SKETCH OF THE THEORY OF GROWTH OF HOLOMORPHIC FUNCTIONS IN A MULTIDIMENSIONAL TORUS

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Abstract. We develop an approach to the theory of growth of the class $H(\mathbb{T}^n)$ of holomorphic functions in a multidimensional torus $\mathbb{T}^n$ based on the structure of elements of this class and well-known results of the heory of growth of entire functions of several complex variables. This approach is illustrated in the case where the growth of the function $g \in H(\mathbb{T}^n)$ is compared with the growth of its maximum modulus on the skeleton of the polydisk. The properties of the corresponding characteristics of growth of the functions in the class $H(\mathbb{T}^n)$ are studied with their relation to coefficients of the corresponding Laurent series. A comparative analysis of these results and similar assertions of the theory of growth of entire functions of several variables is given.

***Keywords and phrases*:** entire function of several variables, holomorphic function in multidimensional torus, convex function, characteristics of growth, multiple Laurent series, carrier, strictly convex cone.

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1. Introduction. In [7, 8] we considered the class $H(\mathbb{T}^n)$ of holomorphic functions in the multidimensional torus $\mathbb{T}^n = \mathbb{C}_*^n$, which is the extension of the class $H(\mathbb{C}^n)$ of entire functions of n complex variables. It contains a proper subclass $\mathcal{A}(\mathbb{T}^n)$ consisting of functions that are equivalent to entire functions in the following sense: a function g belongs to the class $\mathcal{A}(\mathbb{T}^n)$ if there exists a monomial holomorphic mapping $\mathcal{F}$ such that $f = g \circ \mathcal{F}$ is an entire function. The interest in these classes is due to the modern research on the analysis of toric varieties (see [1, 2]).

The theory of entire functions of several variables is well developed (see, e.g., [3, 12, 13]). The aim of this paper is to give an impetus to the development of the theory of growth of holomorphic functions in multidimensional torus. The structure of functions of the class $H(\mathbb{T}^n)$ was studied in [7, 8], where indicators of their growth were considered. In [9], an approach was presented to the development of theory of growth of functions of the class $\mathcal{A}(\mathbb{T}^n)$ by the example of studying the properties of their Laplace–Borel transformation based on the corresponding results in the case of entire functions.

In this paper, we study the growth of functions in $H(\mathbb{T}^n)$ in the case where the growth of a function $g \in H(\mathbb{T}^n)$ is compared with the growth of its maximum modulus

$$M_g(r) = \max\big\{|g(z)| : |z_k| = r_k,\ k = 1, \dots, n\big\}, \quad r \in \mathbb{R}_0^n,\ \mathbb{R}_0 = \big\{r \in \mathbb{R} : r > 0\big\}, \tag{1}$$

on the skeleton of the polydisk. We rely on the known results on the theory of growth in the class $H(\mathbb{C}^n)$ (see [6]). In particular, we study the relationship between the growth and the coefficients of the function g in its Laurent series. Evidence of the obtained statements is given only when it significantly differs from the proof of similar facts in the case of the class $H(\mathbb{C}^n)$.

2. Functions of the class $H(\mathbb{T}^n)$ that are equivalent to entire functions. A natural analog of entire functions of one variable are functions analytic on the Riemann sphere, except for one point. A similar analog in the case of several variables cannot be expected: analytic functions of many variables do not have isolated singularities. The "layout" of the multidimensional analog of this result is suggested by the following version of the Hadamard–Valiron theorem.

Theorem 2.1. *Let $g(z) = g(z_1, \dots, z_n)$ be a function that is holomorphic in the torus $\mathbb{T}^n$, $n > 1$; $M_g(r)$ be its maximum modulus on the skeleton of the polydisk* (see (1)). *Then both $M_g(r)$ and $\ln M_g(r)$ are convex functions of $\ln r_1, \dots, \ln r_n$.*

In the case where g is the trace of an entire function on $\mathbb{T}^n$, $M_g(r)$ is an increasing function with respect to each variable. Hence

$$V_g(u) = \ln M_g(e^u) := \ln M_g\big(e^{u_1}, \dots, e^{u_n}\big), \quad u \in \mathbb{R}^n, \tag{2}$$

is a convex function with a convex cone K_V of decrease directions, and K_V contains $\mathbb{R}^n_- \setminus 0$, where $\mathbb{R}_- = \{u \in \mathbb{R} : u \le 0\}$.

2.1. Convex functions equivalent to increasing functions. For a rigorous definition of the cone K_V, we need the following facts of convex analysis.

Theorem 2.2 (see [11, Theorem 8.5])**.** *Let $V = V(u)$, $u \in \mathbb{R}^n$ be a finite convex function. For any $a \in \mathbb{R}^n$, there exists the limit*

$$\gamma_V(u) = \lim_{t\to\infty} \frac{V(tu+a) - V(a)}{t} = \sup_{t>0} \left\{ \frac{V(tu+a) - V(a)}{t} \right\} \quad \forall u \in \mathbb{R}^n, \tag{3}$$

which is independent of a. In addition, the function γ_V is a positive-homogeneous convex function, which takes, possibly, a value of $+\infty$.

The function γ_V is called the *asymptotic* (or recessive) function of V. In [11, Sec. 8], this definition is given in an equivalent form (see [6, Chap. 1, Sec. 6]).

Theorem 2.3 (see [11, Theorem 8.6, Corollary 8.6.1])**.** *The trace $\varphi(t) = V(ut+a)$, $t \in \mathbb{R}$, of a convex function $V(x)$, $x \in \mathbb{R}^n$, $n > 1$, on the line $\Gamma(u,a) = \{ut + a,\ t \in \mathbb{R}\}$, where $u \in \mathbb{R}^n \setminus \{0\}$, $a \in \mathbb{R}^n$, is a nonconstant decreasing function if and only if the condition $\gamma_V(u) \le 0$, $\gamma_V(-u) > 0$ is satisfied.*

In particular, if we have $\gamma_V(u) = \gamma_V(-u) = 0$ for given $u \in \mathbb{R}^n \setminus \{0\}$, then the function V is constant on the line $\Gamma(u,a)$. This result stimulated the following definition.

Definition 2.1 (see [7] and [8, Definition 1.6])**.** Let $V(u)$, $u \in \mathbb{R}^n$, $n > 1$, be a convex function, and γ_V be its asymptotic function. The set

$$K_V = \big\{u \in \mathbb{R}^n : \ \gamma_V(u) \le 0\big\} \tag{4}$$

is called the *cone of decrease directions* of the function V.

Now we state the criterion of equivalence of a given convex function to a convex function increasing in each variable (a generalization of Theorem 1.9 in [7, 8]). For this we need the following assertion.

Lemma 2.1. *Under the conditions and notation of Definition 2.1, the set*

$$L_V = \big\{u \in K_V : \ \gamma_V(u) = \gamma_V(-u) = 0\big\} \tag{5}$$

is a linear subspace of $\mathbb{R}^n$, and $V(u) \equiv V(0)$ for all $u \in L_V$.

Proof. 1. Take $u \in L_V$, $\tau \in \mathbb{R}$. From the property of positive homogeneity of the function γ_V we have $\tau u \in L_V$ (see Theorem 2.2 and (5)). Since, in addition, γ_V is a convex function, we have

$$\gamma_V[\pm(x+y)] \le \gamma_V(\pm x)] + \gamma_V(\pm y) = 0 \quad \forall x, y \in L_V.$$

On the other hand,

$$0 = \gamma_V(0) \le \gamma_V(x+y) + \gamma_V[(-x-y)] \quad \forall x, y \in L_V.$$

This implies that $\gamma_V(x+y) = \gamma_V[(-x-y)] = 0$, i.e., $x + y \in L_V$. Therefore, L_V is a linear subspace $\mathbb{R}^n$.

