Research Article

Yulia Shefer and Alexander Shlapunov*

On regularization of the Cauchy problem for elliptic systems in weighted Sobolev spaces

https://doi.org/10.1515/jiip-2018-0010
Received February 7, 2018; revised March 3, 2019; accepted April 2, 2019

Abstract: We consider the ill-posed Cauchy problem in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$ for an elliptic differential operator $\mathcal{A}(x, \partial)$ with data on a relatively open subset S of the boundary $\partial\mathcal{D}$. We do it in weighted Sobolev spaces $H^{s,\gamma}(\mathcal{D})$ containing the elements with prescribed smoothness $s \in \mathbb{N}$ and growth near ∂S in $\mathcal{D}$, controlled by a real number γ. More precisely, using proper (left) fundamental solutions of $\mathcal{A}(x, \partial)$, we obtain a Green-type integral formula for functions from $H^{s,\gamma}(\mathcal{D})$. Then a Neumann-type series, constructed with the use of iterations of the (bounded) integral operators applied to the data, gives a solution to the Cauchy problem in $H^{s,\gamma}(\mathcal{D})$ whenever this solution exists.

Keywords: The Cauchy problem, ill-posed problems, elliptic operators, Green's formulas, weighted Sobolev spaces

MSC 2010: Primary 35JXX; secondary 35NXX

1 Introduction

The ill-posed Cauchy problem for elliptic systems of linear partial differential equations is a long standing problem connected with numerous applications in physic, electrodynamics, fluid mechanics etc. (see, for instance, [10, 11]). It appears that the regularization methods (see, for instance, [31]) are most effective for studying the problem. However, there are many different ways to realize the regularization; see, for instance, [1, 4] for the problem of holomorphic continuation in complex analysis or [8, 13, 14] for the Cauchy problem related to the second-order elliptic equations. The book [30] gives a rather full description of solvability conditions for the problem and the ways of its regularization in the Sobolev spaces and Hardy spaces.

Recently, a new approach was developed; cf., for instance, [22, 26]. It is based on the simple observation that the calculus of Cauchy problems for solutions to elliptic equations just amounts to the calculus of (possibly non-coercive) mixed boundary value problems of Zaremba type for elliptic equations with a parameter. On this way, it is possible to obtain a suitable regularization of the Cauchy problem for elliptic equations in the Sobolev spaces; see, for example, [18] for the Cauchy–Riemann system or [24] in the general case. However, local analysis of formal solutions to a partial differential equation (especially in domains with non-smooth boundaries) immediately shows that there are solutions with typical behavior adequately described in weighted spaces only. There is no wonder that the most substantial results on mixed problems were perhaps

***Corresponding author: Alexander Shlapunov,** Institute of Mathematics and Computer Science, Siberian Federal University, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia, e-mail: ashlapunov@sfu-kras.ru
Yulia Shefer, Institute of Mathematics and Computer Science, Siberian Federal University, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia, e-mail: yuliyacherepanova@mail.ru

achieved in weighted Sobolev spaces; see, for instance, [3, 17]. One chooses a weight function to appropriately control the behavior of solutions near interface surface on the boundary where the boundary conditions change their character; see recent advances in [28, 29]. Another motivation to introduce weights consists in possible geometric singularities of the boundary of the manifold where the problem is posed. This is why we would like to revisit the regularization of the ill-posed Cauchy problem in proper weighted spaces.

Namely, let $\mathcal{A}$ be a non-zero bounded linear operator that maps a Hilbert space H_1 to a Hilbert space H_2. As usual, we denote by $\ker(\mathcal{A})$ the null-space of $\mathcal{A}$. Let also $\mathcal{A}^\star$ stand for the Hilbert space adjoint operator for $\mathcal{A}$ that maps H_2 to H_1, and let I_{H_j} stand for the identity operator in the space H_j. Since the range of the map $\mathcal{A}$ may be non-closed, solving the operator equation

$$\mathcal{A}u = f \tag{1.1}$$

might prove to be an ill-posed problem (see [11, 31]). The following theorem provides one of the possible way of its regularization; see, for instance, [11, 23].

Theorem 1.1. *Let* $a = \max(\|\mathcal{A}\|, 1)$. *Given* $f \in H_2$, *there is a solution* $u \in H_1$ *of the operator equation* (1.1) *if an only if the following assertions hold:*
(i) $(f, g) = 0$ *for all* $g \in \ker(\mathcal{A}^\star)$.
(ii) *The Neumann series* $u(f) = a^{-2}\sum_{\nu=0}^{\infty}(I_{H_1} - a^{-2}\mathcal{A}^\star\mathcal{A})^\nu\mathcal{A}^\star f$ *converges in the space* H_1.
Moreover, under these conditions, $u(f)$ *is the unique solution to* (1.1), *orthogonal to* $\ker(\mathcal{A})$.

In the sequel we reduce the Cauchy problem in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$ for an elliptic differential operator $\mathcal{A} = \mathcal{A}(x, \partial)$ (possibly possessing singular low-order coefficients) with data on a relatively open subset S of the boundary $\partial\mathcal{D}$ to the operator equation (1.1) in proper weighted Sobolev spaces $H^{s,\gamma}(\mathcal{D})$ containing the elements with prescribed smoothness $s \in \mathbb{N}$ and growth near ∂S in $\mathcal{D}$, controlled by a real number γ. Finally, we obtain a nice regularization operator for (1.1) indicating a reasonable integral formula for the adjoint operator $\mathcal{A}^\star$ under the consideration.

Actually, the basic idea for constructing an integral formula related to the adjoint operator $\mathcal{A}^\star$ was invented in [19] in the situation where $\mathcal{A}$ is the n-dimensional Cauchy–Riemann operator $\overline{\partial}$ in $\mathbb{C}^n$ acting from the Sobolev space $H^1(\mathcal{D})$ to the space of n-vector functions with components from the Lebesgue space $L^2(\mathcal{D})$ over a sufficiently smooth bounded domain $\mathcal{D} \subset \mathbb{C}^n$. Note that no boundary conditions were imposed in this case, but nevertheless the problem was ill-posed for $n > 1$ because of the subellipticity of the $\overline{\partial}$-operator on the scale of the Sobolev spaces; cf. [7]. In fact, Romanov [19] constructed an inner product on the Sobolev space $H^1(\mathcal{D})$, providing the same topology as the original one and such that the corresponding adjoint is given by the following improper integral:

$$(\overline{\partial}^\star f)(z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n}\int_{\mathcal{D}}\sum_{j=1}^{n}\frac{f_j(\zeta)(\zeta_j - z_j)\,d\overline{\zeta}\wedge d\zeta}{|\zeta - z|^{2n}}$$

where $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ are complex coordinates in $\mathbb{C}^n$ and $\overline{\zeta} = (\overline{\zeta}_1, \dots, \overline{\zeta}_n)$ is the corresponding complex adjoint vector for ζ.

The scheme was adopted later to study solvability conditions of (1.1) in the situation where $\mathcal{A}$ is an (overdetermined) $(l \times k)$-matrix elliptic operator of order $m \geq 1$ acting from the space of k-vector functions with components in the Sobolev space $H^m(\mathcal{D})$ to the space of l-vector functions with components in the Lebesgue space $L^2(\mathcal{D})$, $l \geq k$, over a bounded domain $\mathcal{D} \subset \mathbb{R}^n$; see [16]. Again, no boundary conditions were imposed in this case.

The case of the Cauchy problem for an (overdetermined) $(l \times k)$-matrix elliptic operator $\mathcal{A}$, $l \geq k$, of order $m \geq 1$ acting from the Sobolev space $H^m(\mathcal{D})$ to the Lebesgue space $L^2(\mathcal{D})$, was considered in [22]. As the Cauchy data were given on a relatively open subset S of the boundary $\partial\mathcal{D}$, the corresponding adjoint operator was constructed with the use of the Green function for the Dirichlet problem in the Sobolev space $H^m(\mathcal{X} \setminus \overline{S})$ related to the Laplacian $\mathcal{A}^*\mathcal{A}$ in a domain $\mathcal{X} \setminus \overline{S}$ with the crack S; here $\mathcal{X}$ is a domain in $\mathbb{R}^n$ containing the closure $\overline{\mathcal{D}}$ of the domain $\mathcal{D}$, and $\mathcal{A}^*$ is the formal adjoint differential operator for $\mathcal{A}$. In the sequel we are going to spread this scheme to proper weighted Sobolev spaces.

2 Weighted Sobolev–Slobodetskii spaces

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\mathcal{D}$, i.e. the surface $\partial\mathcal{D}$ is locally the graph of a Lipschitz function: for each boundary point $p \in \partial\mathcal{D}$ there is a neighborhood U of p in $\mathbb{R}^n$, such that, after a possible rotation, $\mathcal{D} \cap U = \{(x', x^n) \in U : x^n > f(x')\}$, where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, i.e. $|f(x') - f(y')| \le L|x' - y'|$ for all $x', y' \in \mathbb{R}^{n-1}$. The smallest L for which the estimate holds is called the bound of the Lipschitz constants. By choosing finitely many balls $\{U_\nu\}$ covering $\partial\mathcal{D}$, the Lipschitz constant for a Lipschitz domain is the smallest L with the property that the Lipschitz constant is bounded by L for every ball U_ν.

Any bounded Lipschitz domain has actually a global Lipschitz defining function ρ, i.e. $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfies $\rho < 0$ in $\mathcal{D}$, $\rho > 0$ outside $\overline{\mathcal{D}}$, and $c_1 < |\rho'| < c_2$ almost everywhere at $\partial\mathcal{D}$, where c_1, c_2 are positive constants (here ρ' means gradient of ρ). The geometric interpretation of this description is that both $\mathcal{D}$ and $\mathbb{R}^n \setminus \overline{\mathcal{D}}$ are locally situated on exactly one side of the boundary $\partial\mathcal{D}$.

We say that the surface $\partial\mathcal{D}$ is C^k-smooth if ρ is k-times continuously differentiable in a neighborhood of $\partial\mathcal{D}$ and $\nabla\rho \neq 0$ on $\partial\mathcal{D}$. By a piece-wise smooth surface we mean a finite union of C^1-smooth surfaces intersecting transversally.

We consider complex-valued functions defined in the domain $\mathcal{D}$.

For a closed set $\sigma \subset \overline{\mathcal{D}}$ denote by $C^s(\overline{\mathcal{D}}, \sigma)$ the space of all s times continuously differentiable functions on $\overline{\mathcal{D}}$ with compact support in $\overline{\mathcal{D}} \setminus \sigma$. The space $C^\infty(\overline{\mathcal{D}}, \partial\mathcal{D})$ is usually denoted by $C^\infty_0(\mathcal{D})$.

For $1 \le q < \infty$, we write $L^q(\mathcal{D})$ for the space of all (equivalence classes of) measurable functions u in $\mathcal{D}$ such that the Lebesgue integral of $|u|^q$ over $\mathcal{D}$ is finite. When endowed with the norm

$$\|u\|_{L^q(\mathcal{D})} = \Big(\int_{\mathcal{D}} |u|^q \, dx\Big)^{1/q},$$

the space $L^q(\mathcal{D})$ is Banach. As usual, this scale continues to include the case $q = \infty$, too.

More generally, for $s = 1, 2, \ldots$, we denote by $H^s(\mathcal{D})$ the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\int_{\mathcal{D}} \sum_{|\alpha|\le s} |\partial^\alpha u|^2 \, dx\Big)^{1/2},$$

where the sum is over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of norm $|\alpha| := \alpha_1 + \cdots + \alpha_n$ not exceeding s, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x^j$. It is convenient to define $H^s(\mathcal{D}) := L^2(\mathcal{D})$ for $s = 0$. Obviously, every $H^s(\mathcal{D})$ with $s = 0, 1, \ldots$ specifies within $L^2(\mathcal{D})$. In this way we get a scale of Hilbert spaces $H^s(\mathcal{D})$ endowed with inner product

$$(u, v)_{H^s(\mathcal{D})} = \int_{\mathcal{D}} \sum_{|\alpha|\le s} \partial^\alpha u \overline{\partial^\alpha v} \, dx$$

for $u, v \in H^s(\mathcal{D})$.

In order to extend the scale $H^s(\mathcal{D})$ to the fractional values of $s > 0$, one can use an interpolation procedure. There is also a direct construction along more classical lines developed in [27]. Given any non-integer $s > 0$, the so-called Sobolev–Slobodetskii space $H^s(\mathcal{D})$ is defined to be the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\|u\|^2_{H^{[s]}(\mathcal{D})} + \iint_{\mathcal{D}\times\mathcal{D}} \sum_{|\alpha|=[s]} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} \, dx \, dy\Big)^{1/2},$$

where $[s]$ is the integer part of s. The space $H^s(\mathcal{D})$ is endowed with obvious inner product under which it is a Hilbert space.

In the sequel, for a closed subset Γ of $\overline{\mathcal{D}}$, we denote by $H^s(\mathcal{D}, \Gamma)$ the closure of the linear subset $C^\infty(\overline{\mathcal{D}}, \Gamma)$ in $H^s(\mathcal{D})$. When endowed with the induced norm, $H^s(\mathcal{D}, \Gamma)$ is obviously a Hilbert space. If Γ is the whole boundary $\partial\mathcal{D}$, we get what is usually referred to as $H^s_0(\mathcal{D})$.

