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Group Analysis of Hydrostatic Model Equations for a Viscous Fluid

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The group properties of three-dimensional hydrostatic model equations of a viscous fluid are investigated. The examples of several exact solutions are presented. The free surface of fluid and the pressure on this surface are determined.

Keywords: viscous fluid, hydrostatic model, group analysis, exact solutions.

The group analysis of differential equations is a powerful instrument for studying non-linear equations and boundary value problems. It was an outstanding mathematician Sophus Lie who introduced this scientific method in his works in the 19th century. The interest to the group analysis was revived by L.V. Ovsyannikov, who pointed out in [1], [2] a method for describing the properties of differential equations.

One of the main problems in the group analysis of differential equations is the study of the permissible group of transformations of the system of equations on the set of solutions of these equations.

Lie theory group properties of differential equations are studied by L.V. Ovsyannikov [2], N.H. Ibragimov [3], V.V. Pukhnachov, their students and followers S.V. Habirov, Y.N. Pavlovsky, A.A. Buchnev, O.V. Bytev, V.K. Andreev, etc. At present the group properties of equations in liquid mechanics are investigated by V.K. Andreyev, O.V. Kaptsov, V.V. Pukhnachev, A.A. Rodionov [4].

The Navier-Stokes equations are a system of differential equations partially describing the motion of the viscous liquid. The aim of this study is to perform a group analysis of the hydrostatic model of Navier-Stokes equations in three – dimensional case and to find exact solutions of this model.

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1. Problem Statement. Basic Equations

The three-dimensional Navier-Stokes equations for the motion of a viscous incompressible fluid are considered

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + \tfrac{1}{\rho}p_x &= \nu(u_{xx} + u_{yy} + u_{zz}), \\ v_t + uv_x + vv_y + wv_z + \tfrac{1}{\rho}p_y &= \nu(v_{xx} + v_{yy} + v_{zz}), \\ w_t + uw_x + vw_y + ww_z + \tfrac{1}{\rho}p_z &= \nu(w_{xx} + w_{yy} + w_{zz}) - g, \\ u_x + v_y + w_z &= 0. \end{aligned} \tag{1}$$

Here u, v, w - are the components of the velocity vector along the x, y, z directions, p - is the pressure, t - is the time, $g = \text{const} > 0$ - is the acceleration of gravity in the z axis direction, ν - is the dynamic viscosity coefficient, $\rho = const$ - fluid density (can be regarded as $\rho = 1$).

Let's suppose, that the pressure in the fluid is linearly dependent on the depth,

$$p_z = -g. \tag{2}$$

The assumption is often used to describe processes in oceanology [5]. Then

$$p(x, y, z, t) = -gz + q(x, y, t), \tag{3}$$

$q(x, y, t)$ — is a new function. In this case, the system (1) will be rewritten as

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + q_x &= \nu(u_{xx} + u_{yy} + u_{zz}), \\ v_t + uv_x + vv_y + wv_z + q_y &= \nu(v_{xx} + v_{yy} + v_{zz}), \\ w_t + uw_x + vw_y + ww_z &= \nu(w_{xx} + w_{yy} + w_{zz}), \\ u_x + v_y + w_z = 0, &\quad q_z = 0. \end{aligned} \tag{4}$$

Let $\Gamma : z = \eta(x, y, t)$ — be the equation for the free boundary of a fluid on which the following kinematic and dynamic conditions are satisfied:

$$\eta_t + u(x, y, \eta(x, y, t), t)\eta_x + u(x, y, \eta(x, y, t), t)\eta_y = w(x, y, \eta(x, y, t), t), \tag{5}$$

$$(p_a - p)\overrightarrow{n} + 2\nu D \cdot \overrightarrow{n} = 2\sigma H \overrightarrow{n}, \tag{6}$$

where $p_a(x, y, t)$ - is the atmospheric pressure, $p = -g\eta(x, y, t) + q(x, y, t)$, the normal to the free surface $\overrightarrow{n}$ and the mean curvature of H depend on the position of the point on the surface, $\sigma = const$ - is the surface tension coefficient, $D = D(u, v, w)$ - is the deformation rate tensor [4].