For $a = 0$, from the formula (3) we conclude that

$$V(ut) \le V(0) + \gamma_V(ut) \quad \forall t \ge 0,\ u \in \mathbb{R}^n.$$

Hence, $V(u) \le V(0)$ for all $u \in L_V$ (see (5)). Since the function V is convex and bounded from above on L_V, it is constant on L_V (see [11, Corollary 8.6.2]). □

Theorem 2.4. *Let $V(x)$, $x \in \mathbb{R}^n$ be a nonconstant convex function, γ_V be its asymptotic function, K_V be the cone of decrease directions of V, and L_V be a subset of K_V* (see (3)–(5)). *The equalities* $\dim K_V = n$ *and* $\dim L_V = m$, *where $0 \le m < n$, are satisfied if and only if there exists a nondegenerate linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $u = Av$, possessing the following property*: $W(v) := V(Av)$, $v \in \mathbb{R}^n$, *is a convex function increasing in each of the variables $v_1, \dots, v_p$, where $p = n - m$, and, under the condition $m > 0$, is independent of the remaining variables $v_{p+1}, \dots, v_n$.*

Proof. Necessity. Assume that $m > 0$. By the condition of the theorem, $V \ne \text{const}$ and $\dim K_V = n$. Therefore, the representation

$$\mathbb{R}^n = L_V^{\perp} \oplus L_V, \quad K_V = \mathcal{K}_V \oplus L_V,$$

is valid and $\dim L_V^{\perp} = \dim \mathcal{K}_V = p > 0$. Note that $L_V = K_V \cap (-K_V)$ is the maximum linear subspace contained in K_V (see [11, Theorem 2.7]). Hence, $\mathcal{K}_V$ is a strictly convex cone (i.e., a cone that does not contain straight lines). We choose a basis $b^{(k)} \in K_V$, $k = 1, \dots, n$, such that in the notation used in the formulas (4) and (5), the following condition holds:

$$b^{(k)} \in \mathcal{K}_V \subset L_V^{\perp}, \quad k = 1, \dots, p; \quad b^{(k)} \in L_V, \quad k = p+1, \dots, n.$$

We define a linear mapping $D : \mathbb{R}^n \to \mathbb{R}^n$ with by the conditions $D[b^{(k)}] = -e_k$, $k = 1, \dots, n$, where $\{e_k\}_1^n$ is the standard basis in $\mathbb{R}^n$. It possesses the property $D(I_V) = \mathbb{R}^n_-$, where $I_V \subset K_V$ is the convex cone with vertex at 0 generated by the n-dimensional simplex with vertices $\{0, \{b_k\}_1^n\}$ and $\mathbb{R}^n_-$ is the negative octant in $\mathbb{R}^n$. In addition, $D(L_V) = \mathbb{R}^m$ is the linear span of the vectors $\{e_k,\ k = p+1, \dots, n\}$ since, due to Lemma 2.1, L_V is a linear subspace $\mathbb{R}^n$. This and Lemma 2.1 imply that the desired mapping is $A = D^{-1}$. In the case $m = 0$, the proof is much simpler.

Sufficiency is proved by the same arguments in the reverse order. □

Thus, a convex function V given in $\mathbb{R}^n$ is "equivalent" to a convex function W, which increases in each variable, under the condition $\dim K_V = n$; moreover, W may depend on less number of variables. In the case where, in the notation used in the formula (2), $V = V_g$ for the function g from a certain subclass of holomorphic functions in $\mathbb{T}^n$, $n > 1$, all coefficients of the mapping A can be chosen to be integers.

2.2. Structure of functions that are equivalent to entire functions and their properties. Consider the proper subclass $\mathcal{A}(\mathbb{T}^n)$ of the class $H(\mathbb{T}^n)$ consisting of holomorphic functions in the space $\mathbb{T}^n$, which are equivalent to entire functions in the following sense.

Definition 2.2 (see [7] and [8, Definition 3.1])**.** We say that a function $g \in H(\mathbb{T}^n)$, $n > 1$, *is equivalent to an entire function f* if there exists a monomial mapping

$$\mathcal{F} = \mathcal{F}_g : \mathbb{T}^n \to \mathbb{T}^n; \quad z = \mathcal{F}(w), \quad z_k = \prod_{j=1}^{n} w_j^{s_{kj}}, \quad k = 1, \dots, n, \tag{6}$$

where $B := \|s_{kj}\|$ is an integer nondegenerate square $(n \times n)$-matrix such that the function $f(w) = [g \circ \mathcal{F}](w)$ admits an analytic continuation $F(w)$ to $\mathbb{C}^p$, where $p \le n$. This means that F is an entire function of p complex variables $w_1, \dots, w_p$, which is independent[1] of $w_{p+1}, \dots, w_n$ for $p < n$. In this case, we write $g \sim f$.

[1]This can be assumed without loss of generality.

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Remark 2.1 (see [9, Sec. 1])**.** The inverse mapping $\mathfrak{F}^{-l}$ is a monomial mapping with fractional exponents, i.e., it is multivalued in $\mathbb{T}^n$. However, $g_1(z) := [f \circ \mathcal{F}^{-1}](z)$, $z \in \mathbb{T}^n$, is a single-valued function such that $g_1(z) \equiv g(z)$, $z \in \mathbb{T}^n$. In order to verify this, we need the notion of the *supports* of the Laurent series

$$g(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k, \quad z \in \mathbb{T}^n; \quad z^k = z_1^{k_1} \dots z_n^{k_n}, \tag{7}$$

for the function g and the power series

$$f(w) = \big[g \circ \mathcal{F}\big](w) = \sum_{m \in \mathbb{Z}_+^p} c_m w^m, \quad w \in \mathbb{T}^p,$$

converging to an entire function. These sets are defined as follows:

$$S_g = \big\{k \in \mathbb{Z}^n : \ a_k \neq 0\big\}, \quad S_f = \big\{m \in \mathbb{Z}_+^p : \ c_m \neq 0\big\}.$$

Comparing coefficients of these series, we obtain the relations

$$S_f = B'[S_g], \quad c_m = a_k, \quad m = B'k \in S_f, \quad k \in S_g,$$

where B' is the transposed matrix for $B = \|s_{kj}\|$ (see (6)). It is convenient to assume that $S_f \subset \mathbb{Z}_+^n$, since in Definition 2.2 the condition on the dependence of the function f only on the variables $w_1, \dots, w_p$ for $p < n$ means that $m_{p+1} = \dots = m_n = 0$ for all $m \in S_f$. In this notation, the following expansion holds:

$$g_1(z) = \big[f \circ \mathcal{F}^{-1}\big](z) = \sum_{m \in S_f} c_m \big[\mathcal{F}^{-1}(z)\big]^m, \quad z \in \mathbb{T}^n.$$

From the relations for the supports S_f and S_g we deduce

$$\langle m, B^{-1}(\ln z)\rangle = \langle B'k, B^{-1}(\ln z)\rangle = \langle k, \ln z\rangle, \quad k = (B')^{-1} m \in S_g, \quad m \in S_f; \quad z \in \mathbb{T}^n.$$

Here and below, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{C}^n$. This and the relations for the coefficients of the series for f and g imply the following:

$$c_m \big[\mathcal{F}^{-1}(z)\big]^m = a_k z^k, \quad k = (B')^{-1} m \in S_g, \quad m \in S_f; \quad z \in \mathbb{T}^n.$$

The equality $g_1 = g$ follows from these arguments.

The entire function $f = g \circ \mathcal{F}$ is said to be *equivalent* for a function $g \in \mathcal{A}(\mathbb{T}^n)$.

Example 2.1 (see [8, example 1′])**.** Let K be an obtuse angle with vertex $0 \in \mathbb{R}^3$ whose sides have the directing vectors $c_1 = (1, 2, -3)$ and $c_2 = (-3, -1, 4)$ and let $g(z)$ be a holomorphic function of three complex variables in $\mathbb{T}^3$ represented by the Laurent series (7) with $n = 3$ whose support is $S_g = K \cap \mathbb{Z}^3$. Based on Definition 2.2, we show that $g \in \mathcal{A}(\mathbb{T}^3)$.