To define the spaces $H^s(\mathcal{D})$ for all negative $s \in \mathbb{R}$, too, we exploit the standard duality construction that will be used also for the weighted spaces below. More precisely, let H^+ and H^0 be complex Hilbert spaces with inner products $(\cdot,\cdot)_+$ and $(\cdot,\cdot)_0$, respectively. Suppose that H^+ is a subspace of H^0 and the natural inclusion

$\iota : H^+ \to H^0$ is continuous. We also assume that there is a space $\Sigma \subset H^+$ such that Σ is dense in H^+ and $\iota(\Sigma)$ is dense in H^0. Write H^- for the completion of Σ with respect to the norm

$$\|u\|_- = \sup_{\substack{v\in\Sigma \\ v\neq 0}} \frac{|(v, u)_0|}{\|v\|_+}.$$

Then H^- is topologically isomorphic to the dual $(H^+)'$ and the duality is induced by the pairing

$$\langle v, u\rangle_0 = \lim_{\nu\to\infty} (v, u_\nu)_{H^0}, \quad u \in H^-,\ v \in H^+, \tag{2.1}$$

where $\{u_\nu\}$ is a sequence convergent to u in the space H^+; see, for instance, [20, Lemma 3.3] or [2, Chapter 1, Section 1]. Thus, let $H^{-s}(\mathcal{D})$ to be the dual to $H^s(\mathcal{D})$ with respect to the paring induced by $(\cdot,\cdot)_{L^2(\mathcal{D})}$.

Various types of weighted Sobolev spaces are of a long time use in analysis; see, for instance, [32] in an abstract setting or [3] for cone-type singularities and [17, 28] for edge-type singularities in problems related to (pseudo-)differential operators. This theory can be easily spread to the crack domains $\mathcal{D} \setminus \Gamma$ in the case where Γ is a closure of a relatively open connected piece of a sufficiently smooth oriented hyper-surface in $\mathcal{D}$ with piece-wise smooth boundary $\partial\Gamma$; cf. [22, Section 2] for the spaces of type $H^{s,0}(\mathcal{D} \setminus \Gamma)$.

Fix a closed set $\Xi \subset \overline{\mathcal{D}}$ situated on an $(n-1)$-dimensional surface. Let us introduce special weighted Sobolev spaces associated with Ξ. Let ρ be a continuous non-negative function in $\overline{\mathcal{D}}$ such that $0 \le \rho(x) \le 1$ for all $x \in \overline{\mathcal{D}}$ and $\rho(x) = 0$ if and only if $x \in \Xi$. We assume that ρ is sufficiently smooth away from Ξ. More precisely, we require

$$\rho^{|\alpha|-1}\partial^\alpha \rho \in L^\infty(\mathcal{D}) \tag{2.2}$$

for all multi-indices $\alpha \in \mathbb{Z}^n_{\ge 0}$ satisfying $|\alpha| \le k$ with some $k \in \mathbb{N}$. By working on the whole scale of the weighted Sobolev-type spaces parametrized by the smoothness index $s \in \mathbb{R}$, it is natural to assume that ρ is C^∞-smooth away from Ξ and that (2.2) is valid for any $k \in \mathbb{N}$, while by handling one particular space of finite smoothness s, it is always enough to assume that $k \ge |s|$. Estimates (2.2) guarantee various important properties of weighted Sobolev spaces with weight function ρ. One may think of $\rho(x)$ as the distance from x to Ξ locally near Ξ in $\overline{\mathcal{D}}$. If the set Ξ is empty, we choose $\rho \equiv 1$.

Let s be a non-negative integer and $\gamma \in \mathbb{R}$. On smooth functions with compact support in $\overline{\mathcal{D}} \setminus \Xi$ we introduce the inner product

$$(u, v)_{\mathcal{H}^{s,\gamma}(\mathcal{D})} = \int_{\mathcal{D}} \rho^{-2\gamma} \sum_{|\alpha|\le s} \rho^{2|\alpha|}\partial^\alpha u \overline{\partial^\alpha v}\, dx$$

and denote by $\mathcal{H}^{s,\gamma}(\mathcal{D})$ the completion of $C^\infty(\overline{\mathcal{D}}, \Xi)$ with respect to the corresponding norm. Of course, this definition essentially depends on the choice of the function ρ (see Theorem 1.4) and we should better write $\mathcal{H}^{s,\gamma}_\rho(\mathcal{D})$. However, we keep the symbol $\mathcal{H}^{s,\gamma}(\mathcal{D})$ for the sake of simplicity of the notations. By the very construction, $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is a Hilbert space.

From the definition of $\mathcal{H}^{s,\gamma}(\mathcal{D})$ it follows readily that if $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, then

$$\partial^\alpha u \in \mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})$$

for all multi-indices α with $|\alpha| \le s$. Although $\mathcal{H}^{0,0}(\mathcal{D}) = H^0(\mathcal{D})$, the space $\mathcal{H}^{s,0}(\mathcal{D})$ does not coincide with $H^s(\mathcal{D}, \Xi)$ for integer $s > 0$. Besides, $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$ for all $\gamma \ge 0$.

To introduce weighted Sobolev spaces of fractional smoothness $s > 0$, i.e. for functions $u \in C^\infty(\overline{\mathcal{D}}, \Xi)$, we consider the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} = \big(\|u\|^2_{\mathcal{H}^{[s],\gamma}(\mathcal{D})} + \|\rho^{s-\gamma}u\|^2_{H^s(\mathcal{D})}\big)^{1/2}.$$

Write $\mathcal{H}^{s,\gamma}(\mathcal{D})$ for the completion of the space $C^\infty(\overline{\mathcal{D}}, \Xi)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$.

If $0 < s < 1$, then on this way the space $\mathcal{H}^{s,\gamma}(\partial\mathcal{D})$ can be naturally defined by using the Lebesgue space $L^2(\partial\mathcal{D})$ instead of $L^2(\mathcal{D})$. More precisely, we assume that $\Xi \cap \partial\mathcal{D}$ is situated on a piece-wise smooth $(n-2)$-dimensional surface in the C^1-smooth surface $\partial\mathcal{D}$. Let ds stand for the area form on $\partial\mathcal{D}$ induced by the Lebesgue measure in $\mathbb{R}^n$. We introduce the inner product

$$(u, v)_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} = \int_{\partial\mathcal{D}} \rho^{-2\gamma} u\overline{v}\, ds$$

for $u, v \in C(\partial\mathcal{D}, \Xi)$. Denote by $\mathcal{H}^{0,\gamma}(\partial\mathcal{D})$ the completion of $C(\partial\mathcal{D}, \Xi)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} = \Big(\int\limits_{\partial\mathcal{D}} \rho^{-2\gamma} |u|^2 \, ds \Big)^{1/2}.$$

Then, for $0 < s < 1$, we write $\mathcal{H}^{s,\gamma}(\partial\mathcal{D})$ for the completion of $C^{0,1}(\partial\mathcal{D}, \Xi)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\partial\mathcal{D})} = \big(\|u\|^2_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} + \|\rho^{s-\gamma} u\|^2_{H^s(\partial\mathcal{D})}\big)^{1/2}.$$

For $s \geq 1$ this can be done with the use of a proper partition of unity.

The following re-indexation allows one to distinguish important natural embeddings. Namely, for each $s \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \mathbb{R}$, we have $\mathcal{H}^{s,\gamma}(\mathcal{D}) = \mathcal{H}^{s,(\gamma-s)+s}(\mathcal{D})$. Then we set

$$H^{0,\gamma}(\mathcal{D}) =: \mathcal{H}^{0,\gamma}(\mathcal{D}), \quad H^{s,\gamma}(\mathcal{D}) =: \mathcal{H}^{s,s+\gamma}(\mathcal{D})$$

for $s \in \mathbb{Z}_{\geq 0}$. The similar re-indexation will be used for the spaces on $\partial\mathcal{D}$.

According to the explanations above, $\mathcal{H}^{-s,-\gamma}(\mathcal{D})$ and $H^{-s,\gamma}(\mathcal{D})$ are the completions of $C^\infty(\overline{\mathcal{D}}, \Xi)$ in the norms

$$\|u\|_{\mathcal{H}^{-s,-\gamma}(\mathcal{D})} = \sup_{\substack{v \in \mathcal{H}^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}},$$

$$\|u\|_{H^{-s,\gamma}(\mathcal{D})} = \sup_{\substack{v \in H^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma}(\mathcal{D})}},$$

respectively. The re-indexing relation between the weighted spaces $H^{s,\gamma}(\mathcal{D})$ and $\mathcal{H}^{s,s+\gamma}(\mathcal{D})$ still holds for negative s; see [28, Lemma 4.5].

For particular configurations of singularities Ξ, if we choose as ρ the distance to Ξ in a suitable coordinate system, then the scale of Hilbert spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ coincides with that used in [3] for cone-type singularities and in [17] for edge-type singularities, the only difference being in indexing the spaces. Let us briefly list the most important properties of the weighted Sobolev scale $H^{s,\gamma}(\mathcal{D})$ that we will mostly use; see [28].

First of all, we note that the multiplication on the weight function induces a bounded operator on the scale; see [28, Theorem 3.6 and Corollary 4.3].

Theorem 2.1. *Let $s \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{N}$, $k \geq s$, and let $\rho \in C^k(\overline{\mathcal{D}} \setminus \Xi)$ satisfy* (2.2) *for all multi-indices α with $|\alpha| \leq k$. Then, for any $\delta \in \mathbb{R}$, the correspondence*

$$\operatorname{Op}(\rho^\delta) : u \mapsto \rho^\delta u$$

induces topological isomorphisms

$$H^{s,\gamma}(\mathcal{D}) \to H^{s,\gamma+\delta}(\mathcal{D}), \quad H^{-s,\gamma}(\mathcal{D}) \to H^{-s,\gamma+\delta}(\mathcal{D}).$$

Next we see that there are natural classes of differential operators acting on the scale; see [28, Lemma 3.10].

Lemma 2.2. *Let $\rho \in C^s(\overline{\mathcal{D}} \setminus \Xi)$ and* (2.2) *hold for $|\alpha| \leq s$, where $s \in \mathbb{Z}_{\geq 0}$. Then any differential operator*

$$\sum_{|\alpha| \leq m} a_\alpha(x) \rho^{|\alpha|-m}(x) \partial^\alpha$$

of order $m \leq s$ with coefficients a_α of class $C^s(\overline{\mathcal{D}})$ maps $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously to $\mathcal{H}^{s-m,\gamma-m}(\mathcal{D})$ and $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-m,\gamma}(\mathcal{D})$.

Similarly to the usual Sobolev spaces, the weighted ones admit adequate theorems on continuous and compact embedding including theorems on traces (see [28, Section 5]). In particular, we have the following basic embedding theorem; see [28, Theorem 5.5]

Theorem 2.3. *Let $s \geq s' > 0$, $k \in \mathbb{N}$, $k \geq s$, and let $\rho \in C^k(\overline{\mathcal{D}} \setminus \Xi)$ satisfy* (2.2) *for all multi-indices α with $|\alpha| \leq k$. Then the normed space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s',\gamma}(\mathcal{D})$. The embedding is compact if $s > s'$.*

However, the most advanced statements correspond to the very natural case where the geometry of the domain is involved; see [28, Section 6].

Indeed, we are to impose additional conditions on the function ρ that lead us to typical situations of analysis on manifolds with singularities. Namely, assume that there is a neighborhood U of the set Ξ in $\overline{\mathcal{D}}$ and C^1-smooth functions $\rho_1, \dots, \rho_{n-1}$ in U such that

$$|d\rho_1(x) \wedge \dots \wedge d\rho_{n-1}(x) \wedge d\rho(x)| \geq c \tag{2.3}$$

for all $x \in U \setminus \Xi$, where $c > 0$ is a constant independent of x. Note that the differential form in (2.3) has the form $(\det J(x))\, dx$, where $J(x)$ is the Jacobi matrix of the functional system $\rho_1, \dots, \rho_{n-1}, \rho$. Hence, condition (2.3) means that the modulus of $\det J$ is bounded away from zero in $U \setminus \Xi$. Thus, ρ can be completed to a coordinate system in U as a new singular coordinate. By the very nature, the analytical condition (2.3) corresponds to a (possible) singularity Ξ of $\partial\mathcal{D}$ similar to transversal intersection (like conic points or edges). The proof of the following theorem can be found in [22, Corollary 2.2] or [28, Theorem 6.1]. However, the surface $\partial\mathcal{D}$ can be also C^1-smooth and the singularity Ξ can be artificial.