If the solution of the system (4)is known, then from the equality (5) the free surface the equation can be found, and $p_a(x, y, t)$ can be explained from (6).

The task of group analysis [2] is set for the equations of the system (4). It is required to find the Lie algebra of admissible operators for this system and to construct exact solutions.

A similar study of the hydrostatic model of an ideal fluid was carried out in [6].

2. Group properties of the equations

Let us consider the group properties of equations (4). Let introduce the following index designations: $u^1 = u$, $u^2 = v$, $u^3 = w$, $u^4 = q$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$. In these new designations, equations (4) supplemented by the requirement (4) assume the following form:

$$
\begin{gathered}
u_4^1 + u^1 u_1^1 + u^2 u_2^1 + u^3 u_3^1 + u_1^4 - \nu(u_{11}^1 + u_{22}^1 + u_{33}^1) = 0,\\
u_4^2 + u^1 u_1^2 + u^2 u_2^2 + u^3 u_3^2 + u_2^4 - \nu(u_{11}^2 + u_{22}^2 + u_{33}^2) = 0,\\
u_4^3 + u^1 u_1^3 + u^2 u_2^3 + u^3 u_3^3 - \nu(u_{11}^2 + u_{22}^2 + u_{33}^2) = 0,\\
u_1^1 + u_2^2 + u_3^3 = 0, \quad u_3^4 = 0.
\end{gathered} \tag{7}
$$

The lower index is the differentiation.

The admissible operator for system (7) is found in the following way:

$$X = \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \eta^k(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^k},$$

Here, the summation is done over $i, k = 1, 2, 3, 4$. Let us extend the operator to the first derivatives

$$\underset{1}{X} = X + \zeta_i^k \frac{\partial}{\partial u_i^k} \qquad \zeta_i^k = \frac{\partial \eta^k}{\partial x^i} + u_i^l \frac{\partial \eta^k}{\partial u^l} - u_j^k \left(\frac{\partial \xi^j}{\partial x^i} + u_i^l \frac{\partial \xi^j}{\partial u^l} \right)$$

and to the second derivatives

$$\underset{2}{X} = \underset{1}{X} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} = \underset{1}{X} + \zeta_{11}^1 \frac{\partial}{\partial u_{11}^1} + \zeta_{22}^1 \frac{\partial}{\partial u_{22}^1} + \zeta_{11}^2 \frac{\partial}{\partial u_{11}^2} + \zeta_{22}^2 \frac{\partial}{\partial u_{22}^2} + \ldots,$$

where

$$\zeta_{ij}^\alpha = \frac{\partial \zeta_i^\alpha}{\partial x^j} + u_i^l \frac{\partial \zeta_i^\alpha}{\partial u^l} + u_{jk}^l \frac{\partial \zeta_i^\alpha}{\partial u_k^l} - u_{ik}^\alpha \left(\frac{\partial \xi^k}{\partial x^l} + u_j^l \frac{\partial \xi^k}{\partial x^l} \right),$$

here, the summation is done over l, k.

Note, in system (7) the values u_{12}^1, u_{13}^1, u_{23}^1, u_{12}^2, u_{13}^2, u_{23}^2, u_{12}^3, u_{13}^3, u_{23}^3 are absent. Thus, from the invariance criterion [2], acting via the operator $\underset{2}{X}$ onto equations (7), the determining equations are odtained. At the same time let's pay attention to (7), exchanging u_4^1, u_4^2, u_4^3, u_1^1 for the remaining variables. Splitting determining equations as related to independent variables, the coordinates of the operator X are odtained

$$
\begin{gathered}
\xi^1 = C_4 x^1 + f(x^4), \quad \xi^2 = C_4 x^2 + C_5 x^1 + h(x^4),\\
\xi^3 = C_2 + C_3 x^4 + C_4 x^3, \quad \xi^3 = C_1 + 2C_4 x^4,\\
\eta^1 = -C_4 u^1 - C_5 u^2 + f'(x^4), \quad \eta^2 = -C_4 u^2 + C_5 u^1 + h'(x^4),\\
\eta^3 = C_3 + 2C_4 x^4, \quad \eta^4 = -2C_4 u^4 - x^1 f''(x^4) - x^2 h''(x^4) + \varphi(x^3),
\end{gathered}
$$

where $C_1, \ldots, C_5$ are constants, $f(x^4)$, $h(x^4)$, $\varphi(x^4)$ are arbitrary functions.