It is easy to see that the angle K is located in the plane $k_1 + k_2 + k_3 = 0$, $k \in \mathbb{R}^3$. Therefore, in this case, the Laurent series of g in (7) admits the representation

$$g(z) = \sum_{k \in S_g} a_k \left(\frac{z_1}{z_3}\right)^{k_1} \cdot \left(\frac{z_2}{z_3}\right)^{k_2}, \quad z \in \mathbb{T}^3; \quad k_3 = -k_1 - k_2.$$

Consider the mapping

$$z = \mathcal{F}w, \quad w \in \mathbb{T}^3, \quad z_1 = \frac{w_3}{w_1 w_2^2}, \quad z_2 = w_1^3 w_2 w_3, \quad z_3 = w_3$$

and the series

$$f(w) = \big[g \circ \mathcal{F}\big](w) = \sum_{m \in S_f} b_m w_1^{m_1} w_2^{m_2}, \quad b_m = a_k, \quad m_1 = -k_1 + 3k_2, \quad m_2 = -2k_1 + k_2, \quad m_3 = 0.$$

Here $k \in S_g \subset K$, with $k_3 = -k_1 - k_2$, and $(k_1, k_2) \in \hat{K}$, where $\hat{K}$ is the projection of the angle K on $\mathbb{R}^2$. Hence we deduce $(k_1, k_2) = m_1 h_1 + m_2 h_2$, $m \in S_f$, where $h_1 = (1, 2)/5$ and $h_2 = (-3, -1)/5$ are the directing vectors of the sides of the angle $\hat{K}$. From elementary geometric fact we obtain that $m_i \geq 0$, $i = 1, 2$, i.e., for any element m of the support S_f we have $(m_1, m_2) \in \mathbb{Z}^2_+$. Thus, $g \sim f$.

Theorem 2.4 implies the following property of functions of the class $\mathcal{A}(\mathbb{T}^n)$, $n > 1$.

Corrolary 2.1. *Let $g \in \mathcal{A}(\mathbb{T}^n)$ and $M_g(r)$ be its maximum modulus on the skeleton of the polydisk,*

$$V(u) := V_g(u) = \ln M_g(e^u), \quad u \in \mathbb{R}^n$$

(see Definition 2.2 and (1)–(2)). Then, in the notation of the formulas (4) and (5), the following equalities hold:

$$\dim K_V = n, \quad \dim L_V = m := n - p,$$

where $0 \leq m < n$.

Proof. Let f be an entire function equivalent to g. According to Definition 2.2 (see (6)), there is a mapping

$$E : \mathbb{R}^n_0 \to \mathbb{R}^n_0; \quad r = E(q), \quad r_k = \prod_{j=1}^{n} q_j^{s_{kj}}, \quad k = 1, \ldots, n, \tag{8}$$

where E is the trace of the mapping $\mathcal{F}$ on $\mathbb{R}^n_0$, $B = \|s_{kj}\|$ is an integer, nondegenerate $(n \times n)$-matrix such that $M_f(q) = M_g \circ E(q)$, $q \in \mathbb{R}^n_0$, M_f is the maximum modulus of f on the skeleton of the polydisk. From (8) we conclude that the linear mapping $B : \mathbb{R}^n \to \mathbb{R}^n$, $u = Bv$, defined by the matrix B, possesses the property

$$V_f(v) := \ln M_f(e^v) = V_g(Bv), \quad v \in \mathbb{R}^n, \tag{9}$$

and the convex function V_f (see Theorem 2.1) increases in each of the variables $v_1, \ldots, v_p$, and under the condition of $p < n$, it is independent of the remaining variables $v_{p+1}, \ldots, v_n$. To complete the proof, it remains to apply Theorem 2.4. □

Based on geometric properties of coefficients of the Laurent series of a function g from the class $H(\mathbb{T}^n)$, we state a criterion of its belonging to the class $\mathcal{A}(\mathbb{T}^n)$.

Theorem 2.5 (see [7] and [8, Theorem 3.2]). *Assume that a function $g(z)$ belongs to the class $H(\mathbb{T}^n)$, $n > 1$, and let its expansion in a multiple Laurent series of the form (7) be given. Let K_g be the smallest closed convex cone with vertex $0 \in \mathbb{R}^n$ containing the support S_g of the series (7). The function G belongs to the class $\mathcal{A}(\mathbb{T}^n)$ (i.e., in terms of Definition 2.2, $f(w) = \big[g \circ \mathcal{F}\big](w)$ is an entire function; see (6)) if and only if the set K_g is a strictly convex cone. In addition, the following statements are valid:*

(1) *the support S_f of the power series of the function f belongs to a certain sublattice of $\mathbb{Z}^n_+$ and possesses the property $S_f = B'[S_g]$, where S_g is the support of the Laurent series (7), and $B' : \mathbb{R}^n \to \mathbb{R}^n$ is the mapping generated by the matrix transposed to $B = \|s_{kj}\|$ (see Definition 2.2 and the remark to it);*
(2) *if $\dim K_g = p \leq n$, then f is an entire function of p complex variables.*

Functions of the class $\mathcal{A}(\mathbb{T}^n)$ play the key role in the structure of elements of the class $H(\mathbb{T}^n)$. The following *multidimensional analog of the Laurent theorem* holds.

Theorem 2.6 (see [7] and [8, Theorem 3.3]). *Let $g \in H(\mathbb{T}^n)$. Then*

$$g(z) = \sum_{i=1}^{m} g_i(z), \quad z \in \mathbb{T}^n, \quad m = m(g) \leq n + 1, \tag{10}$$

where $\{g_i\}_1^m \subset H(\mathbb{T}^n)$ *are functions equivalent to entire functions* (*see Definition* 2.2). *Moreover, for any function* g_i *its support* $S(g_i)$ *possesses the property* $S(g_i) \subset K_i$ *for all* $i = 1, \dots, m$, *where* K_i *is a strictly convex cone in* $\mathbb{R}^n$ *with vertex* 0 *such that* $\dim K_i = n$. *Moreover,*

$$K_i \cap K_j = \varnothing, \quad i \neq j, \quad (i,j) \subset \{1, \dots, m\}.$$

3. Characteristics of growth of functions of the class $H(\mathbb{T}^n)$ and their properties. In [7, 8], by analogy with the growth indices of entire functions of several variables (see [4–6]), the asymptotic growth characteristics of functions of class $H(\mathbb{T}^n)$ are given. Let us recall their definitions, necessary for the further presentation.

3.1. Order function. Let $g \in H(\mathbb{T}^n)$ and M_g be the maximum modulus of the function g on the skeleton of the polydisk (see (1)).

Definition 3.1 (see [7], [8, Definition 3.5])**.** The *order function* of a function $g \in H(\mathbb{T}^n)$ is the function

$$\rho_g(u) = \limsup_{t\to\infty} \frac{\ln^+ \ln^+ M_g(t^u)}{\ln t}, \quad t^u = t^{u_1}, \dots, t^{u_n}, \quad t > 0, \quad u \in \mathbb{R}^n, \tag{11}$$

where $\ln^+ S = \max |\ln S, 1|$, $S > 0$. A function g is called a function *of finite order* if its order function $\rho_g(u)$, $u \in \mathbb{R}^n$, is a finite function.

As was noted in [8, Proposition 3.6], the order function ρ_g of a function $g \in H(\mathbb{T}^n)$ is a nonnegative sublinear function in $\mathbb{R}^n$, possibly not finite everywhere. If g is not an entire function, then ρ_g is not a function increasing in each variable; moreover, the cone of decrease directions (see Definition 2.1) can be an empty set. However, the geometric meaning of the order function ρ_g for $g \in H(\mathbb{T}^n)$ remains the same as in the case of entire functions. To ascertain this, we need the following notion from convex analysis.