Theorem 2.4. *Let $s > 0$, $k \in \mathbb{N}$, $k \geq s$, and let $\rho \in C^k(\overline{\mathcal{D}} \setminus \Xi)$ satisfy (2.2) for all multi-indices α with $|\alpha| \leq k$. Then the space $\mathcal{H}^{s,s}(\mathcal{D}) = H^{s,0}(\mathcal{D})$ is continuously embedded into $H^s(\mathcal{D}, \Xi)$. If, in addition, (2.3) holds, then the normed spaces $\mathcal{H}^{s,s}(\mathcal{D})$, $H^{s,0}(\mathcal{D})$ and $H^s(\mathcal{D}, \Xi)$ are isomorphic.*

Theorems 2.1 and 2.4 suggest that under condition (2.3) any bounded linear operator B acting from the space $H^s(\mathcal{D}, \Xi)$ to the space $H^{s'}(\mathcal{D}, \Xi)$ induces a bounded linear operator $B^{(\gamma,\gamma')}$ acting from the space $H^{s,\gamma}(\mathcal{D})$ to the space $H^{s',\gamma'}(\mathcal{D})$:

$$B^{(\gamma,\gamma')} : \mathrm{Op}(\rho^{\gamma'}) \circ B \circ \mathrm{Op}(\rho^{-\gamma}).$$

Because in this case, $\mathrm{Op}(\rho^{-\gamma})$ maps $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s,0}(\mathcal{D}) = H^s(\mathcal{D}, \Xi)$ and $\mathrm{Op}(\rho^{\gamma'})$ maps

$$H^{s',\gamma'}(\mathcal{D}, \Xi) = H^{s',0}(\mathcal{D})$$

continuously to the space $H^{s',\gamma'}(\mathcal{D})$. Thus, these operators are just translations from the space of bounded operators on the scale of the usual Sobolev spaces to the space of bounded operators on the scale of weighted spaces. It is convenient to set

$$B^{(\gamma)}u := \mathrm{Op}(\rho^{\gamma}) \circ B \circ \mathrm{Op}(\rho^{-\gamma}). \tag{2.4}$$

For the Sobolev spaces $H^s(\mathcal{D})$ with $s > \frac{1}{2}$, there is a well-defined bounded linear trace operator

$$t_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^{s-1/2}(\partial\mathcal{D});$$

see for instance [27]. We will omit the index $\mathcal{D}$ and write just t_s if it causes no misunderstandings. Thus, if $\Xi \cap \partial\mathcal{D}$ is situated on an $(n-2)$-dimensional surface in $\partial\mathcal{D}$, then t_1 maps $H^1(\mathcal{D}, \Xi)$ continuously to $H^{1/2}(\partial\mathcal{D}, \Xi)$ and the trace operator

$$t_1^{(\gamma)} : H^{1,\gamma}(\mathcal{D}) \to H^{1/2,\gamma}(\partial\mathcal{D})$$

is well-defined by (2.4). Note that $H^{1/2}(\partial\mathcal{D}, \Xi) = H^{1/2}(\partial\mathcal{D})$ if Ξ is situated on an $(n-2)$-dimensional surface.

Theorem 2.5. *Suppose that $\partial\mathcal{D}$ is a Lipschitz surface, $\Xi \cap \partial\mathcal{D}$ is situated on an $(n-2)$-dimensional surface in $\partial\mathcal{D}$, and $\rho \in C^1(\overline{\mathcal{D}} \setminus \Xi)$ and $\rho' \in L^\infty(\mathcal{D})$. If the norms of the spaces $H^{1,0}(\mathcal{D})$ and $H^1(\mathcal{D}, \Xi)$ are equivalent, then formula (2.4) induces the bounded trace operator $t_1^{(\gamma)} : H^{1,\gamma}(\mathcal{D}) \to H^{1/2,\gamma}(\partial\mathcal{D})$ possessing a bounded right inverse $(t_1^{(\gamma)})_r^{-1} : H^{1/2,\gamma}(\partial\mathcal{D}) \to H^{1,\gamma}(\mathcal{D})$.*

Similarly to the usual Sobolev spaces, for a closed set Γ on $\partial\mathcal{D}$ we denote by $H^{s,\gamma}(\mathcal{D}, \Gamma)$ the closure of the subspace $C^\infty(\overline{\mathcal{D}}, \Gamma)$ in $H^{s,\gamma}(\mathcal{D})$.

Corollary 2.6. *Let the norms of the spaces $H^{1,0}(\mathcal{D})$ and $H^1(\mathcal{D}, \Xi)$ be equivalent. If Γ is a closure of a relatively open connected piece of a C^1-smooth oriented hypersurface in $\partial\mathcal{D}$ with piece-wise smooth boundary $\partial\Gamma$, then*

$$H^{1,\gamma}(\mathcal{D}, \Gamma) = \{u \in H^{1,\gamma}(\mathcal{D}) : t_1^{(\gamma)} u = 0 \text{ on } \Gamma\}.$$

In particular, $H^{1,\gamma}(\mathcal{D}, \partial\mathcal{D})$ is the closure of smooth functions with compact support in $\mathcal{D}$.

Proof. According to the results of [6],

$$H^1(\mathcal{D},\Gamma)=\{u\in H^1(\mathcal{D}) : t_1 u=0 \text{ on } \Gamma\}.$$

Thus, it is left to apply Theorem 2.1 to conclude that the statement of the theorem is true. □

In the sequel we will always assume that (2.3) is fulfilled. Now we are ready to formulate the Cauchy problem in weighted Sobolev spaces.

3 The Cauchy problem in weighted Sobolev spaces

Let $\mathcal{X}$ be a domain with C^∞-smooth boundary $\partial\mathcal{X}$ in $\mathbb{R}^n$, containing the closure $\overline{\mathcal{D}}$ of a bounded domain $\mathcal{D}$ with piece-wise smooth boundary $\partial\mathcal{D}$. Let also $E=\mathcal{X}\times\mathbb{C}^k$ and $F=\mathcal{X}\times\mathbb{C}^l$ be trivial vector bundles over $\mathcal{X}$. The space of sections of the bundle E of a function class $\mathfrak{C}(\mathcal{D})$ will be denoted by $\mathfrak{C}_E(\mathcal{D})$. As the bundle E is trivial, $\mathfrak{C}_E(\mathcal{D})$ consists of $(k\times 1)$-columns with components from $\mathfrak{C}(\mathcal{D})$.

Consider a first-order (weighted) differential operator

$$\mathcal{A}=\mathcal{A}(x,\partial)=\sum_{j=1}^{n}\mathcal{A}_j(x)\partial_j+\mathcal{A}_0(x)\rho^{-1}(x) \tag{3.1}$$

mapping sections of E to sections of F. This means that $\mathcal{A}_j$ are $(l\times k)$-matrices with C^∞-smooth components over $\overline{\mathcal{X}}$. We assume that the operator $\mathcal{A}$ is (overdetermined) elliptic, i.e. $l\geq k$ and the map

$$\sigma(\mathcal{A})(x,\xi)=\sum_{j=1}^{n}\mathcal{A}_j(x)\xi_j : \mathbb{C}^k\to\mathbb{C}^l \tag{3.2}$$

is injective for all $x\in\overline{\mathcal{X}}$ and all $\xi\in\mathbb{R}^n\setminus\{0\}$.

Easily, the operator $\mathcal{A}$ maps $H^{1,\gamma}_E(\mathcal{D})$ continuously to $H^{0,\gamma}_F(\mathcal{D})$. Moreover, Theorem 2.5 gives the possibility to consider the Cauchy problem with data on a subset $S\subset\partial\mathcal{D}$: given $g\in H^{0,\gamma}_F(\mathcal{D})$ and $v_0\in H^{1/2,\gamma}_E(\partial\mathcal{D})$, find (if possible) $u\in H^{1,\gamma}_F(\mathcal{D})$ satisfying

$$\begin{cases}\mathcal{A}v=g & \text{in } \mathcal{D},\\ t_1^{(\gamma)}(v)=v_0 & \text{on } S.\end{cases} \tag{3.3}$$

We tacitly assume that S is a relatively open subset of $\partial\mathcal{D}$ with piece-wise smooth boundary ∂S and that ∂S coincides with the singular set Ξ.

First, we easily indicate a class of operators with the Uniqueness Theorem for the Cauchy problem.

Lemma 3.1. *Let the entries of the matrices $\mathcal{A}_i$, $0\leq i\leq n$, be real analytic in a neighborhood of $\overline{\mathcal{D}}$ and let ρ be real analytic outside of Ξ. If S contains a relatively open subset of the piece-wise smooth surface $\partial\mathcal{D}$ then problem* (3.3) *has no more than one solution in $H^{1,\gamma}_E(\mathcal{D})$.*

Proof. As the entries of the matrices $\mathcal{A}_i$, $0\leq i\leq n$, and the function ρ are real analytic in $\mathcal{D}$, the Holmgren Theorem implies that any solution v to (3.3) with zero data is actually real analytic in $\mathcal{D}$, too.

Next, since S contains a relatively open subset of $\partial\mathcal{D}$, there is a domain $\tilde{\mathcal{D}}\subset\mathcal{D}$ with piece-wise smooth boundary and such that $\overline{\tilde{\mathcal{D}}}\cap\Xi=\emptyset$ and the set $S_1=\partial\tilde{\mathcal{D}}\cap S$ is relatively open in $\partial\tilde{\mathcal{D}}$. Then any solution v to (3.3) with zero data actually belongs to $H^1_E(\tilde{\mathcal{D}})$ and satisfies

$$\begin{cases}\mathcal{A}v=0 & \text{in } \tilde{\mathcal{D}},\\ t_1(v)=0 & \text{on } S_1.\end{cases}$$

Hence v is identically zero in $\tilde{\mathcal{D}}$ according to the Uniqueness Theorem for the Cauchy problem for elliptic operators in the Sobolev spaces; see [25, Theorem 2.8]. Then v equals zero identically in $\mathcal{D}$ because v is real analytic there. □

Next we indicate a reasonable situation where the Cauchy problem is densely solvable.

Lemma 3.2. *Let $l = k$, let the entries of the matrices $\mathcal{A}_i$, $0 \le i \le n$, be real analytic and let ρ be real analytic outside of Ξ. If $\partial\mathcal{D}$ is C^∞-smooth and $\partial\mathcal{D} \setminus \overline{S}$ contains a relatively open subset of $\partial\mathcal{D}$, then the range of the bounded linear map*

$$\mathcal{A} : H^{1,\gamma}_E(\mathcal{D}, \overline{S}) \to H^{0,\gamma}_F(\mathcal{D}). \tag{3.4}$$

is dense in $H^{0,\gamma}_F(\mathcal{D})$.

Proof. We shall have established the theorem if we prove that the orthogonal complement $H^\perp$ of the range of the map (3.4) in $H^{0,\gamma}_F(\mathcal{D})$ is zero. Indeed, if $f \in H^\perp$, then

$$(\mathcal{A}v, f)_{H^{0,\gamma}_F(\mathcal{D})} = 0 \tag{3.5}$$

for all $v \in H^{1,\gamma}_E(\mathcal{D}, \overline{S})$. As $C^\infty_0(\mathcal{D})$ lies in $H^{1,\gamma}(\mathcal{D}, \overline{S})$, we see that $\mathcal{A}^*(\rho^{-2\gamma} f) = 0$ in $\mathcal{D}$, where

$$\mathcal{A}^* = -\sum_{j=1}^n \mathcal{A}^*_j(x)\partial_j - \sum_{j=1}^n (\partial_j \mathcal{A}^*_j)(x) + \mathcal{A}^*_0(x)\rho^{-1}(x)$$

is the formal adjoint for $\mathcal{A}$ and $\mathcal{A}^*_j$ is the adjoint matrix for $\mathcal{A}_j$. Since $l = k$, the operator $\mathcal{A}^*$ is elliptic in $\mathcal{D}$. Moreover, as the entries of the matrices $\mathcal{A}_i$, $0 \le i \le n$, and the function ρ are real analytic, we see that the function f is real analytic in D because of the Holmgren Theorem. Then the function $(\rho^{-2\gamma} f)$ is real analytic in $\mathcal{D}$, too, because $\rho \neq 0$ in $\mathcal{D}$.

Next, since $\partial\mathcal{D} \setminus \overline{S}$ contains a relatively open subset of $\partial\mathcal{D}$, there is a domain $\hat{\mathcal{D}} \subset \mathcal{D}$ with C^∞-smooth boundary and such that $\hat{\mathcal{D}} \cap \Xi = \emptyset$ and the set $S_2 = \partial\hat{\mathcal{D}} \cap (\mathcal{D} \setminus \overline{S})$ is relatively open in $\partial\hat{\mathcal{D}}$. Then f belongs to $L^2_F(\hat{\mathcal{D}})$.

Let η be a defining function for the domain $\hat{\mathcal{D}}$:

$$\hat{\mathcal{D}} = \{x \in \mathbb{R}^n : \eta(x) < 0\}.$$

As $\partial\hat{\mathcal{D}} \in C^\infty$, we may choose $\eta \in C^\infty(U)$ for a neighborhood U of $\partial\hat{\mathcal{D}}$ with $|\nabla\eta| \neq 0$ on $\partial\hat{\mathcal{D}}$ such that the vector field $\nu = \frac{\nabla\eta}{|\nabla\eta|}$ is the unit exterior normal vector field to $\partial\hat{\mathcal{D}}$. For sufficiently small $\delta > 0$, consider domains

$$\hat{\mathcal{D}}_\delta = \{x \in \mathbb{R}^n : \eta(x) < -\delta\}.$$

Clearly, $\hat{\mathcal{D}}_\delta \Subset \hat{\mathcal{D}}$ and $\eta_\delta = \eta + \delta$ is a defining function for the domain $\hat{\mathcal{D}}_\delta$ with $\nabla\eta = \nabla\eta_\delta$ and $\nu_\delta = \frac{\nabla\eta}{|\nabla\eta|}$ being the unit exterior normal vector field to $\partial\hat{\mathcal{D}}_\delta$. Then the vector function f has weak boundary value $f_0 \in \mathcal{D}'_F(\partial\hat{\mathcal{D}})$ on $\partial\mathcal{D}$ in the sense that

$$\lim_{\delta \to +0} \int_{\partial\hat{\mathcal{D}}_\delta} \sum_{j=1}^n f(y)\mathcal{A}_j(y)v(y)\, dy[j] = \langle f_0, v\rangle \quad \text{for all } C^\infty_E(\overline{\hat{\mathcal{D}}});$$

see [25, Theorem 4.4] (here the space $\mathcal{D}'(\partial\hat{\mathcal{D}})$ is the space of distributions on $\partial\hat{\mathcal{D}}$, $\langle f_0, v\rangle$ denotes the action of the distribution f_0 on a test function v, $dy[j] = dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n$ and $\wedge$ is the exterior product defined on differential forms).

