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Symmetry Analysis of Ideal Fluid Equations in Terms of Trajectories and Weber's Potential

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The 2D perfect fluid motions equations in Lagrangian coordinates are considered. If body forces are potential one, then there is the general integral called Weber's integral and the resulting system includes initial data which in fact make the problem of group-theoretical classification actual. It is established that the basic group becomes infinite-dimensional with respect to the space variable too. The exceptional values of arbitrary initial vorticity are obtained at which we can be observed further extension of the group. Group properties of Euler equations in arbitrary Lagrangian coordinates are also considered and some exact solutions are constructed.

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Introduction

Group analysis of differential equations (Lie – Ovsyannikov symmetry method [1–3]) became a powerful tool for studying nonlinear equations and boundary value problems. This analysis is especially fruitful in application to the basic equations of mechanics and physics because the invariance principles are already involved in their derivation. Symmetry methods are successfully used to study the mathematical models in hydrodynamics and to construct exact solutions of non-linear problems, e.g. in [4–6]. An admissible group characterizes symmetry properties of differential equations and used for complete integration or construction of certain classes of exact solutions as well as qualitative investigation of the equations.

In describing the motion of an ideal incompressible fluid that has a free boundary the problem, in particular waves problem, is reduced to finding the solution of the Euler equations with

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the fulfilment of the kinematic and the dynamic condition at the free surface. The kinematic condition allows us to convert this problem to another one in which the domain is fixed. It is achieved by the transition to the Lagrangian coordinates which are introduced as the coordinate values of the fluid particles at the initial time [7]. The new system of differential equations arised very is often used for the investigation solvability of this problem [8–11].

The paper purpose is to study group properties of equations which describe 2D perfect liquid flows. It is well known [12] that, there are two integrals of such equations in Lagrangian coordinates. First of them is the Cauchy one which implies that the particle vorticity is preserved under planar motions. The second integral is appeared due to Weber transformation which applies to Euler equations [12]. The differential equations system corresponding Cauchy integral is the closed system for only trajectories. In this case the problem of group-theoretic classification was solved in [13, 14]. Another differential equations system is the closed system for trajectories and Weber's potential. The passage from one system to another mentioned above is a nonlocal transformation, therefore there should be no isomorphism between the basic Lie group of the equations under consideration. So, we consider the last differential equations system in terms of trajectories and Weber's potential. It is established that the basic group is infinite-dimensional with respect to the space variables. The exceptional values of initial vorticity at which we observe further extension of the group are obtained. Group properties in arbitrary Lagrangian coordinates are considered and two exact solution are constructed.

1. Governing equations

A nonstationary 3D flow of an ideal incompressible fluid is described by the following system of equations (Euler equations)

$$\mathbf{u}_t + \mathbf{u}\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{g}(\mathbf{x}, t), \quad \operatorname{div}\mathbf{u} = 0, \tag{1.1}$$

where $\rho = \text{const}$ is the fluid density and $\mathbf{g}$ is the vector of body forces, $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ are the velocity vector and the pressure, respectively.

Let us introduce the Lagrangian coordinates $\boldsymbol{\xi}$ by solving the Cauchy problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x}\big|_{t=0} = \boldsymbol{\xi}. \tag{1.2}$$

Without loss of generality we can assume that the fluid density equals unity. Then the system (1.1) can be rewritten in the following form [7]

$$M^*(\mathbf{x}_{tt} - \mathbf{g}(\mathbf{x}, t)) + \nabla p = 0, \quad \operatorname{div} M^{-1}\mathbf{x}_t = 0. \tag{1.3}$$

Here M is the Jacobi matric

$$M = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{pmatrix} = \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\xi})},$$

and M^* is inverse matrix to M. The second equations of (1.3) is equivalent to equation $\det M = 1$ [7].

If body forces have a potential $\mathbf{g} = \nabla_x h$ then from the first eq'n of (1.3) we derive [7]

$$M^*\mathbf{x}_t = \nabla\varphi + \mathbf{u}_0(\boldsymbol{\xi}), \tag{1.4}$$

$$\varphi_t + p = \frac{1}{2}\,|\mathbf{x}_t|^2 + h + \chi(t), \tag{1.5}$$

where $\mathbf{u}_0(\boldsymbol{\xi})$ is the initial value of $\mathbf{u}(\mathbf{x}, t)$ and $\chi(t)$ is arbitrary function. The relation (1.5) is the Weber's integral. If flow is potential one then this integral coincides with Cauchy–Lagrange integral [6, 7].

Function $\varphi(\boldsymbol{\xi}, t)$ is the solution of second order elliptic eq'n

$$\text{div}\,\left[M^{-1}M^{*-1}(\nabla\varphi + \mathbf{u}_0(\boldsymbol{\xi}))\right] = 0 \tag{1.6}$$

and can be called Weber's potential.

