Group classification of ideal fluid equations in terms of trajectory – Weber's potential

Viktor K. Andreev, Daria A. Krasnova

ICM SB RAS, Krasnoyarsk

1 Statement of the problem

The unsteady motion of an ideal fluid with a free boundary is described by the following system of equations

$$\begin{gathered}\mathbf{u}_t + \mathbf{u}\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{g}(\mathbf{x}, t),\\ \operatorname{div}\mathbf{u} = 0, \quad \mathbf{x} \in \Omega_t,\end{gathered} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ – is the velocity vector, p – the pressure, $\mathbf{x} = (x, y, z)$ – are coordinates in space, t – is time. The solution of the problem involves solving the specified Euler's equations and finding the velocity vector $\mathbf{u}$ and pressure p in a certain region $\Omega(x)$, on the border Γ_t of which the following boundary conditions are satisfied

$$\begin{gathered}f_t + \mathbf{u}\cdot\nabla f = 0,\\ p_0 - p = 2\sigma K,\end{gathered} \tag{2}$$

where $p_0(\mathbf{x}, t)$ – is the known function of external pressure, $\sigma \geqslant 0$ is the constant coefficient of surface tension, K – is the mean curvature of the free boundary. K is negative if the boundary Γ_t is convex to the outside of the fluid. Conditions (2) are called kinematic and dynamic conditions, respectively. If the conditions (2) are satisfied, then the Γ_t boundary is called a free one. If the free boundary Γ_t does not completely coincide with the boundary $\Omega(x)$, the rest of the boundary Σ is satisfied by the condition of nonpermeability

$$\mathbf{u}\cdot\mathbf{n}_\Sigma = 0,$$

where $\mathbf{n}_\Sigma$ – is the outer normal of the boundary Σ.

We finalize our problem with initial conditions

$$\mathbf{u} = \mathbf{u}_0(\mathbf{x}), \quad \operatorname{div}\,\mathbf{u}_0 = 0, \quad x \in \Omega_0, \quad t = 0. \tag{3}$$

The complex of equations (1), boundary (2) and initial (3) conditions gives us the ability to formulate the problem in Euler coordinates.

Weber integral. Let $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ — be the velocity field and the pressure of the ideal fluid which are the desired quantities, that are defined in a certain area Ω_t with the boundary Γ_t, which itself is unknown. On the border Γ_t a kinematic condition is fulfilled (it consists of the same fluid particles at $t \geqslant 0$) and a dynamic condition (the pressure change is determined by the Laplace formula). Such tasks are extremely difficult for mathematical research and are called problems with free borders. This class also includes the problem of waves on the water.It is well known that it is easier to study initial-boundary problems in a fixed region. In this case, this is achieved by moving to the Lagrangian system of coordinates.

Let us introduce the Lagrangian coordinates $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ as the coordinates of the particles of the fluid at the initial moment of time

$$\mathbf{x} = \boldsymbol{\xi}, \quad t = 0, \quad \boldsymbol{\xi} \in \Omega = \Omega_0. \tag{4}$$

Then the law of motion of the particles will be determined by the solution of the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \tag{5}$$

with the initial condition (4) and can be found in the form of $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$, $\boldsymbol{\xi} \in \Omega$.

The formulation of the problem of the unsteady motion of an ideal fluid with a free boundary in Lagrangian coordinates is as follows [1]:

$$M^*(\mathbf{x}_{tt} - \mathbf{g}(\mathbf{x}, t)) + \nabla p = 0, \quad \operatorname{div} M^{-1}\mathbf{x}_t = 0, \quad \boldsymbol{\xi} \in \Omega; \tag{6}$$

$$p_0 - p = 2\sigma H_1, \quad \boldsymbol{\xi} \in \Gamma; \tag{7}$$

$$\mathbf{x} = \boldsymbol{\xi}, \quad \mathbf{x}_t = \mathbf{u}_0(\boldsymbol{\xi}), \quad \operatorname{div} \mathbf{u}_0 = 0, \quad t = 0, \tag{8}$$