However, the Stokes' formula for differential forms and (3.5) imply that $f_0 = 0$ on S_2, and then

$$\begin{cases} \mathcal{A}^* f = 0 & \text{in } \hat{\mathcal{D}}, \\ f = 0 & \text{on } S_2. \end{cases}$$

Hence f is identically zero in $\hat{\mathcal{D}}$ according to the Uniqueness Theorem for the Cauchy problem for elliptic systems in the Lebesgue spaces; see [25, Theorem 2.8]. Then f equals zero identically in $\mathcal{D}$ because f is real analytic there. □

Clearly, $\mathcal{A}$ maps $H^{1,\gamma}_E(\mathcal{D}, \overline{S})$ continuously to $H^{0,\gamma}(\mathcal{D})$ and we may consider the operator equation (1.1) in the situation where $H_1 = H^{1,\gamma}_E(\mathcal{D}, \overline{S})$ and $H_2 = H^{0,\gamma}_F(\mathcal{D})$: given $f \in H^{0,\gamma}_F(\mathcal{D})$, find (if possible) $u \in H^{1,\gamma}_E(\mathcal{D}, \overline{S})$

satisfying (1.1). Of course, this corresponds to the Cauchy problem (3.3) with zero boundary data. Let us reduce the Cauchy problem (3.3) to this particular case. With this purpose, we note that, according to Theorem 2.1, for any $u_0 \in H^{1/2,\gamma}_E(S)$ the element $\rho^{-\gamma} u_0$ belongs to $H^{1/2}(S, \Xi)$. If S is a relatively open subset on the C^1-smooth hyper-surface $\partial\mathcal{D}$ and $\partial S = \Xi$ is a piece-wise smooth $(n-2)$-dimensional surface, then, according to the Whitney Extension Theorem, there is a bounded extension operator $e : H^{1/2}_E(S) :\to H^{1/2}_E(\partial\mathcal{D})$. Hence the operator $e^{(\gamma)} = \rho^\gamma e \rho^{-\gamma}$ maps $H^{1/2,\gamma}_E(S)$ continuously to $H^{1/2,\gamma}_E(\partial\mathcal{D})$.

Lemma 3.3. *Let S be a relatively open piece of C^1-smooth hyper-surface $\partial\mathcal{D}$ and let $\partial S = \Xi$ be a piece-wise smooth $(n-2)$-dimensional surface.*
(i) *The null-space* $\ker(\mathcal{A})$ *of problem* (3.3) *coincides with the space*

$$\{u \in H^{1,\gamma}(\mathcal{D}, \overline{S}) : \mathcal{A}u = 0 \text{ in } \mathcal{D}\}.$$

(ii) *Problem* (3.3) *is solvable if and only if the operator equation* (1.1) *is solvable for* $f = g - \mathcal{A}(t^{(\gamma)}_1)^{-1}_r e^{(\gamma)}(u_0)$.

Proof. Indeed, the first statement follows from Corollary 2.6 immediately.

Let (1.1) be solvable for $f = g - \mathcal{A}(t^{(\gamma)}_1)^{-1}_r e^{(\gamma)}(u_0)$ and let u be its solution. Then, according to Corollary 2.6, the vector $v = u + (t^{(\gamma)}_1)^{-1}_r e^{(\gamma)}(u_0)$ is a solution to problem (3.3). Similarly, if problem (3.3) is solvable and v is its solution then, according to Corollary 2.6, the vector $u = v - (t^{(\gamma)}_1)^{-1}_r e^{(\gamma)}(u_0)$ is a solution to the operator equation (1.1) with $f = g - \mathcal{A}(t^{(\gamma)}_1)^{-1}_r e^{(\gamma)}(u_0)$. □

4 A Dirichlet problem in a domain with crack

The aim of this section is to prepare a background for the next steps and to make the operator $(t^{(\gamma)}_1)^{-1}_r$ more visible. With this purpose we invoke the Hodge theory for the Dirichlet problem in the weighted Sobolev space related to strongly elliptic operators on manifolds with a (possibly empty) crack; see, for instance, [21, 22].

Fix a C^1-smooth oriented $(n-1)$-dimensional surface $\tilde{\Gamma}$ in $\mathcal{X}$. Let $\Gamma \subset \mathcal{X}$ be the closure of a relatively open connected piece of $\tilde{\Gamma}$ with piece-wise smooth boundary $\partial\Gamma = \Xi$. We are going to consider the domain $\mathcal{X} \setminus \Gamma$ as a manifold with crack Γ. Next, for each domain $\Omega \subset \mathcal{X}$ such that either $\Gamma \subset \Omega$ or $\Gamma \subset \partial\Omega$ and Ω has Lipschitz boundary (the case $\Omega = \mathcal{X}$ is included), we introduce a first-order Hermitian form

$$\begin{aligned} h_{\Omega,\gamma}(u, v) = &\sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i v)_{H^{0,\gamma}(\Omega)} + (a_{0,0}u, v)_{H^{0,\gamma+1}(\Omega)} \\ &+ \sum_{j=1}^{n} \big((a_{j,0}\partial_j u, \rho^{-1}v)_{H^{0,\gamma}(\Omega)} + (\rho^{-1}u, a_{j,0}\partial_j v)_{H^{0,\gamma}(\Omega)}\big) \end{aligned}$$

on the space $H^{1,\gamma}_E(\Omega)$ where the coefficients $a_{i,j}$ are assumed to be $(k \times k)$-matrices with complex-valued entries of class $L^\infty(\mathcal{X})$.

Let $\mathcal{H}_\gamma(\Omega)$ stand for the subspace in $H^{1,\gamma}_E(\Omega, \partial\Omega)$ consisting of all functions w, satisfying

$$h_{\Omega,\gamma}(w, v) = 0 \quad \text{for all } v \in H^{1,\gamma}_E(\Omega, \partial\Omega). \tag{4.1}$$

Lemma 4.1. *We have $w \in \mathcal{H}_\gamma(\Omega)$ if and only if $h_{\Omega,\gamma}(w, w) = 0$.*

Proof. It follows from (4.1) that $h_{\Omega,\gamma}(w, w) = 0$ for each $w \in \mathcal{H}_\gamma(\Omega)$.

Let $w \in H^{1,\gamma}_E(\Omega, \partial\Omega)$ and $h_{\Omega,\gamma}(w, w) = 0$. As $h_{\Omega,\gamma}(\cdot\,,\cdot)$ is a Hermitian form, we may apply the generalized Cauchy inequality to conclude that

$$|h_{\Omega,\gamma}(v, w)|^2 \le h_{\Omega,\gamma}(w, w) h_{\Omega,\gamma}(v, v) = 0$$

for each $v \in \mathcal{H}_\gamma(\Omega)$, i.e. $v \in \mathcal{H}_\gamma(\Omega)$. □

By definition, $H^{1,\gamma}(\Omega)$ is continuously embedded into $H^{0,\gamma+1}(\Omega)$, and hence the space $\mathcal{H}_\gamma(\Omega)$ is continuously embedded into $H^{0,\gamma}_E(\Omega)$ because of Theorem 2.3. Let $\Pi^{(\Omega)}_\gamma$ denote the orthogonal projection from $H^{0,\gamma}_E(\Omega)$

to $\mathcal{H}_\gamma(\Omega)$. We assume that there is a positive constant m_Ω such that

$$\|u\|^2_{H^{1,\gamma}_E(\Omega)} \leq m_\Omega\big(h_{\Omega,\gamma}(u,u) + \|u\|^2_{H^{0,\gamma}_E(\Omega)}\big) \tag{4.2}$$

for all $u \in H^{1,\gamma}_E(\Omega, \partial\Omega)$. This assumption means that the Hermitian form $h_{\mathcal{X},\gamma}(\cdot,\cdot)$ induces a Fredholm Dirichlet problem for the second-order (weighted) differential operator

$$\Delta_\gamma = -\sum_{i,j=1}^{n} \partial_i(a_{i,j}\partial_j u) + \sum_{j=1}^{n}\Big(\frac{a_{j,0}\partial_j u}{\rho} - \partial_j\Big(\frac{a^*_{j,0}u}{\rho}\Big)\Big) + \frac{a_{0,0}u}{\rho^2} \tag{4.3}$$

in the (possibly cracked) domain Ω that maps $H^{1,\gamma}(\Omega, \partial\Omega)$ continuously to the dual space for the space $H^{1,\gamma}(\Omega, \partial\Omega)$. Of course, there are other ways to provide the Fredholm property of a boundary value problem in weighted spaces (see [17, 21]) but it is difficult to check their abstract assumptions in a particular situation. To be more precise, let $\tilde{H}^{-1,\gamma}(\Omega)$ stand for the completion of the space $H^{1,\gamma}_E(\Omega, \partial\Omega)$ with respect to the norm

$$\|u\|_{\tilde{H}^{-1,\gamma}(\Omega)} = \sup_{\substack{v\in H^{1,\gamma}(\Omega,\partial\Omega)\\ v\neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\Omega)}|}{\|v\|_{H^{1,\gamma}(\Omega)}}.$$

As we have explained in Section 2 the space $\tilde{H}^{-1,\gamma}(\Omega)$ can be identified with the dual for the space $H^{1,\gamma}(\Omega, \partial\Omega)$ (see [20] or [2, Chapter 1, Section 1]). Consider the problem: given $f \in \tilde{H}^{-1,\gamma}_E(\Omega)$, find $u \in H^{1,\gamma}_E(\Omega, \Omega)$ satisfying

$$h_{\Omega,\gamma}(u, v) = \langle f, v\rangle_{\Omega,\gamma} \tag{4.4}$$

for all $v \in \mathcal{H}^{1,\gamma}(\Omega, \partial\Omega)$, where $\langle\cdot,\cdot\rangle_{\Omega,\gamma}$ is the pairing (2.1) in the situation with

$$H^0 = H^{0,\gamma}_E(\Omega) \quad \text{and} \quad H^+ = H^{1,\gamma}_E(\Omega, \partial\Omega).$$

As usual, this generalized setting may be interpreted as follows:

$$\begin{cases} \Delta_\gamma u = f & \text{in}\,\Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 4.2. *Under assumption* (4.2)*, the Dirichlet problem* (4.4) *is a Fredholm one. Its kernel coincides with the finite-dimensional space* $\mathcal{H}_\gamma(\Omega)$ *and it is solvable if and only if*

$$\langle f, v\rangle_{\Omega,\gamma} = 0 \quad \textit{for all } v \in \mathcal{H}_\gamma(\Omega). \tag{4.5}$$

Moreover, there is a bounded linear operator

$$\Phi^{(\Omega)}_\gamma : \tilde{H}^{-1,\gamma}_E(\Omega) \to H^{1,\gamma}_E(\Omega, \partial\Omega)$$

such that $\Phi^{(\Omega)}_\gamma \Pi^{(\Omega)}_\gamma = \Pi^{(\Omega)}_\gamma \Phi^{(\Omega)}_\gamma = 0$ *and*

$$\Phi^{(\Omega)}_\gamma \Delta_\gamma u = u - \Pi^{(\Omega)}_\gamma u \quad \textit{for all } u \in H^{1,\gamma}_E(\Omega, \partial\Omega),$$
$$\Delta_\gamma \Phi^{(\Omega)}_\gamma f = f - \Pi^{(\Omega)}_\gamma f \quad \textit{for all } f \in \tilde{H}^{-1,\gamma}_E(\Omega).$$

Proof. Note that (4.2) yields $\mathcal{H}_\gamma(\Omega) \subset H^{1,\gamma}_E(\Omega)$. Moreover, $\mathcal{H}_\gamma(\Omega)$ is the kernel of problem (4.4) by the very definition of the space, and (4.5) is necessary for the Dirichlet problem to be solvable by the very setting of the problem.

Actually, the compact embedding

$$H^{1,\gamma}_E(\Omega) \to H^{0,\gamma}_E(\Omega), \tag{4.6}$$

granted by Theorem 2.3, allows to keep the same proof for all the statements of Theorem 4.2 as in the classical Hodge Theorem for the Dirichlet problem on manifolds with crack in the usual Sobolev spaces; cf. [22, Lemma 3.2 and Theorem 3.3].