For 2D motions let us put $\mathbf{u}_0 = (u(\xi, \eta), v(\xi, \eta))$, $\boldsymbol{\xi} = (\xi, \eta)$, $\mathbf{x} = (x(\xi, \eta, t), y(\xi, \eta, t))$. Then the equations (1.4), (1.6) imply the closed differential system

$$x_t = y_\eta(\varphi_\xi + u(\xi, \eta)) - y_\xi(\varphi_\eta + v(\xi, \eta)); \tag{1.7}$$

$$y_t = -x_\eta(\varphi_\xi + u(\xi, \eta)) + x_\xi(\varphi_\eta + v(\xi, \eta)); \tag{1.8}$$

$$x_\xi y_\eta - x_\eta y_\xi = 1; \tag{1.9}$$

$$u_\xi + v_\eta = 0, \tag{1.10}$$

where $x(\xi, \eta, t)$, $y(\xi, \eta, t)$ are the trajectories of liquid particles and $\varphi(\xi, \eta, t)$ is the Weber potential. The relationship (1.10) is the compatibility condition for velocity vector at initial time.

The pressure distribution can be found from Weber's integral (1.5) in the following form

$$p(\xi, \eta, t) = \frac{1}{2}\,(x_t^2 + y_t^2) - \varphi_t + h(\xi, \eta, t) \tag{1.11}$$

and $h = -gy(\xi, \eta, t)$ for water waves, where $g = \text{const}$ is the gravitational acceleration.

Remark 1. Equations (1.7), (1.8) are equivalent to the following

$$\varphi_\xi = x_t x_\xi + y_t y_\xi - u, \quad \varphi_\eta = x_t x_\eta + y_t y_\eta - v.$$

The compatibility condition of these equations implies the eq'n

$$x_\xi x_{\eta t} - x_\eta x_{\xi t} + y_\xi y_{\eta t} - y_\eta y_{\xi t} = \omega(\xi, \eta) \equiv v_\xi - u_\eta \neq 0. \tag{1.12}$$

It is the conservation vorticity law (Cauchy integral [7]). The equations (1.9) and (1.12) are the closed system for functions $x(\xi, \eta, t)$, $y(\xi, \eta, t)$. The group properties for this system were studied in [13, 14]. Due to inequality (1.12) an unsteady flow is rotational one for all time.

2. Symmetry properties of system (1.7)–(1.9)

Firstly, let us compute the equivalence transformation. We can demonstrate that the equivalence transformation is

$$\begin{gathered}\bar{t} = a_1 t, \quad \bar{x} = a_1(x\cos a_2 + y\sin a_2) + a_3, \quad \bar{y} = a_1(-x\sin a_2 + y\cos a_2) + a_4,\\ \bar{\xi} = a_1(\xi\cos a_2 + \eta\sin a_2) + a_3,\ \bar{\eta} = a_1(-\xi\sin a_2 + \eta\cos a_2) + a_4,\\ \bar{\varphi} = a_1\varphi + b_1\xi + b_2\eta + d(t);\end{gathered} \tag{2.1}$$

$$\bar{\omega}(\bar{\xi}, \bar{\eta}) = a_5\omega(\bar{\xi}, \bar{\eta}) \neq 0. \tag{2.2}$$

Here $a_1 \neq 0$, a_2, a_3, a_4, $a_5 \neq 0$, b_1, b_2 are arbitrary constants and $d(t)$ is arbitrary function. Of course compatibility condition (1.10) is invariant with respect to transformation (2.1). At last, we should supplement (2.1) with the discrete transformation

$$\bar{x} = -x, \quad \bar{y} = -y. \tag{2.3}$$

Generator for system (1.7)–(1.9) is sought in the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial \xi} + \xi^3 \frac{\partial}{\partial \eta} + \eta^1 \frac{\partial}{\partial x} + \eta^2 \frac{\partial}{\partial y} + \eta^3 \frac{\partial}{\partial \varphi}, \tag{2.4}$$

where ξ^i, η^i depend on all variables t, ξ, η, x, y, φ. To derive the determining equations it is necessary to apply the first prolongation $Y_{(1)}$ of generator Y to system (1.7)–(1.9). After a large member of calculations the determining system can be reduced to the following one

$$\begin{gathered}
\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(\xi,\eta), \quad \xi^3 = \xi^3(\xi,\eta), \\
\eta^1 = C_1 x + C_2 y + n(t), \quad \eta^2 = C_1 y - C_2 x + m(t), \quad \xi^2_\xi + \xi^3_\eta = 2C_1, \\
\eta^3 = (2C_1 - \xi^1_t)\varphi + n_t x + m_t y + h(\xi,\eta) + d(t), \\
u(\xi^3_\eta - \xi^1_t) - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v - h_\xi = 0, \\
v(\xi^2_\xi - \xi^1_t) - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u - h_\eta = 0,
\end{gathered} \tag{2.5}$$

here $n(t)$, $m(t)$, $d(t)$ are arbitrary smooth functions, C_1, C_2 are constants.