where M — is the Jacobi matrix of $\mathbf{x}(\boldsymbol{\xi}, t)$ with respect to a variable $\boldsymbol{\xi}$; $M(\boldsymbol{\xi}, 0) = E$; M^*, M^{-1} — transposed and inverse to the M matrix; $\mathbf{g}$ — is the specified vectortheir strengths; p_0 — is the known pressure (of air) above the free boundary; H_1 — is the mean curvature of the free border Γ_t, transformed to the Lagrange coordinates; $\sigma \geqslant 0$ — is the surface tension coefficient. In addition to the boundary and initial conditions, there are no impermeability conditions on the solid wall (bottom) and Dupre — Young conditions on the contact lines of the liquid and the solid wall [1].

Let's suppose that external forces have a potential, $\mathbf{g} = \nabla_x h$. Then it is easy to prove that the momentum equation (6) can be integrated. Indeed, applying the operation rot to this equation, the equality $M^*\mathbf{x}_t = \nabla\varphi + \mathbf{u}_0(\boldsymbol{\xi})$ is odtained and, therefore, the mentioned integral has the form of

$$\varphi_t + p = \frac{1}{2}|\mathbf{x}_t|^2 + h + \chi(t) \tag{9}$$

with an arbitrary function $\chi(t)$. The function φ the elliptic equation of the 2nd order

$$\operatorname{div}\left[M^{-1}M^{*-1}(\nabla\varphi + \mathbf{u}_0(\boldsymbol{\xi}))\right] = 0, \quad \boldsymbol{\xi} \in \Omega. \tag{10}$$

Integral (9) was first obtained by G. Weber in 1868; in particular, for potential motions, it reduces to the well-known Cauchy – Lagrange integral.

Let $\sigma = 0$, $p_0 = \text{const}$. Differentiating expression (9) by time t and replacing $\mathbf{x}_{tt}$ from the equation of momentum (7), the following is obtained [1]

$$\begin{gathered}\varphi_{tt} + \left(-\frac{\partial p}{\partial n}\mathbf{n} + \nabla_{\xi} h\right) \cdot M^{-1}M^{*-1}(\nabla\varphi + \mathbf{u}_0) - h_t = 0, \\ \varphi(\boldsymbol{\xi}, 0) = 0, \quad \varphi_t(\boldsymbol{\xi}, 0) = \frac{1}{2}|\mathbf{u}_0(\boldsymbol{\xi})| + h(\boldsymbol{\xi}, 0),\end{gathered} \tag{11}$$

where the equality $\nabla p = \mathbf{n}\partial p/\partial n$is taken into account. From this representation of conditions on the free boundary one can clearly see the role of the sign of the derivative along the normal to pressure. When $\partial p/\partial n < 0$ prodlem (10), (22) is correctly posed, and if $\partial p/\partial n > 0$ it is incorrect according to Hadamard. Let's also note here that the direction of the pressure gradient on the free boundary plays a decisive role in the correctness of the

Cauchy–Poisson problem in classes of functions of finite smoothness (V. I. Nalimov, 1974; L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov et al., 1985, M. A. Bimenov, 1992).

Group classification of ideal fluid equations in terms of trajectories – Weber's potential

For two-dimensional motions of an ideal fluid in trajectory variables, the Weber potential is the system of equations (9), (10) which is

$$x_t = y_\eta(\varphi_\xi + u(\xi,\eta)) - y_\xi(\varphi_\eta + v(\xi,\eta)); \tag{12}$$

$$y_t = -x_\eta(\varphi_\xi + u(\xi,\eta)) + x_\xi(\varphi_\eta + v(\xi,\eta)); \tag{13}$$

$$x_\xi y_\eta - x_\eta y_\xi = 1; \tag{14}$$

$$u_\xi + v_\eta = 0, \quad v_\xi - u_\eta = \omega(\xi,\eta) \neq 0; \tag{15}$$

$x(\xi,\eta,t)$, $y(\xi,\eta,t)$ —are the trajectories, $\varphi(\xi,\eta,t)$ — is the Weber potential. The pressure is restored via the formula (9):

$$p(\xi,\eta,t) = \frac{1}{2}\,(x_t^2 + y_t^2) - \varphi_t + h(\xi,\eta,t) \tag{16}$$

and $h = -gy(\xi,\eta,t)$ for waves on the water.

