{\omega_T} b_\nu(z,\tilde\tau)\Phi(z-y,\tilde\tau-\tau)dzd\tilde\tau \Big) / \|b_\nu\|^2_{L^2_{\mathcal{L}}(\omega_T)}.$$

Also, let $c_\nu(\tilde{\mathcal{F}})$ be the Fourier coefficients of the function $\tilde{\mathcal{F}}$ with respect to the orthogonal basis $\{b_{\nu|\omega_T}\}$ in the space $L^2_{\mathcal{L}}(\omega_T)$:

$$c_\nu(\tilde{\mathcal{F}}) = \Big( \int_{\omega_T} \overline{b_\nu(z,\tilde\tau)} \tilde{\mathcal{F}}(z,\tilde\tau)dzd\tilde\tau \Big) / \|b_\nu\|^2_{L^2_{\mathcal{L}}(\omega_T)}. \tag{23}$$

**Corollary 3.** *Let $\omega$ be a relatively compact subset of $\Omega^+$ with piece-wise smooth boundary. Under assumptions of Corollary 2, Problem (2), (4) is solvable in the space $C^{2,1,\lambda,\lambda/2}(\Omega_T) \cap C^{1,0,\lambda,\lambda/2}(\Omega_T \cup \Gamma_T) \cap H^{2,1}(\Omega_T)$ if and only if*

$$\sum_{\nu=1}^{\infty} |c_\nu(\tilde{\mathcal{F}})|^2 < \infty.$$

*Besides, the solution, if exists, is given by*

$$u(x,t) = \lim_{N\to+\infty} \Bigg( \int_0^t \int_\Omega \mathfrak{C}^{(\omega)}_N(x,y,t,\tau) f(y,\tau)dyd\tau + \int_0^t \int_{\partial\Omega} \mathfrak{C}^{(\omega)}_N(x,y,t,\tau)\tilde u_2(y,\tau)ds(y)d\tau + \tag{24}$$

$$\int_0^t \int_{\partial\Omega} \partial_\nu(y)\mathfrak{C}^{(\omega)}_N(x,y,t,\tau)\tilde u_1(y,\tau)ds(y)d\tau \Bigg).$$

*Proof.* Again, we note that the heat operator is hypoelliptic, i.e. all weak $L^2(\Omega_T)$-solutions to the homogeneous heat equations are in fact the smooth ones in $\Omega_T$. This means that $H^{2,1}_{\mathcal{L}}(\Omega_T) = H^{2,1}_{\mathcal{L}}(\Omega_T) \cap C^\infty(\Omega_T)$. Then [17, Ch. VI, §1, Theorem 1] implies that conditions 1'), 2') of Corollary 2 are equivalent to the following one: $\tilde F = \tilde{\mathcal{F}}$ in $\omega_T$, or, the same, $R_{D,\omega}\tilde F = \tilde{\mathcal{F}}$. Thus, the first statement of the corollary follows immediately from Lemma 5 and [7, Example 1.9]. On the other hand, according [7, Example 1.9], if $\tilde F \in H^{2,1}_{\mathcal{L}}(D_T)$ is the extension of the function $\tilde{\mathcal{F}}$ from $\omega_T$ to $D_T$ then

$$\tilde F = \sum_{\nu=0}^{\infty} c_\nu(\tilde{\mathcal{F}}) b_\nu(x,t), \quad (x,t) \in D_T.$$

Therefore, (22) yields

$$u(x,t) = \lim_{N\to+\infty} \Big(\tilde{\mathcal{F}}(x,t) - \sum_{\nu=0}^{N} c_\nu(\tilde{\mathcal{F}}) b_\nu(x,t)\Big), \quad (x,t) \in \Omega_T. \tag{25}$$