From the last two equations (2.5) for h we get that $\xi^1 = C_3 t + C_4$ and

$$\xi^2 \omega_\xi + \xi^3 \omega_\eta + C_3 \omega = 0, \tag{2.6}$$

It is the classifying equation and now we can find the function h in the form

$$h = \int \left[u(\xi^3_\eta - C_3) - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v\right] d\xi + \left[v(\xi^2_\xi - C_3) - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u\right] d\eta. \tag{2.7}$$

Basis for the Lie algebra is

$$\begin{gathered}
L_0 \,:\, Y_1 = \partial_t, \quad Y_2 = y\partial_x - x\partial_y, \; Y_n = n(t)\partial_x + n_t(t) x \partial_\varphi, \\
Y_m = m(t)\partial_y + m_t(t) y \partial_\varphi, \quad Y_d = d(t)\partial_\varphi.
\end{gathered} \tag{2.8}$$

It is convenient to use the new unknowns $\bar{\xi}^2(\xi,\eta)$, $\bar{\xi}^3(\xi,\eta)$ by the formulae

$$\xi^2 = C_1 \xi + \bar{\xi}^2(\xi,\eta), \quad \xi^3 = C_1 \eta + \bar{\xi}^3(\xi,\eta), \tag{2.9}$$

then the classifying part of system (2.5) of determining equations can be rewritten as

$$\begin{gathered}
\bar{\xi}^2_\xi + \bar{\xi}^3_\eta = 0, \quad h_\xi = u(C_1 - C_3 + \bar{\xi}^3_\eta) - (C_1\xi + \bar{\xi}^2)u_\xi - (C_1\eta + \bar{\xi}^3)u_\eta - \bar{\xi}^3_\xi v, \\
h_\eta = v(C_1 - C_3 + \bar{\xi}^2_\xi) - (C_1\xi + \bar{\xi}^2)v_\xi - (C_1\eta + \bar{\xi}^3)v_\eta - \bar{\xi}^2_\eta u, \\
(C_1\xi + \bar{\xi}^2)\omega_\xi + (C_1\eta + \bar{\xi}^3)\omega_\eta + C_3\omega = 0.
\end{gathered} \tag{2.10}$$

If $\omega = \omega_0 = \text{const}$, then $C_3 = 0$ and L_0 is prolonged by the generators

$$Y_\psi = \psi_\eta \partial_\xi - \psi_\xi \partial_\eta + h\partial_\varphi, \quad Y_3 = \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + (2\varphi + h)\partial_\varphi \tag{2.11}$$

with arbitrary function $\psi(\xi,\eta)$. Function $h(\xi,\eta)$ is determined from (2.7), where $C_3 = 0$, $\xi^2 = \bar{\xi}^2 = \psi_\eta$, $\xi^3 = \bar{\xi}^3 = -\psi_\xi$ with arbitrary function $\psi(\xi,\eta)$, $u = \bar{u}(\xi,\eta) - \omega_0\eta$ for generator Y_ψ and $C_3 = 0$, $\xi^2 = \xi$, $\xi^3 = \eta$, $u = \bar{u}(\xi,\eta) + \omega_0\eta$ for generator Y_3. The functions $\bar{u}(\xi,\eta)$, $v(\xi,\eta)$ are the solution of Cauchy–Riemann system: $\bar{u}_\xi + v_\eta = 0$, $\bar{u}_\eta - v_\xi = 0$.

If $\omega(\xi,\eta) \neq$ const basis for the Lie algebra extended L_0 is given by the generators

$$Y_F = \frac{\partial F(\omega)}{\partial \eta}\,\partial_\xi - \frac{\partial F(\omega)}{\partial \xi}\,\partial_\eta, \quad Y_4 = \delta t\partial_t + \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + (2-\delta)\varphi\partial_\varphi. \tag{2.12}$$

Function F is arbitrary for Y_F and $h(\xi,\eta)$ is found from (2.10), where $C_1 = C_3 = 0$, $\bar{\xi}^2 = \partial F(\omega)/\partial\eta$, $\bar{\xi}^3 = -\partial F(\omega)/\partial\xi$. For generator Y_4 δ is a constant and the vorticity $\omega(\xi,\eta)$ has the form

$$\omega(\xi,\eta) = \eta^{-\delta} f(\xi/\eta), \tag{2.13}$$

with arbitrary function $f(\zeta)$, $\zeta = \xi/\eta$. In this case $h = 0$.

3. Invariance of the initial conditions

Solving the system of equations (1.7)–(1.9), which is not normal with respect to time, it is necessary to take into account the initial conditions

$$x = \xi, \quad y = \eta, \quad t = 0. \tag{3.1}$$

We require the relations (3.1) are invariant under action of the generator Y from (2.5). The system of determining equations has the simpler form

$$\begin{gathered}\xi^1 = C_3 t, \quad \xi^2 = C_1\xi + C_2\eta + n(0), \quad \xi^3 = C_1\eta - C_2\xi + m(0),\\ \eta^1 = C_1 x + C_2 y + n(t), \quad \eta^2 = C_1 y - C_2 x + m(t),\\ \eta^3 = (2C_1 - C_3)\varphi + n_t x + m_t y + h(\xi,\eta) + d(t),\\ h_\xi = (C_1 - C_3)u - \xi^2 u_\xi - \xi^3 u_\eta + C_2 v,\\ h_\eta = (C_1 - C_3)v - \xi^2 v_\xi - \xi^3 v_\eta - C_2 u;\end{gathered} \tag{3.2}$$

$$(C_1\xi + C_2\eta + n(0))\omega_\xi + (C_1\eta - C_2\xi + m(0))\omega_\eta + C_3\omega = 0. \tag{3.3}$$