Definition 3.2 (see [11, Sec. 8])**.** Let T be an unbounded closed convex set in $\mathbb{R}^n$ and $\dim T = n$. The *asymptotic cone* $A(T)$ of the set T is the maximal cone with vertex 0 such that

$$A(T) = \{u \in \mathbb{R}^n : \ T + u \subset T\}.$$

Proposition 3.1 (see [11, Sec. 8], [6, Chap. 1, Theorem 6.5])**.** *Let* $V(u)$, $u \in \mathbb{R}^n$, *be a convex function, and*

$$\operatorname{epi} V = \{(u, u_{m+1}) \in \mathbb{R}^m \times \mathbb{R} : \ u_{m+1} \geq V(u)\}$$

be its epigraph. Then the asymptotic cone $A(\operatorname{epi} V)$ *of the epigraph of* V *coincides with the epigraph* $\operatorname{epi} \gamma_V$ *of its asymptotic function* γ_V (*see Theorem* 2.2).

Now we discuss the geometric property of the function ρ_g (see (11)). We set

$$\Phi_g(r) = \ln^+ M_g(r), \quad r \in \mathbb{R}_0^n; \qquad W_g(u) = \ln^+ \Phi_g(e^u), \quad u \in \mathbb{R}^n. \tag{12}$$

If W_g is a convex function, then the function ρ_g is its asymptotic function (see (2), Theorems 2.1 and 2.2), and according to Proposition 3.1, the epigraph $\operatorname{epi} \rho_g$ is the asymptotic cone $A(\operatorname{epi} W_g)$ of the epigraph of the function W_g. A similar property of ρ_g is also valid in the general case, where $V := W_g$ is a quasi-convex function satisfying the inequality

$$V(\lambda x + (1-\lambda)y) \leq \max\{V(x), V(y)\} \quad \forall \lambda \in [0,1], \ x, y \in \mathbb{R}^n$$

(see [6, Chap. 6, Sec. 2]).

We show that the definition of the order function ρ_g for $g \in H(\mathbb{T}^n)$ is independent of the choice of a parabolic ray of a more general form

$$L(r,u) = \{rt^u = (r_1 t^{u_1}, \ \dots, \ r_n t^{u_n}) : \ t > 0\}, \quad u \in \mathbb{R}^n \setminus \{0\}, \quad r \in \mathbb{R}_0^n, \tag{13}$$

as in the case of entire functions (see [6, Chap. 6, lemma 2.8]) and functions of the class $\mathcal{A}(\mathbb{T}^n)$ (see [7] and [8, Theorem 3.9]).

Theorem 3.1. *In the notation accepted, for any given $x \in \mathbb{R}^n \setminus \{0\}$, the trace of the function $\Phi_g = \ln^+ M_g$ on each parabolic half-ray of $\{L(r,x),\ r \in \mathbb{R}^n_0\}$ (see (13)) has the growth order*

$$\psi_x(r) = \limsup_{t\to\infty} \frac{\ln^+ \Phi_g(rt^x)}{\ln t} \equiv \rho_g(x), \quad r \in \mathbb{R}^n_0, \tag{14}$$

where ρ_g is the order function for $g \in H(\mathbb{T}^n)$.

Proof. If $\psi_x(r) \not\equiv \infty$ for $r \in \mathbb{R}^n_0$, then there exists a point $a \in \mathbb{R}^n_0$ such that $0 \le \psi_x(a) < \infty$. Take $r \in \mathbb{R}^n_0 \setminus \{a\}$. According to Theorem 2.1, $\Phi_g(r) = \ln^+ M_g(r)$ is a convex function, and $\ln^+ \Phi_g(r)$ is a quasi-convex function of $\ln r_1, \dots, \ln r_n$. Therefore, the following inequality holds:

$$\Phi_g\big(t_1^\lambda s_1^\mu,\ \dots,\ t_n^\lambda s_n^\mu\big) \le \max\big\{\Phi_g(t),\ \Phi_g(s)\big\} \quad \forall \{t,s\} \in \mathbb{R}^n_0, \tag{15}$$

where $\lambda \in (0,1)$, $\mu = 1-\lambda$. Assuming that $t_i = a_i \cdot t^{x_i/\lambda}$ and $s_i = a_i^{-\lambda/\mu} \cdot r_i^{1/\mu}$, $i = 1, \dots, n$, we find from (15) and (14):

$$\Phi_g(rt^x) \le \max\big\{\Phi_g(at^{x/\lambda}),\ A\big\} \le \max\big\{t^{\varepsilon+\psi_x(a)/\lambda},\ A\big\} \quad \forall t > t_0(\varepsilon), \tag{16}$$

where $A = \Phi\big(r_1^{1/\mu} \cdot a_1^{-\lambda/\mu},\ \dots,\ r_n^{1/\mu} \cdot a_n^{-\lambda/\mu}\big)$, $\varepsilon > 0$. Hence we deduce

$$\psi_x(r) \le \varepsilon + \frac{\psi_x(a)}{\lambda}$$

or (passing to the limit as $\lambda \to 1$ and $\varepsilon \to 0$)

$$\psi_x(r) \le \psi_x(a).$$

Thus, $\psi_x(r)$ is a bounded function in $\mathbb{R}^n_0$. Changing the places of a and r with each other in these arguments, we conclude that $\psi_x(a) \le \psi_x(r)$. Consequently,

$$\psi_x(r) \equiv \psi_x(a), \quad r \in \mathbb{R}^n_0, \qquad \rho_g(x) = \psi_x(\mathbb{I}) = \psi_x(a) \quad \forall a \in \mathbb{R}^n_0, \quad \mathbb{I} = (1, \dots, 1).$$

The theorem is proved. □

We mention the following criterion for a finite-order function of the class $\mathcal{A}(\mathbb{T}^n)$.

Proposition 3.2 (see [7] and [8, Proposition 3.8])**.** *Assume that a function g belongs to the class $\mathcal{A}(\mathbb{T}_n)$ and K_V is the cone of decrease directions of the function $V_g(u) = \ln M_g(e^u)$, $u \in \mathbb{R}^n$ (see Definition 2.1) with vertex at 0. Assume that x is an arbitrary fixed element of $\mathbb{R}^n \setminus \{0\}$ such that $-x \in \operatorname{int} K_V$ and ρ_g is the order function for g. If $\rho_g(x) < \infty$, then g is a finite-order function.*

The positive homogeneity of the order function ρ_g for $g \in H(\mathbb{T}^n)$ implies that its (finite) positive values are determined by the set

$$T_g = \{u \in \mathbb{R}^n :\ \rho_g(u) = 1\}, \tag{17}$$

which is called the *order hypersurface* of the function g.

Example 3.1. For $n = 1$, the Laurent theorem on the expansion of a holomorphic function in a ring implies that for any function $g \in H(\mathbb{T})$, the following representation is valid:

$$g(z) = g_+(z) + g_-\left(\frac{1}{z}\right), \quad z \in \mathbb{T} = \mathbb{C} \setminus \{0\},$$

where g_+ and g_- are entire functions. If they have finite nonzero orders ρ_+ and ρ_-, respectively, then the order function is

$$\rho_g(u) = \begin{cases} \rho_+ u, & u > 0, \\ \rho_- u, & u \leq 0. \end{cases}$$

Therefore, the set T_g consists of two points $1/\rho_+$ and $1/\rho_-$.

3.2. Type function in a given direction of growth. Now let us consider a more subtle indicator of growth of functions from the class $H(\mathbb{T}^n)$.

Definition 3.3 (see [7], [8, Definition 3.10])**.** Let $g \in H(\mathbb{T}^n)$ and let ρ_g be the order function for the function g. We assume that $\rho_g(x) \in (0, \infty)$ for fixed $x \in \mathbb{R}^n \setminus \{0\}$. The function

$$\sigma_g(r; x) = \limsup_{t \to \infty} \frac{\ln^+ M_g(rt^x)}{t^{\rho_g(x)}}, \quad r \in \mathbb{R}_0^n, \tag{18}$$

is called the *type function in the direction* x or the x*-type function* for the function g, and the value $\sigma_g(x) := \sigma_g(\mathbb{I}; x)$, where $\mathbb{I} = (1, \dots, 1)$, is called its *type in the direction* x.