Indeed, as the coefficients $a_{i,j}$ belong to $L^\infty(\mathcal{X})$, we see that $h_\Omega(u,u)$ is dominated by $\|u\|_{H^{1,\gamma}(\Omega)}$. Next, we may argue by contradiction. If the dimension of the space $\mathcal{H}_\gamma(\Omega)$ is infinite, then it admits a countable orthogonal system, say, $\{b_\nu\} \subset \mathcal{H}_\gamma(\Omega)$ satisfying

$$\|b_\nu\|_{H^{1,\gamma}_E(\Omega)} = 1.$$

As any orthonormal system weakly converges to zero and the embedding (4.6) is compact, $\{b_\nu\}$ converges to zero in $H^{0,\gamma}_E(\Omega)$. This contradicts with (4.2) because then

$$1 = \|b_\nu\|_{H^{1,\gamma}_E(\Omega)} \leq m\|b_\nu\|_{H^{0,\gamma}_E(\Omega)} \to 0 \quad \text{as } \nu \to \infty.$$

Thus, the space $\mathcal{H}_\gamma(\Omega)$ is finite-dimensional.

Let $\mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$ consist of all the functions $u \in H^{1,\gamma}_E(\Omega, \partial\Omega)$ satisfying

$$(u, v)_{H^{0,\gamma}_E(\Omega)} = 0 \quad \text{for all } v \in \mathcal{H}_\gamma(\Omega).$$

A familiar argument shows that there is a constant $\tilde{m}_\Omega > 0$ with the property that

$$\|u\|^2_{H^{1,\gamma}_E(\Omega)} \leq \tilde{m}_\Omega h_{\Omega,\gamma}(u,u) \quad \text{for each } \mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega).$$

Indeed, we argue by contradiction again. If there is no such constant, then we can find a sequence $\{u_\nu\}$ in $\mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$ such that

$$\|u_\nu\|_{H^{1,\gamma}_E(\Omega)} = 1, \quad h_{\Omega,\gamma}(u,u) < 2^{-\nu}.$$

As the unit ball in a separable Hilbert space is weakly compact, we can assume that $\{u_\nu\}$ converges weakly to a vector $u_\infty \in \mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$. It follows that

$$h_{\Omega,\gamma}(u_\infty, v) = \lim_{\nu\to\infty} h_{\Omega,\gamma}(u_\nu, v) = 0$$

for all $v \in H^{1,\gamma}_E(\Omega, \partial\Omega)$, i.e. $u_\infty \in \mathcal{H}_\gamma(\Omega)$. We thus conclude that $u_\infty = 0$. But inequality (4.2) yields

$$1 \leq m_\Omega\big(2^{-\nu} + \|u_\nu\|^2_{H^{0,\gamma}_E(\Omega)}\big)$$

for all ν. Since the embedding (4.6) is compact, and thus u_ν converges strongly to u_∞ in $H^{0,\gamma}_E(\Omega)$, we get

$$\|u_\infty\|_{H^{0,\gamma}_E(\Omega)} \geq \frac{1}{m_\Omega},$$

which contradicts $u_\infty = 0$.

We have thus proved that the Hermitian form $h_{\Omega,\gamma}(\cdot,\cdot)$ defines an inner product in the Hilbert space $\mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$, the corresponding norm being equivalent to the original one. Now, as the space $\tilde{H}^{-1,\gamma}_E(\Omega)$ is dual to $H^{1,\gamma}_E(\Omega, \partial\Omega)$ with respect to the pairing (2.1), the Riesz Theorem enables us to assert that for every $f \in \tilde{H}^{-1,\gamma}_E(\Omega)$ there exists a unique vector-function

$$u \in \mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$$

satisfying (4.4) for all $v \in \mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega)$.

Obviously, every $v \in H^{1,\gamma}_E(\Omega, \partial\Omega)$ can be written in the form $v = v_1 + v_2$, with

$$v_1 \in \mathcal{H}_\gamma(\Omega), \quad v_2 \in \mathcal{H}^\perp_\gamma(\Omega) \cap \mathcal{H}_\gamma(\Omega).$$

It follows that if $f \in \tilde{H}^{-1,\gamma}_E(\Omega)$ satisfies (4.5), then u satisfies (4.4) for all $v \in H^{1,\gamma}_E(\Omega, \partial\Omega)$, as desired.

Fix an (finite) orthogonal basis (b_ν) for $\mathcal{H}_\gamma(\Omega)$. Then

$$(\Pi^{(\Omega)}_\gamma F)(x) = \sum_\nu (F, b_\nu)_{H^{0,\gamma}_E(\Omega)} b_\nu(x)$$

for all $F \in H^{0,\gamma}_E(\Omega)$. Thus, the operator $\Pi^{(\Omega)}_\gamma$ naturally extends to $\tilde{H}^{-1,\gamma}_E(\Omega)$ by

$$(\Pi^{(\Omega)}_\gamma f)(x) = \sum_\nu \langle f, b_\nu\rangle_{\Omega,\gamma} b_\nu(x)$$

for all $f \in \tilde{H}^{-1,\gamma}_E(\Omega)$; obviously, it maps $\tilde{H}^{-1,\gamma}_E(\Omega)$ continuously to $H_\gamma(\Omega) \subset H^{1,\gamma}_E(\Omega, \partial\Omega)$.

Easily, for each $F \in \tilde{H}_E^{-1,\gamma}(\Omega)$ the distribution $(F - \Pi_\gamma^{(\Omega)} F)$ satisfies (4.5). Then, as we have proved above, there is a solution

$$u(f) \in H_E^{1,\gamma}(\Omega, \partial\Omega)$$

to (4.5) with data $f = F - \Pi_\gamma^{(\Omega)} F$. The rest of the proof obviously follows if we set

$$\Phi_\gamma^{(\Omega)} f = u(f) - \Pi_\gamma^{(\Omega)} u(f).$$

The continuity of the operator Φ_γ follows from the Banach Inverse Theorem. □

Corollary 4.3. *Under assumption* (4.2), *the Hermitian form*

$$\tilde{h}_{\Omega,\gamma}(\cdot\,,\cdot) = h_{\Omega,\gamma}(\cdot\,,\cdot) + (\Pi_\gamma \cdot\,, \Pi_\gamma \cdot)_{H_E^{0,\gamma}(\Omega)}$$

is an inner product on the space $H_E^{1,\gamma}(\Omega, \partial\Omega)$, *and the corresponding norm is equivalent to the original one of this space.*

Proof. By the definition of the space $\mathcal{H}_\gamma(\Omega)$ we have

$$\tilde{h}_{\Omega,\gamma}(u, v) = h_{\Omega,\gamma}((I - \Pi_\gamma)u, (I - \Pi_\gamma)v) + (\Pi_\gamma u, \Pi_\gamma v)_{H_E^{0,\gamma}(\Omega)}$$

for all $u, v \in H_E^{1,\gamma}(\Omega, \partial\Omega)$. Hence it is inner product on $H_E^{1,\gamma}(\Omega, \partial\Omega)$. As we already proved in Theorem 4.2, the functional

$$\sqrt{h_{\Omega,\gamma}(u, u)}$$

provides a norm on $\mathcal{H}_\gamma^\perp(\Omega) \cap H_E^{1,\gamma}(\Omega, \partial\Omega)$, equivalent to the original one of the space $H_E^{1,\gamma}(\Omega, \partial\Omega)$. Finally, the statement follows because on the finite-dimensional space $\mathcal{H}_\gamma(\Omega)$, any two norms are equivalent. □

Then, using the right inverse $(t_1^{(\gamma)})_r^{-1}$ as in Theorem 2.5, one easily gets the solution to the non-homogeneous Dirichlet problem

$$\begin{cases} \Delta_\gamma u = f & \text{in } \Omega, \\ t_1^{(\gamma)} u = u_0 & \text{on } \partial\Omega; \end{cases}$$

see [22, Theorem 4.1]. Note that if $\Gamma \subset \Omega$, then Γ has two sides Γ^+ and Γ^- in Ω and the Dirichlet data on Γ should be given on both Γ^+ and Γ^-.

Corollary 4.4. *Under assumption* (4.2), *there is a bounded linear operator*

$$P_\gamma^{(\Omega)} : H_E^{1/2,\gamma}(\partial\Omega) \to H_E^{1,\gamma}(\Omega)$$

such that

$$\begin{aligned} P_\gamma^{(\Omega)} \Pi_\gamma^{(\Omega)} = P_\gamma^{(\Omega)} \Pi_\gamma^{(\Omega)} &= 0, \quad \Delta_\gamma P_\gamma^{(\Omega)} = 0, \\ \Phi_\gamma^{(\Omega)} \Delta_\gamma u + P_\gamma^{(\Omega)} t_1^{(\gamma)} u = u - \Pi_\gamma^{(\Omega)} u & \quad \textit{for all } u \in H_E^{1,\gamma}(\Omega), \\ t_1^{(\gamma)} P_\gamma^{(\Omega)} u_0 = u_0 & \quad \textit{for all } u_0 \in H_E^{1/2,\gamma}(\partial\Omega). \end{aligned}$$

Thus Φ_γ and P_γ are analogues of the Green function of the Dirichlet problem (4.4) and the Poisson integral, respectively.

Now let us give some typical examples of the related Dirichlet problems.

An admissible situation comes from the strong Hermitian forms on the space $H_E^{1,\gamma}(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma))$.

Example 4.5. Suppose that the matrices $a_{i,j}(x)$ are Hermitian and satisfy

$$w^* \sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j w \geq 0 \quad \text{for all } (x, \xi, w) \in \overline{\mathcal{X}} \times \mathbb{R}^n \times \mathbb{C}^k, \tag{4.7}$$

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \neq 0 \quad \text{for all } (x, \xi) \in \overline{\mathcal{X}} \times (\mathbb{R}^n \setminus \{0\}). \tag{4.8}$$

This means that the second-order part of the operator Δ_γ is strongly elliptic. If we additionally assume that $a_{0,0}$ is a strictly positive matrix satisfying

$$(a_{0,0}w, w)_{H_E^{0,\gamma+1}(\mathcal{X})} \geq c(w, w)_{H_E^{0,\gamma+1}(\mathcal{X})} \quad \text{for all } w \in H_E^{0,\gamma+1}(\mathcal{X})$$

with a constant c independent on w, then the Hermitian form

$$h_{\Omega,\gamma}(u, v) = \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i v)_{H^{0,\gamma}(\Omega)} + (a_{0,0}u, v)_{H^{0,\gamma+1}(\Omega)}$$

satisfies assumption (4.2), providing Theorem 4.2 and Corollary 4.4; see [28, Lemma 7.12]. In particular, $\mathcal{H}_\gamma(\Omega) = \{0\}$ and $\Pi_\gamma^{(\Omega)} = 0$. Of course, it is possible to set $h_\Omega(\cdot,\cdot) = (\cdot,\cdot)_{H^{1,\gamma}(\Omega)}$.

It should be noted that, since the coefficients of the operator and the functions under consideration are complex-valued, inequalities (4.7) and (4.8) are weaker than the (strong) coercivity of the Hermitian form, i.e. the existence of a constant m such that

$$w^* \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j w \geq m|w|^2|\xi|^2 \quad \text{for all } (x, \xi, w) \in \overline{\mathcal{X}} \times \mathbb{R}^n \times \mathbb{C}^k.$$

The simplest cases come from the translation-type operators introduced in Section 2; see formula (2.4).

Example 4.6. Consider the Dirichlet problem in the usual Sobolev spaces $H^1(\mathcal{X} \setminus \Gamma)$ for a strongly elliptic operator A^*A where

$$A = \sum_{i,j=1}^{n} a_j(x)\partial_j + a_0(x) \tag{4.9}$$

and a_j are $(l \times k)$-matrices with C^∞-smooth entries over $\overline{\mathcal{X}}$, i.e. as before $l \geq k$ and the map

$$\sigma(A)(x, \xi) = \sum_{j=1}^{n} a_j(x)\xi_j : \mathbb{C}^k \to \mathbb{C}^l$$

is injective for all $x \in \overline{\mathcal{X}}$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$. Easily,

$$A^*Au = -\sum_{i,j=1}^{n} \partial_i(a_{i,j}\partial_j u) + \sum_{j=1}^{n}(a_{j,0}\partial_j u - \partial_j(a_{j.0}^* u)) + a_{0,0}u$$

with Hermitian matrices $a_{i,j} = (a_i^* a_j + a_j^* a_i)/2$ and (4.7) and (4.8) hold true because $\sigma(A)(x, \xi)$ is injective.

Set

$$h_{\mathcal{X}\setminus\Gamma}(u, v) = (Au, Av)_{L_F^2(X)}.$$

This form induces a linear continuous map

$$A^*A : H_{0,E}^1(\mathcal{X} \setminus \Gamma) \to \tilde{H}_E^{-1}(\mathcal{X} \setminus \Gamma).$$

As the operator A^*A is strongly elliptic, the classical Gårding inequality holds true for it (see, for instance, [33]):

$$\|u\|_{H^1(\mathcal{X}\setminus\Gamma)}^2 \leq m_{\mathcal{X}\setminus\Gamma}\big(\|Au\|_{L_F^2(X)}^2 + \|u\|_{L_E^2(X)}^2\big)$$

for all $u \in H_{0,E}^1(\mathcal{X} \setminus \Gamma)$ with a positive constant $m_{\mathcal{X}\setminus\Gamma}$ independent on u. Then the space

$$\mathcal{H}(\mathcal{X} \setminus \Gamma) = \{u \in H_0^1(\mathcal{X} \setminus \Gamma) : Au = 0 \text{ in } \mathcal{X} \setminus \Gamma\}$$

is finite-dimensional. For the empty crack Γ this type of results are known since the paper [33] (cf. also [12]); for an non-empty crack Γ, see, for instance, [22, Lemma 3.2].