The basic algebra L_{00} includes the generators

$$\begin{gathered}Z_n = n(t)\partial_x + n_t(t)x\partial_\varphi, \quad Z_m = m(t)\partial_y + m_t(t)y\partial_\varphi,\\ Z_d = d(t)\partial_\varphi, \quad n(0) = m(0) = 0.\end{gathered} \tag{3.4}$$

Here the equivalence transformations have the form (2.1), (2.2). To carry out the group classification let us rewrite the classifying eq'n (3.3) in the form

$$(A\xi + B\eta + C)\omega_\xi + (A\eta - B\xi + D)\omega_\eta + H\omega = 0 \tag{3.5}$$

with some constants A, B, C, D and H. It is easy to verity that eq'n (3.5) in invariant with respect to equivalence transformations (2.1) with the transformations of A, B, C, D, H such as

$$\begin{gathered}\bar{A} = A, \quad \bar{B} = B, \quad \bar{C} = \cos a_2 C - \sin a_2 D - a_3 A - a_4 B,\\ \bar{D} = \cos a_2 D - \sin a_2 C - a_4 A + a_4 B, \quad \bar{H} = H.\end{gathered} \tag{3.6}$$

Under integration of equation (3.6) we consider the following cases:

I. $AB \neq 0$. Then using (3.6) we can set $\bar{C} = \bar{D} = 0$. Eq'n (3.5) has the solution

$$\omega = \exp[\gamma_2 \operatorname{arctg}(\xi/\eta)] f\left((\xi^2 + \eta^2)^{1/2} \exp[\gamma_1 \operatorname{arctg}(\xi/\eta)]\right) \tag{3.7}$$

with constants $\gamma_2 \neq 0$, γ_2 and arbitrary function f.

II. $A \neq 0$, $B = 0$. Then from (3.6) $\bar{C} = 0$ and the solution of eq'n (3.5) has the form

$$\omega = (\eta + \gamma_3)^{\gamma_4} f\left(\frac{\xi}{\eta + \gamma_3}\right) \tag{3.8}$$

with constants γ_3, γ_4 and arbitrary function f.

III. $A = 0$, $B \neq 0$. Then $\bar{C} = 0$ and we obtain the solution of eq'n (3.5) in the form

$$\omega = \exp\left[\gamma_6 \arcsin\left(\frac{\gamma_5 - \xi}{\eta^2 + (\gamma_5 - \xi)^2}\right)\right] f\left(\eta^2 + (\gamma_5 - \xi)^2\right) \tag{3.9}$$

with constants γ_5, γ_6 and arbitrary function f.

IV. $A = 0$, $B = 0$. Then the solution of eq'n (3.5) can be written as

$$\omega = e^{-\gamma_7 \xi} f(\gamma_8 \xi - \gamma_9 \eta), \tag{3.10}$$

if $C \neq 0$, or

$$\omega = e^{-\gamma_7 \eta} f(\gamma_8 \xi - \gamma_9 \eta), \tag{3.11}$$

if $D \neq 0$ with γ_7, γ_8, γ_9 are constants and f is arbitrary function.

Substituting solutions (3.7)–(3.11) to classifying equation (3.3) and using formulae (3.2) and (2.2) we obtain the extension of the Lie symmetry algebras spanned by generators L_{00}. The result of group classification is presented in Tab. 1. There we can find the form of the function ω and basis of generator which are possessed by equations (1.7)–(1.9) and initial conditions (3.1).

Table 1. Results of group classification

ω	Generators	Remarks
(3.7), f is arbitrary function	L_{00}, Z_1	
(3.7), $f = 1$	L_{00}, Z_1, Z_2, Z_3	
(3.7), $f = \zeta^{\gamma_8}$, $\gamma_8 \neq 0$	L_{00}, Z_4, Z_5	$\zeta = (\xi^2 + \eta^2)^{1/2} \exp[\gamma_1 \operatorname{arctg}(\xi/\eta)]$
(3.7), $f = \zeta^{\gamma_2}$, $\gamma_2 \neq 0$, $\gamma_1 = 1$	L_{00}, Z_1, Z_6	$\zeta = (\xi^2 + \eta^2)^{1/2} \exp[\operatorname{arctg}(\xi/\eta)]$
(3.8), f is arbitrary function	L_{00}	
(3.8), $f = 1$	L_{00}, Z_7	
(3.8), $f = (\zeta + \gamma_6)^{\gamma_4}$, $\gamma_4 \neq 0$	L_{00}, Z_8	$\zeta = \xi/(\eta + \gamma_3)$
(3.9), f is arbitrary function	L_{00}	
(3.9), $f = 1$	L_{00}, Z_9	
(3.9), $f = (\zeta - \gamma_5^2)^{\gamma_7}$, $\gamma_7 \neq 0$	L_{00}, Z_{10}, Z_{11}	$\zeta = \eta^2 + (\gamma_5 - \xi)^2$
(3.10), f is arbitrary function	L_{00}	
(3.10), $f = 1$	L_{00}, Z_{12}	
(3.10), $f = e^{\gamma_{10}\zeta}$, $\gamma_{10} \neq 0$	L_{00}, Z_{13}	$\zeta = \gamma_8 \xi - \gamma_9 \eta, \gamma_8^2 + \gamma_9^2 \neq 0$
(3.10), $f = \zeta^{\gamma_{12}}$	L_{00}, Z_{14}, Z_{15}	$\zeta = \gamma_8 \xi - \gamma_9 \eta, \gamma_9 \neq 0$
(3.10), $f = (\zeta + \gamma_{11})^{\gamma_{12}}$, $\gamma_{11} \neq 0$	L_{00}, Z_{16}, Z_{17}	$\zeta = \gamma_8 \xi - \gamma_9 \eta, \gamma_9 \neq 0$