We note that if $y \in \Omega$ and $x \in \omega$ then $x \neq y$ and the kernel $\Phi(x-y,t-\tau)$ is integrable over $\Omega_T \times \Omega_T$. Thus, we may use integral formula (20) for $\tilde{\mathcal{F}}$ and Fubini Theorem in order to change the order of the integrations in (23). Hence it follows from (25) that (24) holds true. □

**Example 2.** What is still lacking is an example of a basis with the double orthogonality property granted by Lemma 5 for a pair $D$ and $\omega$. However we may easily give an example of a system with the double orthogonality. Indeed, for each multi-index $k \in \mathbb{Z}^n$ we set

$$b^{(1)}_k(x,t) = e^{-4\pi^2\mu|k|^2 t} \cos\Big(2\pi \sum_{j=1}^n k_j x_j\Big), \ b^{(2)}_k(x,t) = e^{-4\pi^2\mu|k|^2 t} \sin\Big(2\pi \sum_{j=1}^n k_j x_j\Big)$$

where $|k|^2 = \sum_{j=1}^n k_j^2$. Then $\mathcal{L} b^{(i)}_k = 0$ in $\mathbb{R}^n \times (0,T)$ for each $k \in \mathbb{Z}^n$ and the system $\{b^{(1)}_k, b^{(2)}_k\}_{k\in\mathbb{Z}^n}$ is orthogonal in $L^2(Q_{n,j} \times (0,T))$ for every cube $Q_{n,j}$, $j \in \mathbb{N}$, and $T > 0$ because the trigonometrical system is orthogonal on the cube $Q_{n,1}$ and it is periodic. Thus, $\{b^{(1)}_k, b^{(2)}_k\}_{k\in\mathbb{Z}^n}$ is a system with the double orthogonality property for any pair $D = Q_{n,j}$ and $\omega = Q_{n,l}$, if $j > l$. Unfortunately, this system can not be complete in $H^{2,1}((Q_{n,j})_T)$ because each function $b_k$ has equal values on the opposite faces of the cube $Q_{n,j}$ and the elements of the space $H^{2,1}((Q_{n,j})_T)$ admit traces on $(\partial Q_{n,j})_T$.

We note also that the scheme, realized in the paper, is valid for a more general parabolic operator $\mathcal{L}$, too, if it admits a fundamental solution with the following properties: the analyticity with respect to $x$ for each fixed $t$ and the proper behavior of the corresponding integrals $I_\Omega$, $G_\Omega$, $V_{\partial\Omega}$ and $W_\Omega$ on the scale of the (anisotropic) Hölder spaces (see, for example, [13, 26] for the conditions providing the existence of such kernels).

*The work was supported by the Ministry of Education and Science of the Russian Federation N 1.2604.2017/PCh.*

## References

[1] Lavrent'ev, M.M., On the Cauchy problem for Laplace equation, *Izvestia AN SSSR. Ser. matem.*, (1956), N. 20, 819–842.

[2] Lavrent'ev, M.M., Romanov, V.G., Shishatskii, S.P., Ill-posed problems of mathematical physics and analysis, M., Nauka, 1980.

[3] Tihonov, A.N., Arsenin, V.Ya., Methods of solving ill-posed problems, M., Nauka, 1986.

[4] Aizenberg, L.A., Carleman formulas in complex analysis. First applications, Novosibirsk, Nauka, 1990 (Russian). English transl. in Kluwer Ac. Publ., Dordrecht, 1993.

[5] Aizenberg L.A., Kytmanov, A.M., On the possibility of holomorphic continuation to a domain of functions given on a part of its boundary, *Matem. sb.*, **182**(1991), N. 5, 490–597.

[6] Shlapunov, A.A., On the Cauchy Problem for the Laplace Equation, *Siberian math. j.*, **33**(1992), N. 3, 205–215.

[7] Shlapunov, A.A., Tarkhanov, N., Bases with double orthogonality in the Cauchy problem for systems with injective symbols, *Proc. London. math. soc.*, **71**(1995), N. 1, 1–54.