Let Π be the orthogonal projection from $L_E^2(X)$ into $\mathcal{H}(\mathcal{X} \setminus \Gamma)$. Then, according to [22, Theorem 3.3], there is a bounded linear operator

$$\Phi : \tilde{H}_E^{-1}(\mathcal{X} \setminus \Gamma) \to H_{0.E}^1(\mathcal{X} \setminus \Gamma), \quad P : H_E^{1/2}(\partial(\mathcal{X} \setminus \Gamma)) \to H_E^1(\mathcal{X} \setminus \Gamma)$$

such that $\Phi\Pi = \Pi\Phi = 0$, $P\Pi = \Pi P = 0$ and

$$\begin{aligned} \Phi A^* A u &= u - \Pi u && \text{for all } u \in H^1_{0,E}(\mathcal{X} \setminus \Gamma), \\ A^* A \Phi f &= f - \Pi f && \text{for all } f \in \tilde{H}^{-1}_E(\mathcal{X} \setminus \Gamma), \\ \Phi A^* A u + P t_1 u &= u - \Pi u && \text{for all } u \in H^1_E(\mathcal{X} \setminus \Gamma), \\ t_1 P u_0 &= u_0 && \text{for all } u_0 \in H^{1/2}_E(\partial(\mathcal{X} \setminus \Gamma)), \end{aligned}$$

where $\tilde{H}^{-1}_E(\mathcal{X} \setminus \Gamma)$ is the dual to $H^1_{0,E}(\mathcal{X} \setminus \Gamma)$ with respect to the pairing (2.1) in the situation where $H^0 = L^2_E(\mathcal{X})$ and $H^+ = H^1_{0,E}(\mathcal{X} \setminus \Gamma)$.

Now using formula (2.4) and Theorems 2.1 and 2.4, we obtain the equalities

$$H^1_{0,E}(\mathcal{X} \setminus \Gamma) = H^{1,0}_E(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma)), \quad \tilde{H}^{-1}_E(\mathcal{X} \setminus \Gamma) = \tilde{H}^{-1,0}_E(\mathcal{X} \setminus \Gamma)$$

and the bounded operators

$$\begin{aligned} \Phi^{(\gamma)} &: \tilde{H}^{-1,\gamma}_E(\mathcal{X} \setminus \Gamma) \to H^{1,\gamma}_E(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma)), \\ \Pi^{(\gamma)} &: H^{1,\gamma}_E(\mathcal{X} \setminus \Gamma) \to \mathcal{H}^{(\gamma)}_0(\mathcal{X} \setminus \Gamma) = \{u = \rho^\gamma v : u \in \mathcal{H}_0(\mathcal{X} \setminus \Gamma)\}, \\ P^{(\gamma)} &: H^{1/2,\gamma}_E(\partial(\mathcal{X} \setminus \Gamma)) \to H^{1,\gamma}_E(\mathcal{X} \setminus \Gamma) \end{aligned}$$

such that

$$\begin{aligned} \Phi^{(\gamma)}\Pi^{(\gamma)} = \Pi^{(\gamma)}\Phi^{(\gamma)} &= 0, \quad P^{(\gamma)}\Pi^{(\gamma)} = P^{(\gamma)}\Pi^{(\gamma)} = 0, \quad (A^*A)^{(\gamma)}P^{(\gamma)} = 0, \\ \Phi^{(\gamma)}(A^*A)^{(\gamma)}u &= u - \Pi^{(\gamma)}u && \text{for } u \in H^{1,\gamma}_E(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma)), \\ (A^*A)^{(\gamma)}\Phi^{(\gamma)}f &= f - \Pi^{(\gamma)}f && \text{for } f \in \tilde{H}^{-1,\gamma}_E(\mathcal{X} \setminus \Gamma), \\ \Phi^{(\gamma)}(A^*A)^{(\gamma)}u + P^{(\gamma)}t^{(\gamma)}_1 u &= u - \Pi^{(\gamma)}u && \text{for } u \in H^{1,\gamma}_E(\mathcal{X} \setminus \Gamma), \\ t^{(\gamma)}_1 P^{(\gamma)}u_0 &= u_0 && \text{for } u_0 \in H^{1/2,\gamma}_E(\partial(\mathcal{X} \setminus \Gamma)). \end{aligned}$$

By the very construction, these operators correspond to the Dirichlet problem (4.4) induced by the Hermitian form

$$h_{\mathcal{X}\setminus\Gamma,\gamma}(u, v) = (A(\rho^{-\gamma}u), A(\rho^{-\gamma}v))_{L^2_F(\mathcal{X})},$$

satisfying assumptions (4.2) because of [22, Theorem 6.4] and the classical Gårding inequality.

In particular, the results of Example 4.6 mean that the operator $(t^{(\gamma)}_{1,\mathcal{D}})^{-1}_r$ can be chosen to coincide with the operator $P^{(\gamma)}_{\mathcal{D},\Delta} = \rho^\gamma P_{\mathcal{D},\Delta}\rho^{-\gamma}$, where $P_{\mathcal{D},\Delta}$ is the Poisson integral related to the Dirichlet problem to the usual Laplace operator Δ in the domain $\mathcal{D}$ (i.e. in the situation where the crack Γ is empty). Now we give examples of corresponding Green functions in two simple situations.

Example 4.7. Let A be an $(l \times k)$-matrix operator with constant coefficients in $\mathbb{R}^n$ such that $A^*A = -\Delta I_k$, where $\Delta = \sum_{j=1}^n \partial_j^2$ is the usual Laplace operator and I_k is the identity $(k \times k)$-matrix. In this case it is natural to assume that $\mathcal{X} = \mathbb{R}^n$. However, then we can not literally apply the results above in this new situation because the potential operators in Sobolev spaces over unbounded domains behave slightly differently. If we impose some restrictions on the growth of solutions to the Dirichlet problems at infinity instead of considering the problem in the Sobolev spaces, then we can produce a proper Green function Φ. Generally, one should argue in the weighted spaces with additional weight controlling the behavior at infinity; see [15]. In the simplest classical situation, solving the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\ \quad u = u_0 & \text{on } \Gamma, \end{cases}$$

one usually imposes additional conditions at infinity. For instance, if $n = 3$, then we should assume

$$\lim_{|x|\to+0} u(x) = 0.$$

For $n = 2$ one usually assumes that the solution is bounded at infinity or it has a logarithmic growth there. We do not want to describe the right-hand sides for the Poisson equation to avoid technical details.

Of course, from general theory it is clear that $\Phi(x,y) = g_n(x-y) - \psi_\Gamma(x,y)$, where

$$g_n(x) = \begin{cases} \frac{1}{(2-n)\sigma_n}\frac{1}{|x|^{n-2}}, & n > 2, \\ \frac{1}{2\pi}\ln|x-y|, & n = 2, \end{cases}$$

is the standard fundamental solution to the Laplace operator in $\mathbb{R}^n$, σ_n is the square of the unit sphere in $\mathbb{R}^n$ and the additional function ψ_Γ is harmonic with respect to both x and y providing proper behavior near $\partial\Gamma$ and at infinity.

The matter is rather easy if $\partial\Gamma = \emptyset$. For instance, if Γ is a sphere centered at the origin and with radius $0 < R < \infty$, then the kernel of the Green function Φ of the corresponding Dirichlet problem is given as follows:

$$\Phi(x,y) = \begin{cases} \frac{1}{(2-n)\sigma_n}\Big(\frac{1}{|x-y|^{n-2}} - \frac{R^{n-2}|y|^{n-2}}{\big|x|y|^2 - yR^2\big|^{n-2}}\Big), & n > 2, \\ \frac{1}{2\pi}\ln\frac{\big|R^2 y - x|y|^2\big|}{|y|R|x-y|}, & n = 2, \end{cases}$$

for

$$(x,y) \in (B_R \times B_R) \setminus \{x = y\} \quad \text{and} \quad (x,y) \in ((\mathbb{R}^n \setminus \overline{B}_R) \times (\mathbb{R}^n \setminus \overline{B}_R)) \setminus \{x = y\},$$

where B_R is the ball centered at the origin and with radius R. In other words, $\Phi(x,y)$ equals the Green function of the Dirichlet problem for the ball B_R for $x, y \in B_R$ and it equals the Green function of the Dirichlet problem for the complement $\mathbb{R}^n \setminus B_R$ for $x, y \in \mathbb{R}^n \setminus B_R$.

If $n = 2$ and Γ coincide with a segment $[a,b]$ on the abscissa of the plane, then the corresponding Green function and the Poisson kernel were obtained by Gakhov [5, (46.25)–(46.26)] under the continuity conditions [5, (46.18)].

We also may try to act in the weighted spaces similarly to the situation in the usual Sobolev spaces.

Example 4.8. Let us set

$$h_{\mathcal{X}\setminus\Gamma,\gamma}(\cdot,\cdot) = (\mathcal{A}u, \mathcal{A}v)_{H_F^{0,\gamma}(\mathcal{X}\setminus\Gamma)}, \tag{4.10}$$

where $\mathcal{A}$ is the weighted differential operator of type (3.1) satisfying the ellipticity assumption (3.2). In this situation,

$$\mathcal{H}_\gamma(\mathcal{X} \setminus \Gamma) = \{u \in H_E^{1,\gamma}(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma)) : \mathcal{A}u = 0 \text{ in } \mathcal{X} \setminus \Gamma\}. \tag{4.11}$$

However, the summand $\mathcal{A}_0\rho^{-1}$ in formula (3.1) is no longer a "low-order term" on the scale of weighted spaces with respect to the summand $\sum_{j=1}^n \mathcal{A}_j\partial_j$ because the scale admits already two "differentiation" operations: the true differentiation with respect to x and the multiplication on the negative powers of the weight function ρ. This means that the classical Gårding inequality for elliptic operators can not directly provide the weighted inequality (4.2) for the form (4.10).

We only note that for the Hermitian form (4.10) the assumption (4.2) reduces to the following weighted Gårding-type inequality:

$$\|u\|^2_{H_E^{1,\gamma}(\mathcal{X}\setminus\Gamma)} \le m\big(\|\mathcal{A}u\|^2_{H_F^{0,\gamma}(\mathcal{X}\setminus\Gamma)} + \|u\|^2_{H_E^{0,\gamma}(\mathcal{X}\setminus\Gamma)}\big) \tag{4.12}$$

with a positive constant m independent of $u \in H_E^{1,\gamma}(\mathcal{X} \setminus \Gamma, \partial(\mathcal{X} \setminus \Gamma))$.

We finish the section with the following simple but useful lemma. With this purpose, let χ_Ω be the characteristic function of the domain Ω. Then it induces the extension by zero from Ω to $\mathcal{X} \setminus \Gamma$ for any function $u \in L^2(\Omega)$. Moreover, it induces bounded linear operators

$$\chi_\Omega : L^2(\Omega) \to L^2(\mathcal{X} \setminus \Gamma), \quad \chi_\Omega : H_0^1(\Omega) \to H_0^1(\mathcal{X} \setminus \Gamma),$$

satisfying

$$\partial_j(\chi_\Omega u) = \chi_\Omega(\partial_j u), \quad 1 \le j \le n, \tag{4.13}$$

and

$$\|\chi_\Omega v\|_{L^2(\mathcal{X})} = \|v\|_{L^2(\Omega)}, \quad \|\chi_\Omega u\|_{H^1(\mathcal{X}\setminus\Gamma)} = \|u\|_{H^1(\Omega)}$$

for all $v \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$.

Lemma 4.9. *If there is a positive constant $m_{\mathcal{X}\setminus\Gamma}$ such that*

$$\|u\|^2_{H^{1,\gamma}_E(\mathcal{X}\setminus\Gamma)} \le m_{\mathcal{X}\setminus\Gamma}\big(h_{\mathcal{X}\setminus\Gamma,\gamma}(u,u) + \|u\|^2_{H^{0,\gamma}_E(\mathcal{X}\setminus\Gamma)}\big) \tag{4.14}$$

for all $u \in H^{1,\gamma}_E(\mathcal{X}\setminus\Gamma, \partial(\mathcal{X}\setminus\Gamma))$ then (4.2) holds for each domain $\Omega \subset \mathcal{X}\setminus\Gamma$ such that either $\Gamma \subset \Omega$ or $\Gamma \subset \partial\Omega$, where Ω has the Lipschitz boundary. Moreover, for each $v \in \mathcal{H}_\gamma(\Omega)$ we have $\chi^{(\gamma)}_\Omega v \in \mathcal{H}_\gamma(\mathcal{X}\setminus\Gamma)$.