In Tab. 1 the following generators are involved in extension of the algebra L_{00} :

$$Z_1 = \gamma_2 t\partial_t + (\xi-\eta)\partial_\xi + (\xi+\eta)\partial_\eta + (x-y)\partial_x + (x+y)\partial_y + [(2-\gamma_2)\varphi + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (1-\gamma_2)u - (\xi-\eta)u_\xi - (\xi+\eta)u_\eta - v,$$
$$h_\eta = (1-\gamma_2)v - (\xi-\eta)v_\xi - (\xi+\eta)v_\eta + u;$$

$$Z_2 = \xi\partial_\xi + \eta\partial_\eta + y\partial_y + [2\varphi + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = u - \xi u_\xi - \eta u_\eta, \quad h_\eta = v - \xi v_\xi - \eta v_\eta;$$

$$Z_3 = -\gamma_2 t\partial_t + \eta\partial_\xi - \xi\partial_\eta + y\partial_x - x\partial_y + [\gamma_2\varphi + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = \gamma_2 u - \eta u_\xi + \xi u_\eta + v, \quad h_\eta = \gamma_2 v - \eta v_\xi + \xi v_\eta - u;$$

$$Z_4 = -\gamma_8 t\partial_t + \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + [(2+\gamma_8)\varphi + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (1+\gamma_8)u - \xi u_\xi - \eta u_\eta, \quad h_\eta = (1+\gamma_8)v - \xi v_\xi - \eta v_\eta;$$

$$Z_5 = -(\gamma_2+\gamma_8)t\partial_t + \eta\partial_\xi - \xi\partial_\eta + y\partial_x - x\partial_y + [(\gamma_2+\gamma_8)\varphi + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (\gamma_2+\gamma_8)u - \eta u_\xi + \xi u_\eta + v, \quad h_\eta = (\gamma_2+\gamma_8)v - \eta v_\xi + \xi v_\eta - u;$$

$$Z_6 = a(0)\partial_\xi - a(0)\partial_\eta + a(t)\partial_x - a(t)\partial_y + [a_t(x+y) + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = -a(0)u_\xi + a(0)u_\eta, \quad h_\eta = -a(0)v_\xi + a(0)v_\eta;$$

$$Z_7 = -\gamma_4 t\partial_t + \xi\partial_\xi + (\eta+\gamma_3)\partial_\eta + x\partial_x + (y+\gamma_3 a(t))\partial_y + [(2+\gamma_4)\varphi + \gamma_3 a_t y + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (1+\gamma_3)u - \xi u_\xi - (\eta+\gamma_3)u_\eta, \quad h_\eta = (1+\gamma_3)v - \xi v_\xi - (\eta+\gamma_3)v_\eta;$$

$$Z_8 = -\gamma_6\partial_\xi - \gamma_6 a(t)\partial_x + \partial_\eta + a(t)\partial_x + (-\gamma_6 a_t x + a_t y + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = -\gamma_6 u_\xi - u_\eta, \quad h_\eta = -\gamma_6 v_\xi - v_\eta, \quad a(0) = 1;$$

$$Z_9 = (\xi-\gamma_5)\partial_\xi + \eta\partial_\eta + (x-\gamma_5 a(t))\partial_x + y\partial_y + (2\varphi - \gamma_5 a_t x + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = u - (\xi-\gamma_5)u_\xi - \eta u_\eta, \quad h_\eta = v - (\xi-\gamma_5)v_\xi - \eta v_\eta;$$

$$Z_{10} = -2\gamma_7 t\partial_t + (\xi-\gamma_5)\partial_\xi + \eta\partial_\eta + (x-\gamma_5 a(t))\partial_x + y\partial_y + [2(\gamma_7+1)\varphi - \gamma_5 a_t x + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (2\gamma_7+1)u - (\xi-\gamma_5)u_\xi - \eta u_\eta, \quad h_\eta = (2\gamma_7+1)v - (\xi-\gamma_5)v_\xi - \eta v_\eta;$$

$$Z_{11} = \gamma_6 t\partial_t + \eta\partial_\xi - (\xi-\gamma_5)\partial_\eta + y\partial_x - (x-\gamma_5 a(t))\partial_y + (-\gamma_6\varphi + \gamma_5 a_t y + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = \gamma_6 u - \eta u_\xi + \xi u_\eta + v, \quad h_\eta = \gamma_6 v - \eta v_\xi + \xi v_\eta - u, \ a(0) = 1;$$

$$Z_{12} = \gamma_7 t\partial_t + \partial_\xi + a(t)\partial_x + (-\gamma_7\varphi + a_t x)\partial_\varphi,$$
$$h_\xi = -\gamma_7 u - u_\xi, \quad h_\eta = -\gamma_7 v - v_\xi, \ a(0) = 1;$$