[8] Tarkhanov, N., The Cauchy problem for solutions of elliptic equations, Berlin: Akademie-Verlag, 1995.

[9] Fedchenko, D.P., Shlapunov, A.A., On the Cauchy problem for the elliptic complexes in spaces of distributions, *Complex variables and elliptic equations*, **59:5** (2014), 651–679.

[10] Puzyrev, R.E., Shlapunov, A.A., On an ill-Posed problem for the heat equation, *J. Sib. Fed. Univ., Math. and Physics*, **5:3** (2012), 337–348.

[11] Puzyrev, R.E., Shlapunov, A.A., On a mixed problem for the parabolic Lamé type operator, *J. Inv. Ill-posed Problems*, V. 23:6 (2015), 555-570.

[12] Ladyzhenskaya, O.A., Solonnikov V.A., Ural'tseva N.N., Linear and quasilinear equations of parabolic type, M., Nauka, 1967.

[13] Friedman, A., Partial differential equations of parabolic type, Englewood Cliffs, NJ, Prentice-Hall, Inc., 1964.

[14] Krylov, N.V., Lectures on elliptic and parabolic equations in Hölder spaces. Graduate Studies in Math. V. 12, AMS, Providence, Rhode Island, 1996.

[15] Krylov, N.V., Lectures on elliptic and parabolic equations in Sobolev spaces. Graduate Studies in Math. V. 96, AMS, Providence, Rhode Island, 2008.

[16] Landis, E.M., Second order equations of elliptic and parabolic types, M., Nauka, 1971.

[17] Mikhailov, V.P., Partial differential equations, M., Nauka, 1976.

[18] V. P. Mikhailov, The Dirichlet problem for a parabolic equation. I, Mat. Sb. (N.S.), 61(103):1 (1963), 40–64.

[19] Hadamard, J., Lectures on Cauchy's problem in linear partial differential equations, Yale Univ. Press, New Haven-London, 1923.

[20] Sveshnikov, A.G., Bogolyubov, A.N., Kravtsov, V.V., Lectures on mathematical physics, M., Nauka, 2004.

[21] Tarkhanov, N.N., Complexes of differential operators, Kluwer Ac. Publ., Dordrecht, 1995.

[22] Goluzin G.M., Krylov, V.I., Verallgemeinerung einer Formel von Carleman und ihre Anwendung auf analytische Fortsetzung, Mat. Sb. **40** (1933), N. 2, 144–149.

[23] Gilbarg, D., Trudinger, N., Elliptic partial differential equations of second order, Berlin, Springer-Verlag, 1983.

[24] Aronszajn, N., Creese, T., Lipkin, L., Polyharmonic functions. Clarendon Press, Oxford, 1983.

[25] Lions, J.-L., Quelques méthodes de résolution des problèmes aux limites non linéare, Dunod/Gauthier-Villars, Paris, 1969, 588 pp.

[26] Eidel'man S.D., Parabolic equations, Partial differential equations – 6, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 63, VINITI, Moscow, 1990, 201—313.

## О формулах Карлемановского типа для решений уравнения теплопроводности

**Илья А. Куриленко**
**Александр А. Шлапунов**

*Мы применяем метод интегральных предсталений к исследованию некорректной задачи Коши для уравнения теплопроводности. Более точно, используя подходящую формулу Грина, мы восстановливаем комплекснозначную функцию, удовлетворяющую уравнению теплопроводности в цилиндре, по заданным ее значениям и значениям ее нормальной производной на части боковой поверхности цилиндра. Мы показываем, что задача является некорректной в естественных для нее (анизптропных) пространствах (Соболева, Гельдера и т.д.). В итоге, нами получены теорема единственности для задачи Коши, а также необходимые и достаточные условия ее разрешимости и формула карлемановского типа для ее решения.*

*Ключевые слова: уравнение теплопроводности, некорректные задачи, метод интегральных представлений, формулы Карлемана*