The following formula holds (see [7], [8, remark to Definition 3.10]):

$$\sigma_g(\cdot; x) = \sigma_g(\cdot; x\tau) \quad \forall \tau > 0.$$

Therefore, without loss of generality, in the definition of the x-type function, it suffices to impose the condition $x \in T_g$ (see (17) and (18)). We recall the simplest properties of the x-type function (see [7] and [8, Proposition 3.11]).

Proposition 3.3. *The functions* $\sigma_g(e^u; x)$ *and* $\delta_g^x(u) := \ln \sigma_g(e^u; x)$ *are convex functions in their domain* D*, and if* $x \in T_g$*, then*

$$\sigma_g(rt^x; x) = t\sigma_g(r; x) \quad \forall r \in \mathbb{R}_0^n, \quad t > 0; \qquad \delta_g^x(u + x\tau) = \delta_g^x(u) + \tau \quad \forall u \in \mathbb{R}^n, \quad \tau \in \mathbb{R}. \tag{19}$$

As in the case of the order function, $\sigma_g(\cdot; x)$ and δ_g^x belong to a wider class of functions compared with the situation where $g \in H(\mathbb{C}^n)$ (see Sec. 3.1). Let us show that, in spite of this geometric property of the function δ_g^x, the requirement $g \in H(\mathbb{T}^n)$ is completely analogous to a similar property in $H(\mathbb{C}^n)$.

For simplicity, in the notation of Proposition 3.3, we set $D = \mathbb{R}^n$. From (19) we conclude that the epigraph (see Proposition 3.1) $I_g(x) := \operatorname{epi} \delta_g^x$ of the function δ_g^x is a convex cylinder. Let $\Phi_g = \ln^+ M_g$, $W_g(u) = \ln^+ \Phi_g(e^u)$, $u \in \mathbb{R}^n$. The relations (18) and (19) imply the equality

$$\limsup_{\tau \to \infty} \Big[W_g(x\tau + u) - \tau - \delta_g^x(u) \Big] = \limsup_{\tau \to \infty} \Big[W_g(x\tau + u) - \delta_g^x(x\tau + u) \Big] = 0, \quad u \in \mathbb{R}^n.$$

Therefore, by analogy with the case of entire functions (see [6, Chap. 6, Definitions 3.5 and 3.6]), we introduce the following concept.

Definition 3.4. For each fixed $u \in \mathbb{R}^n$, the nonvertical line

$$E = \Big\{ \big(x\tau + u, \ \tau + \delta_g^x(u) \big) \in \mathbb{R}^n \times \mathbb{R} : \ \tau \in \mathbb{R} \Big\}$$

is called the (upper one-dimensional) x*-asymptote of the function* W_g, and the cylinder $I_g(x) = \operatorname{epi} \delta_g^x$ whose boundary (the graph of the function $y = W_g(u)$, $u \in \mathbb{R}^n$) is the ruled surface formed by all x-asymptotes of the function W_g, is called an *asymptotic* x*-cylinder* of the epigraph $\operatorname{epi} W_g$ of the function W_g.

We note an important nontrivial property of the x-type function for functions of the class $H(\mathbb{T}^n)$, which contains the criterion of finite type function in a given direction of growth and its structure.

We denote by $Y_H = \{\rho\}$ the class of all finite, nonnegative, sublinear functions in $\mathbb{R}^n$ except for the function $\rho_0(u) \equiv 0$, $u \in \mathbb{R}^n$. Let $\rho \in Y_H$ and

$$\mathfrak{M}_n(\rho) = \Big\{ g \in H(\mathbb{T}^n) : \ \rho_g(u) \equiv \rho(u), \ u \in \mathbb{R}^n \Big\}, \tag{20}$$

where ρ_g is the order function for g (see (11)); $\mathfrak{M}_n(\rho)$ is a subclass of $H(\mathbb{T}^n)$ with a given order function (or with a given order hypersurface, see (17)) $T^\rho = \{u \in \mathbb{R}^n : \ \rho(u) = 1\}$). Consider the class

$$\mathfrak{N}_n^x(\rho) = \Big\{ g \in \mathfrak{M}_n(\rho) : \ 0 < \sigma_g(x) < \infty \Big\}, \tag{21}$$

where $x \in T^\rho$ and $\sigma_g(x)$ is the type of g in the direction x (see (17)). With a function $\rho \in Y_H$, we associate the convex compact set

$$K_\rho = \Big\{ y \in \mathbb{R}^n : \ \langle y, u \rangle \le \rho(u) \ \forall u \in \mathbb{R}^n \Big\}, \tag{22}$$

whose support function is ρ. Let

$$\partial\rho(x) = \Big\{ y \in K_\rho : \ \langle y, x \rangle = \rho(x) \Big\} \tag{23}$$

be the face of K_ρ orthogonal to the vector $x \in T^\rho$ and

$$\partial\rho(x) = \Big\{ y \in \mathbb{R}^n : \ \rho(u) \ge \rho(x) + \langle u - x, y \rangle \ \forall u \in \mathbb{R}^n \Big\}$$

be the subdifferential of the convex function ρ at the point x (see [11, Chap. 5, Sec. 23]). Based on the method of the proof of Theorem 3.9 in [6], we obtain the following assertion in the notation of the formulas (18)–(23).

Theorem 3.2 (see [6]). *Assume that $g \in \mathfrak{N}_n^x(\rho)$, where $\rho \in Y_H$ and $x \in T^\rho$ (see (21), (17)). With a function $\rho \in Y_H$, we associate a convex compact set. Then the x-type function $\sigma_g(\cdot; x)$ of the function g possesses the following properties*:

(1) $\sigma_g(rt^x; x) \equiv t\sigma_g(r; x)$ *for all* $t \ge 0$, $r \in \mathbb{R}_0^n$;

(2) *there exists a unique convex, lower semicontinuous function* $\psi : \partial\rho(x) \to (-\infty, \infty]$, $\psi \not\equiv \infty$, *such that the following representation is valid*:

$$\sigma_g(r; x) = \sup \Big\{ r_1^{y_1} \dots r_n^{y_n} \exp\{-\psi(y)\} : \ y \in \partial\rho(x) \Big\}, \quad r \in \mathbb{R}_0^n. \tag{24}$$

In particular, if the function $\rho \in Y_H$ is differentiable at a point $x \in T^\rho$, then there exists a constant $A_g = A_g(x) > 0$ with the property

$$\sigma_g(r; x) = A_g \cdot \prod_{i=1}^n r_i^{\partial\rho/\partial x_i}, \quad r \in \mathbb{R}_0^n. \tag{25}$$

Remark 3.1. We give another formulation of Theorem 3.2, which will be useful below.

We denote by $N_n^x(\rho)$ the class of positive functions φ in $\mathbb{R}_0^n$ that are logarithmically convex[2] with respect to $\ln r_1, \dots, \ln r_n$ and possess the properties (1) and (2) of x-type functions specified in Theorem 3.2; here $\rho \in Y_H$ and $x \in T^\rho$. In this notation,

$$\big\{ \sigma_g(\cdot; x), \ g \in \mathfrak{N}_n^x(\rho) \big\} \subset N_n^x(\rho).$$

Let us clarify the geometric meaning of Theorem 3.2 assuming, for simplicity, as in Theorem 3.1, that W_g is a convex function (see the remark to Theorem 3.9 in [6]). From (24), using the notation of Proposition 3.3, we obtain

$$\delta_g^x(u) = \ln \sigma_g(e^u; x) = \sup \Big\{ \langle u, y \rangle - \psi(y), \ y \in \partial\rho(x) \Big\}, \quad u \in \mathbb{R}^n,$$

[2]That is, the function $W_\varphi(u) = \ln \varphi(e^u)$, $u \in \mathbb{R}^n$, is convex for all $\varphi \in N_n^x(\rho)$.

where $\langle u, y\rangle$ is the scalar product in $\mathbb{R}^n$. However,

$$\psi(y) = (\delta_g^x)^*(y) = \sup\Big\{\langle u, y\rangle - \delta_g^x(u),\ u \in \mathbb{R}^n\Big\},\quad y \in \mathbb{R}^n,\quad \Big\{y \in \mathbb{R}^n:\ (\delta_g^x)^*(y) < \infty\Big\} \subset \partial\rho(x)$$

is the Young transformation of the function δ_g^x (see [11, Chap. 3], [6, Chap. 1, Proposition 5.3 and Corollary 5.4]). Therefore, *each supporting hyperplane to the asymptotic x-cylinder $I_g(x) = \operatorname{epi}\delta_g^x$ of the epigraph $\operatorname{epi} W_g$ of the function W_g (see (12)) is parallel to some supporting hyperplane to the asymptotic cone $A(\operatorname{epi} W_g) = \operatorname{epi}\rho_g$ of the convex set $\operatorname{epi} W_g$ passing through the ray $\big\{(xt, t) \in \mathbb{R}^n \times \mathbb{R},\ t > 0\big\}$* (see Definitions 3.4 and 3.2 and explanations to them).