$$Z_{13} = \gamma_{10}\gamma_9 t\partial_t + \partial_\eta + a(t)\partial_y + (-\gamma_{10}\gamma_9\varphi + a_t y + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = -\gamma_{10}\gamma_9 u - u_\eta, \quad h_\eta = -\gamma_{10}\gamma_9 v - v_\eta, \ a(0) = 1;$$

$$Z_{14} = -\gamma_{12} t\partial_t + \xi\partial_\xi + a(t)x\partial_x + [(2+\gamma_{12}\varphi + a_t x + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = (1+\gamma_{12})u - \xi u_\xi - \eta u_\eta, \quad h_\eta = (1+\gamma_{12})v - \xi v_\xi - \eta v_\eta;$$

$$Z_{15} = \partial_\xi - \gamma_9^{-1}\gamma_8\partial_\eta + a(t)\partial_x - \gamma_9^{-1}\gamma_8 a(t)\partial_y + (a_t x - \gamma_9^{-1}\gamma_8 a_t y + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = -u_\xi - \gamma_9^{-1}\gamma_8 u_\eta, \quad h_\eta = -v_\xi - \gamma_9^{-1}\gamma_8 v_\eta, \ a(0) = 1;$$

$$Z_{16} = -\gamma_{12} t\partial_t + \xi\partial_\xi + (\eta - \gamma_9^{-1}\gamma_{11})\partial_\eta + x\partial_x + [y - \gamma_9^{-1}\gamma_{11} a(t)]\partial_y + $$
$$+ (\varphi\partial_\varphi - \gamma_9^{-1}\gamma_{11} a_t y + h(\xi,\eta))\partial_\varphi,$$
$$h_\xi = (1 + \gamma_{12} u - \xi u_\xi - (\eta - \gamma_9^{-1}\gamma_{11}) u_\eta,$$
$$h_\eta = (1 + \gamma_{12} v - \xi v_\xi - (\eta - \gamma_9^{-1}\gamma_{11}) v_\eta, \ a(0) = 1;$$

$$Z_{17} = \partial_\xi + a(t)\partial_x + \gamma_9^{-1}\gamma_8 a(t)\partial_y + \gamma_9^{-1}\gamma_8\partial_\eta + [\gamma_9^{-1}\gamma_8 a_t y + h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi = -u_\xi - \gamma_9^{-1}\gamma_8 u_\eta, \quad h_\eta = -v_\xi - \gamma_9^{-1}\gamma_8 v_\eta, \ a(0) = 1.$$

Remark 2. For potential flows $u = \varphi_{0\xi}$, $v = \varphi_{0\eta}$ and range of $\varphi \to \varphi + \varphi_0$ gives $u = v = 0$. Basic algebra Lie of generators is the following

$$\mu(t)\partial_t - \mu_t(t)\varphi\partial_\varphi, \ y\partial_x - x\partial_y, \ n(t)\partial_x + n_t(t)\partial_\varphi, \ m(t)\partial_y + m_t(t)\partial_\varphi,$$

$$d(t)\partial_\varphi, \ \psi_\eta\partial_\xi - \psi_\xi\partial_\eta, \ \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + 2\varphi\partial_\varphi$$

with arbitrary smooth functions $\mu(t)$, $n(t)$, $m(t)$, $d(t)$, $\psi(\xi,\eta)$.

4. Arbitrary Lagrangian coordinates

The Cauchy problem

$$\frac{dx}{dt} = u(x,y,t), \quad \frac{dy}{dt} = v(x,y,t),$$
$$x\big|_{t=0} = \xi, \quad y\big|_{t=0} = \eta$$

defines the Lagrangian coordinates ξ and η as the Cartesian coordinates of a fluid particle at the initial time. However, instead of ξ and η distinguishing between points, we can take any quantities a and b connected with ξ and η via a one-to-one correspondence, see Fig. 1,

$$\xi = f(a,b), \quad \eta = g(a,b), \quad J = f_a g_b - f_b g_a \neq 0. \tag{4.1}$$