It is proven that the Lie algebra for the system of equations (4) is formed by the operators

$$
\begin{gathered}
X_1 = \partial_t, \quad X_2 = \partial_z, \quad X_3 = t\partial_z + \partial_w,\\
X_4 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2q\partial_q,\\
X_5 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,\\
X_6 = f(t)\partial_x + f'(t)\partial_u - xf''(t)\partial_q,\\
X_7 = h(t)\partial_y + h'(t)\partial_v - yh''(t)\partial_q, \quad X_8 = \varphi(t)\partial_q.
\end{gathered} \tag{8}
$$

The first operator is responsible for the transfer in time t, the second and third are responsible for the transfer and the Galileo transformation along the z axis, the fourth one for the tensile transformations, the fifth one for the rotation around the $z$axis. The sixth, seventh and eighth operators contain arbitrary functions $f(t), h(t), \varphi(t)$ depending on time, and define the infinite-dimensional part of the Lie algebra of admissible operators.

For the first time a group analysis of the equations of the system (1) was carried out in [7] by V.O. Bytev. The difference of its result from the received operators (7) is that two rotation operators are absent in (7) along the x and y axes, and the infinite-dimensional operator along the z axis, a similar one X_6, X_7, is represented as two finite-dimensional operators X_2, X_3.

If the two-dimensional hydrostatic model is considered, then in the equations (4) the variable y and velocity v should be excluded. Then for the equations

$$\begin{gathered} u_t + uu_x + wu_z + q_x = \nu(u_{xx} + u_{zz}), \\ w_t + uw_x + ww_z = \nu(w_{xx} + w_{zz}), \\ u_x + w_z = 0, \quad q_z = 0 \end{gathered} \tag{9}$$

the algebra of admissible operators is determined by the basis of operators

$$\begin{gathered} X_1 = \partial_t, \quad X_2 = \partial_z, \quad X_3 = t\partial_z + \partial_w, \\ X_4 = x\partial_x + z\partial_z + 2t\partial_t - u\partial_u - w\partial_w - 2q\partial_q, \\ X_5 = f(t)\partial_x + f'(t)\partial_u - xf''(t)\partial_q, \quad X_6 = \varphi(t)\partial_q. \end{gathered} \tag{10}$$

3. Exact solutions

Example 1. Let us seek a solution to the equations (4) on the operators from the basis (8)

$$\langle X_3 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial w}; \quad X_7 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, (h(t) = t)\rangle.$$

The invariants of these operators are

$$J = (x, t; u; v - \frac{y}{t}; w - \frac{z}{t}; q).$$

Therefore, the invariant solution of equations must be sought in the form of

$$(u, v, w, q) = (U(x,t); \frac{y}{t} + V(x,t); \frac{z}{t} + W(x,t); Q(x,t)), \tag{11}$$

where U, V, W, Q - are the desired functions of two variables.
Let's substitute the functions in the equations of the system and obtain the factor-system:

$$\begin{gathered} U_t + UU_x + Q_x = \nu U_{xx}, \quad V_t + UV_x + \tfrac{1}{t}V = \nu V_{xx}, \\ W_t + UW_x + \tfrac{1}{t}W = \nu W_{xx}, \quad U_x + \tfrac{2}{t} = 0. \end{gathered} \tag{12}$$

From the last equation of the factor-system (12) it follows that

$$U(x,t) = -\frac{2x}{t} + h_1(t), \tag{13}$$

where $h_1(t)$ - is an arbitrary function. From the first equation in (12) it follows that