In the case of the class $\mathcal{A}(\mathbb{T}^n)$ (see Definition 2.2), there are simple formulas for the relationship between characteristics of the growth of functions of this class and their corresponding equivalent entire functions. In a similar form, they are available in [9, Theorem 2].

Proposition 3.4. *Let $g \in \mathcal{A}(\mathbb{T}^n)$; ρ_g be a finite order function for the function g, f be an entire function equivalent to g, $B = \|s_{kj}\|$ be an integer nondegenerate $(n \times n)$-matrix whose elements are the exponents of a monomial mapping $\mathcal{F}$ such that $f = [g \circ \mathcal{F}]$ (see (6)). We set $\mathcal{B}_p = \pi_p \circ B^{-1}$, where*

$$p \le n,\quad \pi_p : \mathbb{R}^n \to \mathbb{R}^p,\quad \pi_p(v_1, \dots, v_n) = (v_1, \dots, v_p),\quad \mathcal{E}_p = \pi_p \circ E^{-1},$$

and E is the trace of the mapping $\mathcal{F}$ to $\mathbb{R}_0^n$ (see (8)).

1. *We have*

$$\rho_f[\mathcal{B}_p(u)] \equiv \rho_g(u),\quad u \in \mathbb{R}^n,$$

   where ρ_f is a function of orders for f.
2. *If, in addition, for fixed $x \in \mathbb{R}^n\backslash\{0\}$, the conditions $0 < \rho_g(x) < \infty$ and $0 < \sigma_g(x) < \infty$ are satisfied, where $\sigma_g(x)$ is the x-type of g, then the type functions for g and f in the directions x and $y = \mathcal{B}_p(x)$, respectively, are related by the formula*

$$\sigma_f(q; y) \equiv \sigma_g(r; x),\quad q = \mathcal{E}_p(r),\quad r \in \mathbb{R}_0^n.$$

Proof. We use the notation from the proof of Corollary 2.1. The assertion 1 follows from the formulas (9) and Definition 3.1 (see (11)). Here we take into account the fact that the function $V_f(v) := \ln M_f(e^v)$, $v \in \mathbb{R}^n$, is independent of the variables $v_{p+1}, \dots, v_n$ for $p < n$. (Recall that M_f is the maximum modulus of the entire function f on the skeleton of the polydisk.)

Since by the assumption ρ_g is a finite function, $0 < \rho_g(x) < \infty$, and $0 < \sigma_g(x) < \infty$, by Theorem 3.2, the type function $\sigma_g(\cdot; x)$ is also finite. The assertion 2 is a consequence of the equality

$$M_f(q) = [M_g \circ E](q),\quad q \in \mathbb{R}_0^n,$$

and Definition 3.3 (see (18)), if we again take into account the fact that the function $M_f(q)$ is independent of the variables $q_{p+1}, \dots, q_n$ for $p < n$ (see Definition 2.2). □

4. Characteristics of the growth of functions of the class $H(\mathbb{T}^n)$ and the coefficients of their expansion in the Laurent series. This section is devoted to deriving formulas that establish a connection between growth characteristics of functions from the class $H(\mathbb{T}^n)$, $n > 1$, and the coefficients of their Laurent expansions. These formulas are a complete analog of the corresponding results for the class $H(\mathbb{C}^n)$ of entire functions (see [6, Chap. 7, Theorem 1.4]), but their proofs are more complicated.

For each function $g \in H(\mathbb{T}^n)$, the following n-fold Laurent expansion converging everywhere in $\mathbb{T}^n$ holds: (see [14, p.40]):

$$g(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k,\quad z^k = z_1^{k_1} \dots z_n^{k_n},\quad a_k = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_r} \frac{g(z)dz}{z^{k+I}},\quad z^{k+I} = z_1^{k_1+1} \dots z_n^{k_n+1}. \tag{26}$$

Here $dz = dz_1 \dots dz_n$, $\Gamma_r = \{z \in \mathbb{T}^n : |z_j| = r_j,\ j = 1, \dots, n\}$ is the topological product of circles of radii $r_j \in (0, \infty)$, $j = 1, \dots, n$. Further, we assume that the support of the Laurent series is an unbounded set in $\mathbb{Z}^n$, i.e., the function g is not a Laurent polynomial.

The following statement is a convergence criterion of a multiple Laurent series everywhere in $\mathbb{T}^n$.

Theorem 4.1 (see [7] and [8, Theorem 3.4]). *The n-fold Laurent series* (26) *is convergent everywhere in $\mathbb{T}^n$ if and only if its coefficients satisfy the condition*

$$\lim_{\|k\| \to \infty} \sqrt[\|k\|]{|a_k|} = 0, \quad \|k\| = \sum_{j=1}^{n} |k_j|. \tag{27}$$

We consider the following characteristic of the unbounded support S_g of a function g.

Definition 4.1. Let $u \in \mathbb{R}^n \setminus \{0\}$ and $g \in H(\mathbb{T}^n)$. Consider the Laurent series (26) and the function $\lambda = \lambda_u(k) = \langle k, u \rangle$, $k \in S_g$, defined on the support S_g. The set D_g of all vectors $\{u\}$ such that the function λ_u is unbounded from above is called the *growth cone of the support S_g*.

The following assertion shows that the set D_g is the cone of directions of growth which is more rapid than the power growth for the function g.

Proposition 4.1. *Let $g \in H(\mathbb{T}^n)$. An element $x \in \mathbb{R}^n \setminus \{0\}$ (see the notation of Definition* 4.1*) does not belong to the cone D_g if and only if for the maximum modulus of g on the skeleton of the polydisk, the following estimate holds*:

$$M_g(rt^x) \le C(r) t^s, \quad t \ge t_0 > 0. \tag{28}$$

Moreover, $\rho_g(x) = 0$, $x \notin D_g$, where ρ_g is the order function of the function g (see Definition 3.1*).*

Proof. Necessity. As in the case of entire functions, the following inequality (see (26)) holds:

$$M_g(r) \le S_g(r) := \sum_{k \in S_g} |a_k| r^k, \quad r \in \mathbb{R}^n_0.$$

Let $x \notin D_g$, $s < \infty$ be an upper bound of the function $\lambda_x(k) = \langle k, x \rangle$, $k \in S_g$. From the above inequality, for every fixed $r \in \mathbb{R}^n_0$ we obtain the estimate (28), in which $C = S_g(r)$.

Sufficiency. The complete analog of the Cauchy inequality for Taylor coefficients of an entire function is valid for the coefficients of the Laurent expansion of the function g (see (26)):

$$|a_k| r^k \le M_g(r) \quad \forall r \in \mathbb{R}^n_0, \quad k \in S_g.$$

Hence, if the condition (28) is satisfied for some $s \in \mathbb{R}$ and fixed $r \in \mathbb{R}^n_0$, we find

$$|a_k| r^k t^\lambda \le C(r) t^s, \quad \lambda = \langle k, x \rangle, \quad k \in S_g, \quad t \ge t_0 > 0.$$

This means that $\lambda = \lambda_x(k) \le s$, $k \in S_g$, i.e., $x \notin D_g$.