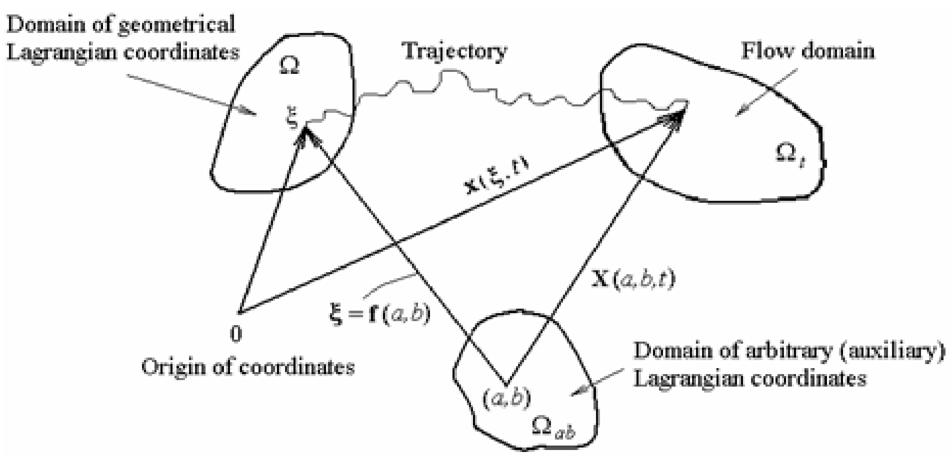

Fig. 1. Arbitrary Lagrangian coordinates

System (1.7)–(1.9) is equivalent to the system

$$JX_t = Y_b(\Phi_a + U_1) - Y_a(\Phi_b + V_1); \tag{4.2}$$

$$JY_t = -X_b(\Phi_a + U_1) + X_a(\Phi_b + V_1); \tag{4.3}$$

$$X_aY_b - X_bY_a = J(a, b) \tag{4.4}$$

with the initial data

$$X\Big|_{t=0} = f(a, b), \quad Y\Big|_{t=0} = g(a, b). \tag{4.5}$$

New "velocities" $U_1(a, b)$, $V_1(a, b)$ are connected with $u(\xi, \eta)$, $v(\xi, \eta)$ by the relations

$$\begin{aligned} U_1(a, b) &= f_a u(f(a, b), g(a, b)) + g_a v(f(a, b), g(a, b)), \\ V_1(a, b) &= f_b u(f(a, b), g(a, b)) + g_b v(f(a, b), g(a, b)). \end{aligned} \tag{4.6}$$

We substitute $X(a, b, t) = x(\xi, \eta, t)$, $Y(a, b, t) = y(\xi, \eta, t)$, $\Phi(a, b, t) = \varphi(\xi, \eta, t)$ in to (4.2)–(4.4) with ξ and η defined by (4.1). It is easy to see that the function x, y, φ satisfy system (1.7)–(1.9), moreover $x = \xi$, $y = \eta$ under $t = 0$. Hence, the Lie group of transformations for the systems (1.7)–(1.9) and (4.2)–(4.4) are similar. The generator admitted by system (4.2)–(4.4) has the form

$$Y_1 = \xi^1(t)\partial_t + \xi_1^2(a, b)\partial_a + \xi_1^3(a, b)\partial_b + \eta^1\partial_X + \eta^2\partial_Y + \eta^3\partial_\Phi,$$

where $\xi^1(t)$, η^1, η^2, η^3 are the same as for generator Y. The coordinates $\xi_1^2(a, b)$, $\xi_1^3(a, b)$ satisfy the determining equations

$$\begin{aligned} (J\xi_1^2)_a + (J\xi_1^3)_b &= 2C_1 J, \\ \xi_1^2(\omega_1/J)_a + \xi_1^3(\omega_1/J)_b + C_3\omega_1/J &= 0, \end{aligned} \tag{4.7}$$

where $\omega_1(a, b) = J(a, b)\omega(f(a, b), g(a, b)) = V_{1a} - U_{1,b}$ is a new "vorticity".

5. Examples of exact solutions

5.1. Unsteady flow in a layer with one free boundary

Let us take $v = v(\eta)$, $v_\eta \neq 0$, then from (1.10) $u(\xi, \eta) = -v_\eta\xi + u_1(\eta)$ with some function $u_1(\eta)$. In such case the initial vorticity has the form $\omega(\xi, \eta) = v_{\eta\eta}\xi - u_{1\eta}$. Because of $u_1(\eta)$ is arbitrary the system (1.7)–(1.9) admits 3D subalgebra $\langle\partial_\xi, \partial_x, \partial_\varphi\rangle$. Since the variables t, η and y are invariant, we seek a partially invariant solution of rank 2 and defect 2 in the form $x = x(\xi, \eta, t)$, $y = y(\eta, t)$ and $\varphi = \varphi(\xi, \eta, t)$. Inserting it into (1.9), we obtain

$$x = a(\eta, t)\xi + b(\eta, t), \quad y = \int_0^\eta \frac{d\eta}{a(z, t)} \tag{5.1}$$

with initial data

$$a(\eta, 0) = 1, \quad a_t(\eta, 0) = -v_\eta, \quad b(\eta, 0) = 0, \quad b_t(\eta, 0) = u_1(\eta). \tag{5.2}$$

The functions $a(\eta, t)$, $b(\eta, t)$ and $\varphi = \varphi(\xi, \eta, t)$ can be found from equations (1.7) and (1.8) as

$$
\begin{gathered}
a = f(t)\left[1 - v_\eta \int_0^t \frac{d\tau}{f^2(\tau)}\right], \quad b = u_1(\eta) f(t) \int_0^t \frac{d\tau}{f^2(\tau)}, \\
\varphi = (a a_t + v_\eta)\frac{\xi^2}{2} + [(a b_t)_\eta - u_{1\eta}]\xi + \int_0^\eta \left(b_t b_\eta + \frac{y_t}{a} - v\right) d\eta,
\end{gathered} \tag{5.3}
$$

where $f(t)$ is arbitrary function satisfying the conditions $f(0) = 1$, $f_t(0) = 0$. Then, from Weber's integral (1.11) and formulae (5.1), (5.3) we find the pressure

$$
p = l(\eta, t) - \frac{k(t)}{2} x^2(\xi, \eta) - g y(\eta, t), \tag{5.4}
$$

where

$$
l(\eta, t) = l(t) - \int_0^\eta \frac{y_{tt}(\eta, t)}{a(\eta, t)} d\eta, \quad k(t) = \frac{f_{tt}}{f(t)} \tag{5.5}
$$

and $l(t)$ is arbitrary function.