$$Q(x,t) = \frac{2x}{t}h_1(t) - xh_1'(t) - \frac{3x^2}{t^2} + h_2(t),$$

with an arbitrary function $h_2(t)$. From the second equation of the system (12) it can be seen that
$$V_t = \nu V_{xx} + \left(\frac{2x}{t} - h_1(t)\right) V_x - \frac{V}{t}. \tag{14}$$
To solve the equation (14) let us turn to the reference book by A.D. Polyanin [8] (section (1.8.6), p. 129) for the equation of the following form
$$w_t = aw_{xx} + [xm(t) + n(t)]w_x + k(t)w.$$
For our equation (14) the following is given:
$$a = \nu, \quad m(t) = \frac{2}{t}, \quad n(t) = -h_1(t), \quad k(t) = -\frac{1}{t}.$$
Let's introduce the designations ($A, B, C = const$):
$$F(t) = Bexp[\int m(t)dt] = Bt^2,$$
$$\tau = \int F^2(t)dt + A = \frac{B^2t^5}{5} + A,$$
$$\delta = xF(t) + \int g(t)F(t)dt + C = xBt^2 - B\int t^2h_1(t)dt + C.$$
Let's make the transition to the new variables $(x, t) \to (\delta, \tau)$. It is evident that
$$V(x,t) = M(\delta,\tau)exp[\int k(t)dt] = \frac{1}{t}M(\delta,\tau).$$
Here $M(\delta, \tau)$ - is a new function.

Let us suppose that $B = 1, A = C = 0$, then $F(t) = t^2$,
$$\tau = \frac{t^5}{5}, \quad \delta = xt^2 - \int t^2h_1(t)dt,$$
$$V_t = -\frac{1}{t^2}M(\delta,\tau) + \frac{1}{t}[M_\delta\frac{\partial\delta}{\partial t} + M_\tau\frac{\partial\tau}{\partial t}] = -\frac{M(\delta,\tau)}{t^2} + M_\delta(2x - th_1(t)) + M_\tau t^3,$$
$$V_x = \frac{1}{t}[M_\delta\frac{\partial\delta}{\partial x} + M_\tau\frac{\partial\tau}{\partial x}] = M_\delta t; V_{xx} = t^3M_{\delta\delta}.$$
This substitution in (14):
$$-\frac{M(\delta,\tau)}{t^2} + M_\delta(2x - th_1(t)) + M_\tau t^3 = \nu t^3M_{\delta\delta} + \left(\frac{2x}{t} + h_1(t)\right) M_\delta t - \frac{M(\delta,\tau)}{t^2},$$
leads to the equation
$$M_\tau = \nu M_{\delta\delta}. \tag{15}$$
The heat equation with constant coefficients was obtained.

Note that if the variable δ is taken in the form $\delta = xt^2 - \int_1^t t^2h_1(t)dt$, then it can be considered a Lagrangian variable, since $\delta = x$ at $t = 1$.

Take the simplest solution for (15) [5]: $M(\delta) = \alpha\delta + \beta$. Then
$$V(x,t) = \frac{\alpha}{t}(xt^2 - \int t^2h_1(t)dt) + \frac{\beta}{t},$$

where $\alpha, \beta = const$.

From the third equation of the factor-system (12) the following is obtained:

$$W_t = \nu W_{xx} + \left(\frac{2x}{t} - h_1(t)\right) W_x - \frac{W}{t}. \tag{16}$$

The solution of the equation (16) is found through the analogy with the equation (14). Consequently,

$$W(x,t) = \frac{\varepsilon}{t}(xt^2 - \int t^2 h_1(t)dt) + \frac{\mu}{t},$$

where $\varepsilon, \mu = const$.

As a result, the exact solution of the equations (4) is obtained:

$$\begin{aligned} u(x,t) &= h_1(t) - \tfrac{2x}{t}, \\ v(x,y,t) &= \tfrac{1}{t}[y + \alpha(xt^2 - \textstyle\int t^2 h_1(t)dt) + \beta], \\ w(x,z,t) &= \tfrac{1}{t}[z + \varepsilon(xt^2 - \textstyle\int t^2 h_1(t)dt) + \mu], \\ q(x,t) &= \tfrac{2xh_1(t)}{t} - xh_1^{'}(t) - \tfrac{3x^2}{t^2} + h_2(t), \end{aligned} \tag{17}$$

where $h_1(t), h_2(t)$-are arbitrary functions; $\alpha, \beta, \varepsilon, \mu$-are constants.