The final statement of Proposition 4.1 follows from Definition 3.1 and Theorem 3.1. □

Remark 4.1. If $s \le 0$ (see the notation for the estimate (28)), then $x \in K_V$, where K_V is the cone of decrease directions of the function $V_g(u) = \ln M_g(e^u)$, $u \in \mathbb{R}^n$ (see (2) and Definition 2.1), i.e., $K_V \subset \mathbb{R}^n \setminus D_g$. (In the general case, $K_V = \varnothing$ is possible.)

Lemma 4.1. *Let $g \in H(\mathbb{T}^n)$, $x \in D_g$, where $x \in D_g$ is the growth cone of the support S_g(see Definition* 4.1*). Assume that for fixed $r \in \mathbb{R}^n_0$, there exist constants Δ and $A > 0$ such that*

$$M_g(rt^x) < \exp\{A t^\Delta\} \quad \forall t > t_0. \tag{29}$$

Then the coefficients of the Laurent series of the function g (see (26)*) satisfy the inequality*

$$|a_k| r^k < \left(\frac{Ae\Delta}{\lambda} \right)^{\lambda/\Delta} \quad \forall \lambda > \lambda_0; \quad \lambda = \langle k, x \rangle. \tag{30}$$

Proof. From the Cauchy inequality for the coefficients of the Laurent expansion of the function g (see the proof of Proposition 4.1) and the inequalities (29), we derive for sufficiently large $\lambda = \langle k, x\rangle$, taking into account the fact that $x \in D_g$, and setting $\tau = \ln t$:

$$-\ln\left(|a_k| r^k\right) > \sup_{\tau > \tau_0}\left[\lambda\tau - Ae^{\Delta\tau}\right] = \frac{\lambda}{\Delta}\ln\frac{\lambda}{Ae\Delta}.$$

After elementary transformations, we obtain the inequality (30). □

Next, we need an additional property of functions that are equivalent to entire functions (see Definition 2.2). Assume that $g \in \mathcal{A}(\mathbb{T}^n)$ and f is an entire function such that $g \sim f$. With the function g represented by the series (7), we associate the following holomorphic function G whose Laurent expansion has the same support as the series (7):

$$G(z) = \sum_{k\in S_g} z^k, \quad z^k = z_1^{k_1}\dots z_n^{k_n}. \tag{31}$$

This series is equivalent to a power series

$$H(w) = \left[G \circ \mathcal{F}\right](w) = \sum_{m \in S_f} w^m, \quad w^m = w_1^{m_1}\dots w_p^{m_p}, \quad p \le n,$$

whose support coincides with the support of the series representing the entire function f, since in the notation of Definition 2.2

$$w^m = z^k, \quad c_m = a_k, \quad z = \mathcal{F}(w), \quad w \in \mathbb{T}^n; \quad k \in S_g, \quad m = B'k \in S_f \subset \mathbb{Z}_+^p \subset \mathbb{Z}_+^n. \tag{32}$$

Here $\mathcal{F} = \mathcal{F}_g$ is a monomial mapping of the form (6), $\{c_m\}$ and $\{a_k\}$ are the Taylor coefficients of the function f and the Laurent coefficients of the function g, respectively, and B' is the matrix transpose to $B = \|s_{kj}\|$ (see (6)). It is obvious that the power series $H(w)$ converges absolutely in $D_p = \{w \in \mathbb{T}^p : \ 0 < |w_i| < 1, \ i = 1,\dots,p\}$. Therefore, we can distinguish in the domain D the convergence set of the series (31) located in the $\mathbb{R}_0^n$.

Proposition 4.2. *In the above notation, the convergence domain of the series* (31) *contains the set*

$$\left\{b(\alpha) := E(\alpha) = E(\alpha_1,\dots,\alpha_n) \in \mathbb{R}_0^n, \ \alpha \in (0,1)^n\right\},$$

where E is the trace of the mapping $\mathcal{F} \to \mathbb{R}_0^n$ (see (8)*), and $b(\alpha) \to I$ as $\alpha \to I$, where $I = (1,\dots,1)$.*

Lemma 4.2. *Let a function $g \in \mathcal{A}(\mathbb{T}^n)$, $n > 1$, be equivalent to an entire function f, $x \in D_g$ (see Definitions* 2.2 and 4.1*). Assume that the coefficients of the Laurent expansion* (7) *of the function g satisfy the inequality* (*see* (30))

$$|a_k|\left(\frac{r}{b}\right)^k < \left(\frac{Ae\Delta}{\lambda}\right)^{\lambda/\Delta} \quad \forall\lambda > \lambda_0; \quad \frac{r}{b} = \frac{r_1}{b_1}\dots\frac{r_n}{b_n}; \quad \lambda = \langle k, x\rangle, \tag{33}$$

for some $r \in \mathbb{R}_0^n$, $\alpha \in (0,1)^n$, $A > 0$, $\Delta > 0$, where (*see the notation of Proposition* 4.2) *$b = b(\alpha) = E(\alpha)$. Then there exists a constant C such that*

$$M_g(rt^x) < Ce^{At^\Delta} \quad \forall t > t_0; \quad rt^x = (r_1t^{x_1},\dots,r_nt^{x_n}). \tag{34}$$

Proof. We perform a change of variables in the inequality (33) as follows. We set $r = E(q)$, where $q \in \mathbb{R}_0^n$ (see (8)). Next, we transform the left-hand side of the inequality (33) using the formula (32):

$$|a_k|\left(\frac{r}{b}\right)^k = |a_k|\left[\frac{E(q)}{E(\alpha)}\right]^k = |a_k|\left[E\left(\frac{q}{\alpha}\right)\right]^k = |c_m|\left(\frac{q}{\alpha}\right)^m, \quad m = B'k, \quad k \in S_g, \tag{35}$$

where $\{c_m,\ m \in S_f\}$ are the Taylor coefficients of the entire function $f = g \circ \mathfrak{F}$ (see (6)). Now we transform the parameter λ on the right-hand side of the inequality (33):

$$\lambda = \langle k, x\rangle = \langle k, By\rangle = \langle B'k, y\rangle = \langle m, y\rangle, \quad k \in S_g, \quad m = B'k \in S_f,$$

where $y := (y_1, \dots, y_n) = B^{-1}x \in \mathbb{R}_0^n$, $B = \|s_{kj}\|$ is a nondegenerate matrix that determine the monomial mapping $\mathcal{F}$ (see (6)), and B' is the transpose matrix for B. But for $p < n$ we obtain $S_f \subset \mathbb{Z}_+^p$, i.e., the coordinates $m_{p+1}, \dots, m_n$ of the vector m are equal to zero (see Definition 2.2 and the remark to it). Therefore, in the general case,

$$\lambda = \langle k, x\rangle = \langle m, y\rangle_p := \sum_{i=1}^{p} m_i y_i, \quad k \in S_g, \quad m = B'k \in S_f. \tag{36}$$

From this and from (35) we conclude that the inequality (33) is transformed to the following inequality for the Taylor coefficients of the function f:

$$|c_m| \left(\frac{q}{\alpha}\right)^m < \left(\frac{Ae\Delta}{\lambda}\right)^{\lambda/\Delta} \quad \forall \lambda > \lambda_0; \quad \lambda = \langle m, y\rangle_p.$$

From the equality

$$M_f(qt^y) = M_g(rt^x), \quad r = E(q) \quad t > 0, \tag{37}$$

taking into account the fact that the function $M_f(v) = M_g \circ E(v)$, $v \in \mathbb{R}_0^n$, is independent of the variables $v_{p+1}, \dots, v_n$ for $p < n$ (see Definition 2.2), we conclude that $(y_1, \dots, y_p) \in D_f$, where D_f is the growth cone of the support S_f of the function $f \in H(\mathbb{C}^p)$, since $x \in D_g$.