Solution (5.1)–(5.5) can be interpreted as the motion of plane layer (or a filled-up fluid layer) with a free boundary (or a moving rigid wall). Really, assume at the initial time there is a liquid layer of thickness $y = h_0 = \text{const}$. The lower rigid wall $y = 0$ is fixed and upper boundary $y = h_0$ is a free one. The initial velocity field is given the formulae $u = u_1(\eta)\xi + u_2(\eta)$, $v = -\int\limits_0^\eta u_1(\eta)\, d\eta$. The further motion is described by formulae (5.1)–(5.5). Of course, the outer pressure must be equals $p_{out} = l(t) - k(t)x^2(\xi, \eta)/2 - g y(\eta, t)$ and the evolution of free boundary $h(t)$ is given by the expression

$$
h(t) = \frac{1}{f(t)} \int_0^{h_0} \left[1 - v_\eta(\eta) \int_0^t \frac{d\tau}{f^2(\tau)}\right]^{-1} d\eta. \tag{5.6}
$$

5.2. Example of Gerstner's waves

Let us take mapping (4.1) in the form

$$
\xi = a + \frac{1}{k} e^{kb} \sin(ka), \quad \eta = b - \frac{1}{k} e^{kb} \cos(ka), \quad J = 1 - e^{2kb}. \tag{5.7}
$$

It is one-to-one under conditions $k > 0$, $b \leqslant b_0 < 0$, but there is not explicit dependence a, b on ξ, η. However, the formulas

$$
\begin{gathered}
X = a + \frac{1}{k} e^{kb} \sin[k(a + ct)], \quad Y = b - \frac{1}{k} e^{kb} \cos[k(a + ct)], \\
\Phi = \frac{c e^{kb}}{k} \{\sin[k(a + ct)] - \sin(ka)\}
\end{gathered} \tag{5.8}
$$

give us the exact solution of system (4.2)–(4.4) (k, c are constant).

Physical vorticity is

$$
\omega = -\frac{2kc e^{kb}}{1 - e^{2kb}}. \tag{5.9}
$$

The fluid pressure can be found from integral (1.5)

$$
p = g(b_0 - b) + \frac{1}{2} c^2 (e^{2kb} - e^{2kb_0}), \quad c^2 = g/k. \tag{5.10}
$$

Formulas (5.7), (5.8), (5.10) determine the water waves called Gerstner's waves [12]. These waves are stationary in coordinate system which moves with velocity $-c$ along the x-axis. Free boundary is a straight line $b = b_0$ (in variables a, b) or it is trochoid with wave number $2\pi/k$. Such waves are always vortex flow and vorticity has a maximum on free boundary and decreases exponentially with the fluid depth.

If $b > 0$ then mapping (5.7) is not one-to-one and such solution has not physical meaning. The case $b_0 = 0$ gives us the cycloid equation

$$x = a + \frac{1}{k}\sin(ka), \quad y = b - \frac{1}{k}\cos(ka).$$

For $ka = (2n+1)\pi$, $n \in N$, this cycloid has a cusps.

In conclusion, Gerstner's waves are the invariant solution of system (4.2)–(4.4) with respect to two-dimensional subalgebra $\langle c^{-1}\partial_t - \partial_a - \partial_X;\ \partial_\Phi\rangle$ when $J = 1 - e^{2kb}$, $f = a + k^{-1}e^{kb}\sin(ka)$, $g = b - k^{-1}e^{kb}\cos(ka)$. New initial "velocities" have the form

$$U_1 = ce^{2kb} + ce^{kb}\cos(ka), \quad V_1 = ce^{kb}\sin(ka),$$

where $c = (g/k)^{1/2}$. Notice that $U_{1a} + V_{1b} = 0$.

Conclusion

The equations describing 2D ideal liquid flows under potential force action are considered. Using Lagrangian coordinates this system is reduced to the system for particle trajectories and Weber's potential. Group classification with respect to the initial vorticity is performed. The equivalence group admitted by the governing equations is calculated. This group is infinite one and is used to reduce arbitrary element to the simpler form. As the example two exact solutions are constructed.

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Симметрийный анализ уравнений идеальной жидкости в терминах траектории – потенциал Вебера

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Рассматриваются уравнения двумерных движений идеальной жидкости в координатах Лагранжа. Для потенциальных внешних сил они имеют общий интеграл Вебера, причем новая система включает в себя начальные данные. Это делает актуальной задачу групповой классификации. Установлено, что основная группа непрерывных преобразований является бесконечномерной по пространственным координатам. Найдены специальные зависимости начальной завихренности, при которых происходит расширение группы. Кроме того, изучены групповые свойства исходной системы в произвольных лагранжевых координатах и найдены точные решения.

Ключевые слова: уравнения Эйлера, симметрийный анализ, преобразование Вебера, преобразование эквивалентности, групповая классификация.