The kinematic condition (5) on the free boundary $\Gamma : z = \eta(x,y,t)$ for a given solution has the form of:

$$\frac{\partial \eta}{\partial t} + \left(h_1(t) - \frac{2x}{t}\right)\frac{\partial \eta}{\partial x} + \left(\frac{1}{t}[y + \alpha(xt^2 - \int t^2 h_1(t)dt) + \beta]\right)\frac{\partial \eta}{\partial y} =$$

$$= \frac{1}{t}[z + \varepsilon(xt^2 - \int t^2 h_1(t)dt) + \mu].$$

While solving this equation, it can be seen $\eta(x,y,t) = t\Phi(J_1, J_2) - \varepsilon J_1 - \mu$, where $\Phi(J_1, J_2)$ - is an arbitrary function of its arguments

$$J_1 = xt^2 - \int t^2 h_1(t)dt, \quad J_2 = \frac{1}{t}(y + \alpha J_1 + \beta).$$

From the dynamic condition (6) $(p_a - p\big|_\Gamma)\overrightarrow{n} + 2\nu D \cdot \overrightarrow{n} = 2\sigma H \overrightarrow{n}$ it's possible to determine the nesessary atmospheric pressure at the free border,

$$p_a = p\big|_\Gamma + \frac{\nu}{t}[1 + \sqrt{9 + t^4(\alpha^2 + \varepsilon^2)}] + 2\sigma H.$$

Here, for the solution of (17) the deformation rate tensor and the pressure on the surface of the liquid are

$$D = \frac{1}{2t}\begin{pmatrix} -4 & \alpha t^2 & \varepsilon t^2 \\ \alpha t^2 & 2 & 0 \\ \varepsilon t^2 & 0 & 2 \end{pmatrix},$$

$$p\big|_\Gamma = -g\eta(x,y,t) + q(x,y,t) = -g[t\Phi(J_1,J_2) - \varepsilon J_1 - \mu] + \frac{2xh_1(t)}{t} - xh_1^{'}(t) - \frac{3x^2}{t^2} + h_2(t).$$

Example 2. Let's get back to the equation (14):

$$M_\tau = \nu M_{\delta\delta}.$$

Let's use solution for this equation[8]:

$$M(\delta,\tau)=\alpha(\delta^2+2\nu\tau)+\beta.$$

Consequently:

$$V(x,t)=\frac{\alpha(\delta^2+2\nu\tau)+\beta}{t}=\frac{\alpha}{t}\left((xt^2-\int t^2h_1(t)dt)^2+\frac{2\nu t^5}{5}\right)+\frac{\beta}{t}. \tag{18}$$

In a similar way $W(x,t)$ is found:

$$W(x,t)=\frac{\varepsilon}{t}\left((xt^2-\int t^2h_1(t)dt)^2+\frac{2\nu t^5}{5}\right)+\frac{\mu}{t}.$$

As a result, one more exact solution of equations is obtained (4):

$$\begin{aligned} u(x,t)&=h_1(t)-\tfrac{2x}{t},\\ v(x,y,t)&=\tfrac{1}{t}[y+\alpha\left((xt^2-\int t^2h_1(t)dt)^2+\tfrac{2\nu t^5}{5}\right)+\beta],\\ w(x,z,t)&=\tfrac{1}{t}[z+\varepsilon\left((xt^2-\int t^2h_1(t)dt)^2+\tfrac{2\nu t^5}{5}\right)+\mu],\\ q(x,t)&=\tfrac{2xh_1(t)}{t}-xh_1'(t)-\tfrac{3x^2}{t^2}+h_2(t), \end{aligned} \tag{19}$$

where $h_1(t),h_2(t)$ - are arbitrary functions; $\alpha,\beta,\varepsilon,\mu$ - are constants. Thus, knowing the set of solutions of the equation (14), the set of solutions of the equations (4)can be constructed.