Under these conditions, there exists a constant $C > 0$ such that the entire function f satisfies the inequality

$$M_f(qt^y) < Ce^{At^{\Delta}} \quad \forall t > t_0; \quad qt^y = (q_1 t^{y_1}, \dots, q_p t^{y_p})$$

(see [6, Chap. 7, Lemma 1.3]). Returning to the former variable, we verify the validity of the lemma (see (36)–(37)). □

Now we derive the relations between growth characteristics of functions of the class $H(\mathbb{T}^n)$, $n > 1$, and the coefficients of their Laurent expansions.

Theorem 4.2 (see [6, Chap. 7, Theorem 1.4]). *Let $g \in H(\mathbb{T}^n)$, D_g be the growth cone of the support S_g (see Definition 4.1). We assume that*

$$\beta_g(u) = \limsup_{\lambda \to \infty} \frac{\lambda \ln \lambda}{-\ln |a_k|}, \quad \lambda = \langle k, u\rangle; \quad u \in D_g, \tag{38}$$

where $\{a_k\}$ are the coefficients of the Laurent expansion of the function g (see (26)), and $(-\ln |a_k|)^{-1} = 0$, if $a_k = 0$. Then the following assertions hold.

(1) *The order function g (see Definition 3.1) possesses the property $\rho_g(u) \equiv \beta_g(u)$, $u \in D_g$.*

(2) *If $\rho \in Y_H$, $g \in \mathfrak{M}_n(\rho)$ (see the notation of the formulas (20)–(21)), then for every $x \in \{u \in \mathbb{R}^n :\ \rho(u) > 0\}$, the type function $\sigma_g(r; x)$ of the function g (see Definition 3.3) is defined by the formula*

$$\Big[\sigma_f(r; x) e \rho(x)\Big]^{1/\rho(x)} = \limsup_{\lambda \to \infty} \lambda^{1/\rho(x)} \cdot (|a_k| r^k)^{1/\lambda}, \quad \lambda = \langle k, x\rangle, \quad \rho(x) = \rho_g(x), \quad r \in \mathbb{R}_0^n. \tag{39}$$

Proof. 1. Fix $x \in D_g$. If $\rho_g(x) < \infty$, then the inequality (29) is valid for $A = 1$, $r = \mathbb{I}$, and $\Delta = \rho_g(x) + \varepsilon$, where $\varepsilon > 0$. Under these restrictions, based on Lemma 4.1, we conclude that the inequality (30) is valid. Hence, after elementary transformations we obtain (see (38))

$$\beta_g(x) \le \rho_g(x) + \varepsilon \quad \forall \varepsilon > 0,$$

i.e., $\beta_g(x) \le \rho_g(x)$. Note that $\beta_g(u) \ge 0$, $u \in D_g$, since $\ln|a_k| \to -\infty$ as $\lambda \to \infty$ (see (38) and (27)). So, $\beta_g(x) = \rho_g(x)$, if $\rho_g(x) = 0$.

2. Now we assume that $\rho_g(x) > 0$. According to Theorem 2.6 for the function $g \in H(\mathbb{T}^n)$, we conclude the formula (10) is valid. We consider the subset $\{g_j \in \mathcal{A}(\mathbb{T}^n),\ j \in A_x\}$ of terms that "form" the function g, where

$$A_x = \Big\{i \in (1,\dots,m):\ S(g_i) \cap \Pi_x \neq \varnothing\Big\}, \quad \Pi_x = \Big\{k \in \mathbb{Z}^n:\ \lambda = \langle k, x\rangle > \lambda_0\Big\} \tag{40}$$

for any sufficiently large values of $\lambda_0 > 0$. We ensure that $A_x \neq \varnothing$. Otherwise, there is a number λ_0 such that $\langle k, x\rangle \le \lambda_0$ for all $k \in S_g$, where S_g is the support of the function g. This means that $x \notin D_g$; a contradiction.

Assume that $\beta_g(x) < \infty$ (see (38)). We denote by $\beta_i(x)$ a number that differs from $\beta_g(x)$ only by the rule that we set $k \in S(g_i)$ in (38) for $u = x$, where $S(g_i)$ is the support of the term $g_i \in \mathcal{A}(\mathbb{T}^n)$ of the function g (see (40)). Let $\mathcal{F}_i$ be a monomial mapping of the form (6), existing for the function g_i, E_i be the trace of the mapping $\mathcal{F}_i$ on $\mathbb{R}^n_0$, and $b^{(i)} = b^{(i)}(\alpha) = E_i(\alpha)$, $\alpha \in (0,1)^n$ (see Definition 2.2 and Proposition 4.1). Theorem 4.1 implies the relation (see (38))

$$\beta_i(x) = \limsup_{\lambda\to\infty} \frac{\lambda(\ln\lambda - \ln e\Delta)}{-\ln|a_k| + \langle k, \ln b^{(i)}\rangle} \le \Delta, \quad \lambda = \langle k, x\rangle, \quad k \in S(g_i); \quad i \in A_x; \quad \Delta = \beta_g(x) + \varepsilon,$$

where $\varepsilon > 0$. Therefore, the estimate for the Laurent coefficients of the term g_i of the function g is

$$|a_k| < [b^{(i)}]^k \left(\frac{e\Delta}{\lambda}\right)^{\lambda/\Delta} \quad \forall\lambda > \lambda_0; \quad \lambda = \langle k, x\rangle, \quad k \in S(g_i), \quad i \in A_x.$$

Hence, applying Lemma 4.2, we arrive at the following inequality for $A = 1$, $r = \mathbb{I}$, $\Delta = \beta_g(x) + \varepsilon$, and $\varepsilon > 0$:

$$M_{g_i}(t^x) < Ce^{t^\Delta} \quad \forall t > t_i; \quad i \in A_x. \tag{41}$$

Finally, from the formula (10) we deduce (see the notation of the relations (40))

$$M_g(r) \le \sum_{i=1}^{m} M_{g_i}(r) = \Sigma_1 + \Sigma_2, \quad \Sigma_1 = \sum_{i\in A_x} M_{g_i}(r), \quad \Sigma_2 = \sum_{i\notin A_x} M_{g_i}(r), \quad r \in \mathbb{R}^n_0.$$

Now, based on the inequality (41) and Proposition 4.1, we conclude that there are constants $C_j > 0$, $j = 1, 2$, and $s \in \mathbb{R}$ such that

$$M_g(t^x) < C_1 e^{t^\Delta} + C_2 t^s, \quad t > 0.$$

Applying the formula (11), we obtain

$$\rho_g(x) \le \Delta = \beta_g(x) + \varepsilon, \quad \rho_g(x) \le \beta_g(x).$$

The opposite inequality was proved above. Moreover, we have proved that the assumption on the finiteness of one of the numbers $\rho_g(x)$ or $\beta_g(x)$ implies the finiteness of the other. Therefore, the equality $\rho_g(x) = \beta_g(x)$ is also valid in the case where one of these numbers is ∞. So, the formula (38) is valid.

3. The assertion (2) of the theorem is proved in the same way. When using Lemmas 4.1 and 4.2 we assume that $\Delta = \rho(x) = \rho_g(x)$. According to Theorem 3.2, the x-type function $\sigma_g(\cdot; x)$ is finite. This determines the finiteness of the function $A_g(r; x)$, $r \in \mathbb{R}^n_0$, defined by the right-hand part of the formula (39), which is convex with respect to $\ln r_1, \dots, \ln r_n$. In particular, if $g \in \mathcal{A}(\mathbb{T}^n)$, then, applying Proposition 4.2, we find the equality

$$A_g(r; x) = \lim_{\alpha\to I} A_g\left[\frac{r}{b(\alpha)}; x\right], \quad I = (1,\dots,1), \quad r \in \mathbb{R}^n_0,$$

used in the proof of the formula (39). □

Remark 4.2. Under the condition $g \in \mathcal{A}(\mathbb{T}^n)$, the proof of the theorem is substantially simplified: it is based on similar formulas for entire functions (see [6, Chap. 7, Theorem 1.4]). This can be verified by applying the method of change of variables used in the proof of Lemma 4.2 (see also [9]).

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