Proof. Indeed, using (2.4) and Theorems 2.1 and 2.4, we obtain the bounded operator

$$\chi^{(\gamma)}_\Omega : H^{1,\gamma}(\Omega, \partial\Omega) \to H^{1,\gamma}(\mathcal{X}\setminus\Gamma, \partial(\mathcal{X}\setminus\Gamma))$$

satisfying for all $u \in H^{1,\gamma}(\Omega, \partial\Omega)$,

$$c_1\|\chi^{(\gamma)}_\Omega u\|_{H^{1,\gamma}(\mathcal{X}\setminus\Gamma)} \le \|u\|_{H^{1,\gamma}(\Omega)} \le c_2\|\chi^{(\gamma)}_\Omega u\|_{H^{1,\gamma}(\mathcal{X}\setminus\Gamma)} \tag{4.15}$$

with a positive constant c_j independent of $u \in H^{1,\gamma}(\Omega, \partial\Omega)$.

Next, as the norm $\|\cdot\|_{H^{1,\gamma}(\mathcal{X}\setminus\Gamma)}$ is not weaker than the norm $\|\cdot\|_{H^{1,\gamma}(\Omega)}$, we conclude that a bounded linear restriction operator

$$R_\Omega : H^{1,\gamma}(\mathcal{X}\setminus\Gamma) \to H^{1,\gamma}(\Omega)$$

is well-defined. Then, using (4.13) and Corollary 2.6, we see that for each

$$u \in H^{1,\gamma}_E(\Omega, \partial\Omega) \quad \text{and} \quad V \in H^{1,\gamma}_E(\mathcal{X}\setminus\Gamma, \partial(\mathcal{X}\setminus\Gamma))$$

we have

$$(a_{i,j}\partial_j(\rho^\gamma\chi_\Omega\rho^{-\gamma}u, \partial_i V))_{H^{0,\gamma}(\mathcal{X}\setminus\Gamma)} = (a_{i,j}\partial_j u, \partial_i R_\Omega V)_{H^{0,\gamma}(\Omega)},$$

and hence

$$h_{\mathcal{X}\setminus\Gamma,\gamma}(\chi^{(\gamma)}_\Omega u, V) = h_{\Omega,\gamma}(u, R_\Omega V). \tag{4.16}$$

Combining (4.14)–(4.16), we conclude that the statement of the lemma holds true.

Finally, the last statement of the lemma follows from Lemma 4.1 and (4.16) because

$$h_{\mathcal{X}\setminus\Gamma,\gamma}(\chi^{(\gamma)}_\Omega w, \chi^{(\gamma)}_\Omega w) = h_{\Omega,\gamma}(w, w) = 0$$

if $w \in \mathcal{H}_\gamma(\Omega)$. □

5 Constructing the adjoint operator

Now we are ready to discuss the regularization of the operator equation (1.1) with the use of Theorem 1.1. First of all, we note that the Hilbert space adjoint operator

$$\mathcal{A}^\star : H^{0,\gamma}(\mathcal{D}) \to H^{1,\gamma}(\mathcal{D}, \overline{S}) \tag{5.1}$$

for the map (3.4) always exists. In order to identify it, for each $f \in H^{0,\gamma}(\mathcal{D})$ one should find the unique solution $w = \mathcal{A}^\star f \in H^{1,\gamma}(\mathcal{D}, \overline{S})$ to the following problem:

$$(w, v)_{H^{1,\gamma}(\mathcal{D})} = (f, \mathcal{A}v)_{H^{0,\gamma}(\mathcal{D})} \quad \text{for all } v \in H^{1,\gamma}(\mathcal{D}, \overline{S}). \tag{5.2}$$

By the discussion above, (5.2) can be treated as a mixed boundary value problem for a second-order elliptic operator generated by the Hermitian form $(\cdot,\cdot)_{H^{1,\gamma}(\mathcal{D})}$; cf. also [22, 24]. However, it is not so easy to find the solution of a mixed problem in a constructive form; cf. [9]. Thus, the idea of finding the adjoint operator $\mathcal{A}^\star : H_2 \to H_1$ for the bounded linear operator $\mathcal{A} : H_1 \to H_2$ was the following: to replace a simple inner product of the space H_1 by (possibly more complicated) another one in such a way that 1) the new inner product induces an equivalent norm to the old one, 2) the corresponding adjoint is given by a (relatively) simple formula (see, for instance, [16, 19, 22] for various differential operators in the usual Sobolev spaces, including the cases without boundary conditions).

We proceed with the situation where $\Gamma = \overline{S}$ and $\Xi = \partial S$.

Fix the operator (4.3) over $\mathcal{X} \setminus \Gamma$, satisfying (4.14) and such that

$$R_{\mathcal{D}} u \in \ker(\mathcal{A}) \quad \text{for all } u \in \mathcal{H}_\gamma(\mathcal{X} \setminus \Gamma), \tag{5.3}$$

where $\ker(\mathcal{A})$ means the null-space of problem (3.3). We denote by $h^{(\mathcal{A})}_{\Omega}(\cdot,\cdot)$ the corresponding forms related to proper $\Omega \subset \mathcal{X} \setminus \Gamma$ (the case $\Omega = \mathcal{X} \setminus \Gamma$ is included). Let also $\Phi^{(\Omega)}_{\gamma,\mathcal{A}}$ and $P^{(\Omega)}_{\gamma,\mathcal{A}}$ stand for the Green function and the Poisson integral related to $h^{(\mathcal{A})}_{\Omega}(\cdot,\cdot)$, respectively.

Then for $u \in H^{1,\gamma}_E(\mathcal{D}, \overline{S})$ we set

$$\mathcal{E}(u) = \begin{cases} u & \text{in } \mathcal{D}, \\ P^{(\mathcal{X}\setminus\overline{\mathcal{D}})}_{\gamma,\mathcal{A}} t^{(\gamma)}_{1,\mathcal{D}} u & \text{in } \mathcal{X} \setminus \overline{\mathcal{D}}. \end{cases}$$

By the construction,

$$\mathcal{E}(u) \in H^{1,\gamma}(\mathcal{X} \setminus \overline{S}, \partial(\mathcal{X} \setminus \overline{S}))$$

for each $u \in H^{1,\gamma}(\mathcal{D}, \overline{S})$. Thus, we may introduce the Hermitian form

$$\mathfrak{h}_{\mathcal{D},\gamma}(u, v) = h^{(\mathcal{A})}_{\mathcal{X}\setminus\overline{S},\gamma}(\mathcal{E}(u), \mathcal{E}(v)) + \big(\Pi^{(\mathcal{X}\setminus\Gamma)}_{\gamma}\mathcal{E}(u), \Pi^{(\mathcal{X}\setminus\Gamma)}_{\gamma}\mathcal{E}(v)\big)_{H^{0,\gamma}_F(\mathcal{X}\setminus\Gamma)}$$

on the space $H^{1,\gamma}_E(\mathcal{D}, \overline{S})$. We also set for each $f \in H^{0,\gamma}_F(\mathcal{D})$,

$$T_{\gamma,\mathcal{A}} f = \Phi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\Big(\rho^{2\gamma} \sum_{j=1}^{n} \partial_j(\rho^{-2\gamma} A^*_j \chi^{(\gamma)}_{\mathcal{D}} f) + \rho^{-1} A^*_0 \chi^{(\gamma)}_{\mathcal{D}} f\Big). \tag{5.4}$$

Theorem 5.1. *Let* (4.14) *and* (5.3) *be fulfilled. Then the Hermitian form* $\mathfrak{h}_{\mathcal{D},\gamma}(\cdot,\cdot)$ *is an inner product on* $H^{1,\gamma}_E(\mathcal{D}, \overline{S})$*, the corresponding norm is equivalent to the norm* $\|\cdot\|_{H^{1,\gamma}_E(\mathcal{D})}$ *on this space and*

$$\mathfrak{h}_{\mathcal{D},\gamma}(R_{\mathcal{D}} T_{\gamma,\mathcal{A}} f, v) = (f, \mathcal{A} v)_{H^{0,\gamma}(\mathcal{D})}$$

for all $f \in H^{0,\gamma}_F(\mathcal{D})$ *and* $v \in H^{1,\gamma}(\mathcal{D}, \overline{S})$*, where*

$$T_{\gamma,\mathcal{A}} : H^{0,\gamma}_F(\mathcal{D}) \to H^{1,\gamma}_E(\mathcal{X} \setminus \overline{S}, \partial(\mathcal{X} \setminus \overline{S})) \tag{5.5}$$

is a bounded linear operator induced by (5.4).

Proof. Indeed, by the construction, $\mathfrak{h}_{\mathcal{D},\gamma}(\cdot,\cdot)$ is a Hermitian form on $H^{1,\gamma}_E(\mathcal{D})$. If $\mathfrak{h}_{\mathcal{D},\gamma}(u, u) = 0$, then $\mathcal{E}(u) = 0$ because of Corollary 4.3. Hence $u = 0$ and $\mathfrak{h}_{\mathcal{D},\gamma}(\cdot,\cdot)$ is an inner product on $H^{1,\gamma}_E(\mathcal{D})$.

As the operators

$$t^{\gamma}_{1,\mathcal{D}} : H^{1,\gamma}_E(\mathcal{D}) \to H^{1/2,\gamma}_E(\partial\mathcal{D}) \quad \text{and} \quad P^{(\mathcal{X}\setminus\overline{\mathcal{D}})}_{\gamma,\mathcal{A}} : H^{1/2,\gamma}_E(\partial\mathcal{D}) \to H^{1,\gamma}_E(\mathcal{X} \setminus \overline{\mathcal{D}})$$

are bounded, the map $\mathcal{E} : H^{1,\gamma}_E(\mathcal{D}) \to H^{1,\gamma}_E(\mathcal{X} \setminus \overline{S})$ is bounded, too. Hence Corollary 4.3 implies that the norm $\mathfrak{h}_{\mathcal{D},\gamma}(u, u)$ is dominated by the standard norm of the space $H^{1,\gamma}_E(\mathcal{D})$. Moreover, it follows from Corollary 4.3 that there is a positive constant c such that

$$\|u\|^2_{H^{1,\gamma}_E(\mathcal{D})} \le \|\mathcal{E}(u)\|^2_{H^{1,\gamma}_E(\mathcal{X}\setminus\Gamma)} \le c\,\mathfrak{h}_{\mathcal{D},\gamma}(u, u)$$

for all $u \in H^{1,\gamma}_E(\mathcal{D})$, i.e. the norm

$$\sqrt{\mathfrak{h}_{\mathcal{D},\gamma}(u, u)}$$

is equivalent to the original norm of the space $H^{1,\gamma}_E(\mathcal{D})$.

Furthermore, by the construction, the operator $\chi^{(\gamma)}_{\mathcal{D}}$ maps $H^{0,\gamma}_F(\mathcal{D})$ continuously to $H^{0,\gamma}_F(\mathcal{X} \setminus \overline{S})$. Fix a function $V \in H^{1,\gamma}_E(\mathcal{X} \setminus \overline{S}, \partial(\mathcal{X} \setminus \overline{S}))$. Using Corollary 2.6, we see that there is a sequence $\{V_\nu\} \subset C^\infty_0(\mathcal{X} \setminus \overline{S})$, converging to V in $H^{1,\gamma}_E(\mathcal{X} \setminus \overline{S}, \partial(\mathcal{X} \setminus \overline{S}))$. Therefore,

$$\begin{aligned}\Big\langle \rho^{2\gamma} \sum_{j=1}^{n} \partial_j(\rho^{-2\gamma} A^*_j \chi^{(\gamma)}_{\mathcal{D}} f + \rho^{-1} A^*_0 \chi^{(\gamma)}_{\mathcal{D}} f), V \Big\rangle_{\mathcal{X}\setminus\overline{S},\gamma} &= \lim_{\nu\to\infty}\Big(\rho^{2\gamma} \sum_{j=1}^{n} \partial_j(\rho^{-2\gamma} A^*_j \chi^{(\gamma)}_{\mathcal{D}} f + \rho^{-1} A^*_0 \chi^{(\gamma)}_{\mathcal{D}} f), V_\nu\Big)_{H^{0,\gamma}(\mathcal{X}\setminus\overline{S})} \\ &= (f, \mathcal{A} R_{\mathcal{D}} V)_{H^{0,\gamma}(\mathcal{D})}.\end{aligned} \tag{5.6}$$

In particular, (5.3) and (5.6) imply that the element

$$F_f = \Big(\rho^{2\gamma}\sum_{j=1}^{n}\partial_j(\rho^{-2\gamma}A_j^*\chi^{(\gamma)}_{\mathcal{D}}f) + \rho^{-1}A_0^*\chi^{(\gamma)}_{\mathcal{D}}f\Big)$$

belongs to $\tilde{H}^{-1,\gamma}(\mathcal{X}\setminus\overline{S})$ for each $f \in H^{0,\gamma}_F(\mathcal{D})$ and it satisfies (4.5). Thus, Theorem 4.2 and (5.6) mean that formula (5.4) induces the bounded linear operator (5.5) and that

$$h_{\mathcal{X}\setminus\overline{S},\gamma}(T_{\gamma,\mathcal{A}}f, V) = \langle F_f, V\rangle_{\mathcal{X}\setminus\Gamma,\gamma} = (f, \mathcal{A}R_{\mathcal{D}}V)_{H^{0,\gamma}(\mathcal{D})} \tag{5.7}$$

for all $V \in H^{1,\gamma}_E(\mathcal{X}\setminus\overline{S}, \partial(\mathcal{X}\setminus\overline{S}))$.