Example 3. Let's consider the operators

$$\langle X_2=\frac{\partial}{\partial t};\quad X_5=\frac{\partial}{\partial z};\quad X_6=\frac{\partial}{\partial x},a_1(t)=1\rangle.$$

The invariants of these operators are $J=\{y;u;v;w;q\}$. Therefore, the invariant solution of equations must be sought in the following form

$$(u,v,w,q)=(U(y);V(y);W(y);Q(y)). \tag{20}$$

Let's insert the systems into the equations, and obtain the factor-system for the stationary solution:

$$VU_y=\nu U_{yy},\quad VV_y+Q_y=\nu V_{yy},\quad VW_y=\nu W_{yy},\quad V_y=0. \tag{21}$$

From (21) it is seen that

$$V=C_1,\quad Q=C_2,\quad U(y)=D_1+D_2e^{\frac{C_1}{\nu}y},\quad W(y)=H_1+H_2e^{\frac{C_1}{\nu}y}.$$

Finally the exact solution of the equations (14) is obtained:

$$u(y)=D_1+D_2e^{\frac{C_1}{\nu}y},\quad v(y)=C_1,\quad w(y)=H_1+H_2e^{\frac{C_1}{\nu}y},\quad q(y)=C_2, \tag{22}$$

where C_1,C_2,D_1,D_2,H_1,H_2 - are constants.

The kinematic condition (5) on the free boundary $z=\eta(x,y,t)$ for the solution (22) has the form of:

$$\frac{\partial\eta}{\partial t}+\left(D_1+D_2e^{\frac{C_1}{\nu}y}\right)\frac{\partial\eta}{\partial x}+C_1\frac{\partial\eta}{\partial y}=H_1+H_2e^{\frac{C_1}{\nu}y}.$$

The solution to this equation is the function

$$\eta(x,y,t) = \frac{1}{C_1}\left(H_1 y + H_2 \frac{\nu}{C_1} e^{\frac{C_1}{\nu} y}\right) + \Phi(J_1, J_2),$$

where $\Phi(J_1, J_2)$ - is an arbitrary function of its arguments

$$J_1 = y - C_1 t, \quad J_2 = D_1 y + D_2 \frac{\nu}{C_1} e^{\frac{C_1}{\nu} y} - C_1 x.$$

In the dynamic condition (6) on the free boundary $z = \eta(x,y,t)$ for the solution (22) the deformation rate tensor on the diagonals of the matrix has zero values. Therefore, it is easy to determine that the external atmospheric pressure at the free boundary should be

$$p_a = -g\eta(x,y,t) + C_2 + \frac{C_1}{2\nu} e^{\frac{C_1}{\nu} y} \sqrt{D_2^2 + H_2^2} + 2\sigma H.$$

Example 4. Let us give an example when, while searching for an exact solution, the factor-system gives a contradiction. Let's consider the operators

$$\langle X_4 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial w}; X_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, a_1(t) = t; X_7 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, a_2(t) = t\rangle$$

.
The invariants of these operators are $J = \{t; u - \frac{x}{t}; v - \frac{y}{t}; w - \frac{z}{t}; q\}$. Therefore an invariant solution of the equations must be sought in the following form

$$(u,v,w,q) = (\frac{x}{t} + U(t); \frac{y}{t} + V(t); \frac{z}{t} + W(t); Q(t)), \tag{23}$$

where U, V, W, Q - are the desired functions of one variable.

Let's substitute the equations of the system (4), to obtain the factor system:

$$\begin{gathered} U_t - \tfrac{x}{t^2} + (\tfrac{x}{t} + U)\tfrac{1}{t} = 0, \quad V_t - \tfrac{y}{t^2} + (\tfrac{y}{t} + V)\tfrac{1}{t} = 0, \\ W_t - \tfrac{z}{t^2} + (\tfrac{z}{t} + W)\tfrac{1}{t} = 0, \quad \tfrac{1}{t} + \tfrac{1}{t} + \tfrac{1}{t} = 0. \end{gathered} \tag{24}$$

The last equality in (24) is contradictory. Therefore, a solution of the form (23) does not exist.

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Групповой анализ уравнений гидростатической модели вязкой жидкости

Александр А. Родионов, Дарья А. Краснова

Рассматриваются групповые свойства уравнений трёх-мерной гидростатической модели вязкой жидкости. Представлено несколько примеров точных решений. Определяются свободная поверхность жидкости и давление на ней . . . .

Ключевые слова: групповой анализ, гидростатическая модель, вязкая жидкость, точные решения.