Now, applying (4.16) with $\Omega = \mathcal{X}\setminus\overline{\mathcal{D}}$, we obtain

$$h_{\mathcal{X}\setminus\overline{\mathcal{D}},\gamma}(R_{\mathcal{X}\setminus\overline{\mathcal{D}}}T_{\gamma,\mathcal{A}}f, v) = h_{\mathcal{X}\setminus\Gamma,\gamma}(T_{\gamma,\mathcal{A}}f, \chi^{(\gamma)}_{\mathcal{X}\setminus\overline{\mathcal{D}}}v) = 0 \tag{5.8}$$

for all $v \in H^{1,\gamma}_E(\mathcal{X}\setminus\overline{\mathcal{D}}, \partial(\mathcal{X}\setminus\overline{\mathcal{D}}))$ because of (5.7).

Finally, using Lemma 4.9, Theorem 4.2 and the properties of the operator $\chi^{(\gamma)}_{\Omega}$, we see that

$$(R_{\mathcal{D}}T_{\gamma,\mathcal{A}}f, w)_{H^{0,\gamma}(\mathcal{X}\setminus\overline{\mathcal{D}})} = (T_{\gamma,\mathcal{A}}f, \chi^{(\gamma)}_{\mathcal{X}\setminus\overline{\mathcal{D}}}w)_{H^{0,\gamma}(\mathcal{X}\setminus\overline{\mathcal{D}})} = 0 \tag{5.9}$$

for all $w \in \mathcal{H}_\gamma(\mathcal{X}\setminus\overline{\mathcal{D}})$ and all $f \in H^{0,\gamma}_F(\mathcal{D})$. As $T_{\gamma,\mathcal{A}}f \in H^{1,\gamma}(\mathcal{X}\setminus\overline{S}, \partial(\mathcal{X}\setminus\overline{S}))$, we see that

$$t^{(\gamma)}_{1,\mathcal{D}}T_{\gamma,\mathcal{A}}f = t^{(\gamma)}_{1,\mathcal{X}\setminus\overline{\mathcal{D}}}T_{\gamma,\mathcal{A}}f \quad \text{on } \partial\mathcal{D}.$$

Thus, formulas (5.8) and (5.9) imply that $\mathcal{E}(R_{\mathcal{D}}T_{\gamma,\mathcal{A}}f) = T_{\gamma,\mathcal{A}}f$ for each $f \in H^{0,\gamma}_F(\mathcal{D})$.

Therefore, according to Theorem 4.2, we have $\Pi^{(\mathcal{X}\setminus\Gamma)}_{\gamma}T_{\gamma,\mathcal{A}} = 0$. Hence, using (5.7), we conclude that

$$\mathfrak{h}_{\mathcal{D},\gamma}(R_{\mathcal{D}}T_{\gamma,\mathcal{A}}f, v) = h_{\mathcal{X}\setminus\overline{S},\gamma}(T_{\gamma,\mathcal{A}}f, \mathcal{E}(v)) = (f, \mathcal{A}v)_{H^{0,\gamma}(\mathcal{D})},$$

which was to be proved. □

Theorem 5.1 gives many ways to identify the adjoint operator (5.1) using Green functions of Dirichlet problems (see Examples 4.5, 4.6 and 4.8 above).

Example 5.2. If the operator $\mathcal{A}$ satisfies (4.12), then the form (4.10) satisfies both (4.14) and (5.3) because of (4.11). In this case we can clarify the nature of the term $(I - \mathcal{A}^\star\mathcal{A})$ in Theorem 1.1.

Indeed, if $u \in H^{1,\gamma}_E(\mathcal{D}, \partial\mathcal{D})$, then $\chi^{(\gamma)}_{\mathcal{D}}\mathcal{A}u = \mathcal{A}\chi^{(\gamma)}_{\mathcal{D}}u$ and

$$T_{\gamma,\mathcal{A}}\mathcal{A}u = \Phi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\Delta_\gamma\chi^{(\gamma)}_{\mathcal{D}}u = \chi^{(\gamma)}_{\mathcal{D}}u - \Pi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\chi^{(\gamma)}_{\mathcal{D}}u \tag{5.10}$$

because of Theorem 4.2.

By Corollary 4.4, for each $u \in H^{1,\gamma}_E(\mathcal{D}, \overline{S})$, the vector $u - P^{(\mathcal{D})}_{\gamma}t^{(\gamma)}_1 u$ belongs to $H^{1,\gamma}_E(\mathcal{D}, \partial\mathcal{D})$. Then (5.10) yields the Green-type formula

$$\chi^{(\gamma)}_{\mathcal{D}}u = T_{\gamma,\mathcal{A}}\mathcal{A}u + \Pi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\chi^{(\gamma)}_{\mathcal{D}}u + Gt^{(\gamma)}_1 u \tag{5.11}$$

with the bounded linear Green-type operator

$$G : H^{1/2,\gamma}_E(\mathcal{D}) \to H^{1,\gamma}_E(\mathcal{D}, \overline{S})$$

given by

$$G = -(\Pi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\chi^{(\gamma)}_{\mathcal{D}} + T_{\gamma,\mathcal{A}}\mathcal{A})P^{(\mathcal{D})}_{\gamma}; \tag{5.12}$$

cf. [22, Theorem 4.1]. Note that the operator $\chi^{(\gamma)}_{\mathcal{D}}$ in formulas (5.11) and (5.12) maps $H^{1,\gamma}_E(\mathcal{D}, \overline{S})$ continuously to $H^{0,\gamma}_E(\mathcal{X}\setminus\overline{S})$ only.

In particular, for each $u \in H^{1,\gamma}_E(\mathcal{D}, \overline{S})$, we have

$$(I - \mathcal{A}^\star\mathcal{A})u = \Pi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\chi^{(\gamma)}_{\mathcal{D}}u + Gt^{(\gamma)}_1 u,$$

where $\Pi^{(\mathcal{X}\setminus\overline{S})}_{\gamma,\mathcal{A}}\chi^{(\gamma)}_{\mathcal{D}}u \in \mathcal{H}_\gamma(\mathcal{X}\setminus\overline{S}) \subset \ker(\mathcal{A})$.

Example 5.3. If (4.12) is not fulfilled for the operator $\mathcal{A}$, then we may take

$$h^{(\mathcal{A})}_{\mathcal{X}\setminus\overline{S}}(\cdot,\cdot) = h_{\mathcal{X}\setminus\overline{S}}(\cdot,\cdot) + (\mathcal{A}\cdot,\mathcal{A}\cdot)_{H^{0,\gamma}_F(\mathcal{D})} \tag{5.13}$$

with any form $h_{\mathcal{X}\setminus\overline{S}}(\cdot,\cdot)$ satisfying (4.14). However, in this case, we need additional assumptions to identify the operator $(I - \mathcal{A}^\star\mathcal{A})$ as a Green-type boundary integral. The strong forms as in Example 4.5 satisfy both (4.14) and (5.3), and hence we do not need to add the second term in (5.13).

Example 5.4. If $\mathcal{A} = \operatorname{Op}(\rho^\gamma)A\operatorname{Op}(\rho^{-\gamma})$ for a non-weighted first-order operator (4.9) from Example 4.6, then Theorem 5.1 is just a translation of results [22] from the usual Sobolev spaces to the weighted ones. Namely, in this case $T_{\gamma,\mathcal{A}} = \operatorname{Op}(\rho^\gamma)(\Phi\chi_D A^*)\operatorname{Op}(\rho^{-\gamma})$, where Φ is the Green function related to the Dirichlet for the Laplacian A^*A in a domain $\mathcal{X}\setminus\overline{S}$ on the scale of the usual Sobolev spaces, and $(I - \mathcal{A}^\star\mathcal{A})$ can still be identified as a Green-type boundary integral; see Example 4.6 and [22].

Funding: The authors were supported by Grant N 1.2604.2017/PCh of the Ministry of Education and Science of the Russian Federation.

References

[1] L. Aizenberg, *Carleman's Formulas in Complex Analysis. First Applications*, Math. Appl. 244, Kluwer Academic, Dordrecht, 1993.

[2] J. M. Berezans'kiĭ, *Expansions in Eigenfunctions of Selfadjoint Operators*, American Mathematical Society, Providence, 1968.

[3] M. Borsuk and V. Kondratiev, *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, North-Holland Math. Libr. 69, Elsevier Science, Amsterdam, 2006.

[4] T. Carleman, *Les fonctions quasianalytiques*, Gauthier-Villars, Paris, 1926.

[5] F. D. Gakhov, *Boundary Value Problems*, Nauka, Moscow, 1977.

[6] L. I. Hedberg and T. H. Wolff, Thin sets in nonlinear potential theory, *Ann. Inst. Fourier (Grenoble)* **33** (1983), no. 4, 161–187.

[7] J. J. Kohn, Subellipticity of the $\bar\partial$-Neumann problem on pseudo-convex domains: Sufficient conditions, *Acta Math.* **142** (1979), no. 1–2, 79–122.

[8] V. A. Kozlov, V. G. Maz'ya and A. V. Fomin, An iterative method for solving the Cauchy problem for elliptic equations, *USSR Comput. Math. Math. Phys.* **31** (1991), no. 1, 45–52.

[9] A. Laptev, A. Peicheva and A. Shlapunov, Finding eigenvalues and eigenfunctions of the Zaremba problem for the circle, *Complex Anal. Oper. Theory* **11** (2017), no. 4, 895–926.

[10] M. M. Lavrent'ev, *On some ill-posed problems of mathematical physics*, Academy of Sciences of USSR, Novosibirsk, 1962.

[11] M. M. Lavrent'ev, V. G. Romanov and S. P. Šišatskiĭ, *Ill-posed Problems of Mathematical Physics and Analysis*, Nauka, Moscow, 1980.

[12] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications. Vol. I*, Springer, New York, 1972.

[13] V. G. Maz'ya and A. V. Fomin, Iterative procedures for solving ill-posed boundary value problems that preserve the differential equations, *Leningrad Math. J.* **1** (1990), no. 5, 1207–228.

[14] V. G. Maz'ya and V. P. Havin, The solutions of the Cauchy problem for the Laplace equation (uniqueness, normality, approximation), *Trudy Moskov. Mat. Obšč.* **30** (1974), 61–114.

[15] R. C. McOwen, The behavior of the Laplacian on weighted Sobolev spaces, *Comm. Pure Appl. Math.* **32** (1979), no. 6, 783–795.

[16] M. Nacinovich and A. Shlapunov, On iterations of the Green integrals and their applications to elliptic differential complexes, *Math. Nachr.* **180** (1996), 243–284.

[17] S. A. Nazarov and B. A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, De Gruyter Exp. Math. 13, Walter de Gruyter, Berlin, 1994.

[18] A. N. Polkovnikov and A. A. Shlapunov, On the construction of Carleman formulas by means of mixed problems with boundary conditions containing a parameter, *Sibirsk. Mat. Zh.* **58** (2017), no. 4, 870–884.

[19] A. V. Romanov, Convergence of iterates of the Bochner–Martinelli operator, and the Cauchy–Riemann equation, *Soviet Math. Dokl* **19** (1978), 1211–1215.

[20] M. Schechter, Negative norms and boundary problems, *Ann. of Math. (2)* **72** (1960), 581–593.

[21] B.-W. Schulze, Crack problems in the edge. Pseudo-differential calculus, *Appl. Anal.* **45** (1992), no. 1–4, 333–360.

[22] B.-W. Schulze, A. Shlapunov and N. Tarkhanov, Green integrals on manifolds with cracks, *Ann. Global Anal. Geom.* **24** (2003), no. 2, 131–160.
[23] A. Shlapunov, On iterations of non-negative operators and their applications to elliptic systems, *Math. Nachr.* **218** (2000), 165–174.
[24] A. Shlapunov and N. Tarkhanov, Mixed problems with parameter, *Russ. J. Math. Phys.* **12** (2005), no. 1, 97–119.
[25] A. A. Shlapunov and N. N. Tarkhanov, Bases with double orthogonality in the Cauchy problem for systems with injective symbols, *Proc. Lond. Math. Soc. (3)* **71** (1995), no. 1, 1–52.
[26] S. R. Simanca, Mixed elliptic boundary value problems, *Comm. Partial Differential Equations* **12** (1987), no. 2, 123–200.
[27] L. N. Slobodeckiĭ, Generalized Sobolev spaces and their application to boundary problems for partial differential equations, *Leningrad. Gos. Ped. Inst. Učen. Zap.* **197** (1958), 54–112.
[28] N. Tarkhanov and A. A. Shlapunov, Sturm–Liouville problems in weighted spaces in domains with nonsmooth edges. I, *Siberian Adv. Math.* **26** (2016), no. 1, 30–76.
[29] N. Tarkhanov and A. A. Shlapunov, Sturm–Liouville problems in weighted spaces in domains with nonsmooth edges. II, *Siberian Adv. Math.* **26** (2016), no. 4, 247–293.
[30] N. N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*, Math. 7, Akademie, Berlin, 1995.
[31] A. N. Tikhonov and V. Y. Arsenin, *Methods of Solving Ill-posed Problems*, Nauka, Moscow, 1986.
[32] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, VEB, Berlin, 1978.
[33] M. I. Višik, On strongly elliptic systems of differential equations, *Mat. Sbornik N. S.* **29(71)** (1951), 615–676.