Equations (12)–(13) are equivalent to the following two:

$$\varphi_\xi = x_t x_\xi + y_t y_\xi - u, \quad \varphi_\eta = x_t x_\eta + y_t y_\eta - v.$$

The conditions for their compatibility are provided by the law of conservation of a vortex in a particle

$$x_\xi x_{\eta t} - x_\eta x_{\xi t} + y_\xi y_{\eta t} - y_\eta y_{\xi t} = \omega(\xi,\eta),$$

which, together with the law of conservation of mass (13) forms a closed system of equations. Its group properties were studied in [2].

2 Group classification of equations

Let us consider the arbitrary smooth transformations that preserve the area on the plane of the variables (ξ,η):

$$\alpha = \alpha(\xi,\eta), \quad \beta = \beta(\xi,\eta), \quad \frac{\partial(\alpha,\beta)}{\partial(\xi,\eta)} = 1. \tag{17}$$

It is easy to see that in this case the system (12)–(14) retains its differential structure, and the functions $u(\xi,\eta)$, $v(\xi,\eta)$ are replaced by

$$u_1(\alpha,\beta) = \beta_\eta u - \beta_\xi v, \quad v_1(\alpha,\beta) = -\alpha_\eta u + \alpha_\xi v, \tag{18}$$

where in the right-hand parts it should be considered, according to (17), $\xi = \xi(\alpha,\beta)$, $\eta = \eta(\alpha,\beta)$.

As for conditions (15), the second of them is always fulfilled, since

$$\begin{aligned} v_\xi - u_\eta =& (u_1\alpha_\eta + v_1\beta_\eta)_\xi - (u_1\alpha_\xi + v_1\beta_\xi)_\eta = v_{1\alpha} - u_{1\beta} = \\ =& \omega(\xi(\alpha,\beta),\eta(\alpha,\beta)) \equiv \omega_1(\alpha,\beta) \neq 0. \end{aligned}$$

A simple calculation, taking into account the formulas (18) gives us the equality

$$
\begin{aligned}
& u_\xi + v_\eta = (u_1\alpha_\eta + v_1\beta_\eta)_\eta + (u_1\alpha_\xi + v_1\beta_\xi)_\xi = \\
& = (v_{1\alpha}\alpha_\eta + v_{1\beta}\beta_\eta)\beta_\eta + v_1\beta_{\eta\eta} + (u_{1\alpha}\alpha_\eta + u_{1\beta}\beta_\eta)\alpha_\eta + u_1\alpha_{\eta\eta} + \\
& + (u_{1\alpha}\alpha_\xi + u_{1\beta}\beta_\xi)\alpha_\xi + u_1\alpha_{\xi\xi} + (v_{1\alpha}\alpha_\xi + v_{1\beta}\beta_\xi)\beta_\xi + v_1\beta_{\xi\xi} = \\
& = u_1(\alpha_{\xi\xi} + \alpha_{\eta\eta}) + v_1(\beta_{\xi\xi} + \beta_{\eta\eta}) + u_{1\alpha}(\alpha_\xi\alpha_\xi + \alpha_\eta\alpha_\eta) + \\
& + u_{1\beta}(\beta_\xi\alpha_\xi + \beta_\eta\alpha_\eta) + v_{1\alpha}(\alpha_\xi\beta_\xi + \alpha_\eta\beta_\eta) + v_{1\beta}(\beta_\xi\beta_\xi + \beta_\eta\beta_\eta) = \\
& = u_1\Delta\alpha + u_2\Delta\beta + |\nabla\alpha|^2 u_{1\alpha} + \\
& + |\nabla\beta|^2 v_{1\beta} + \nabla\alpha\cdot\nabla\beta(u_{1\beta} + v_{1\alpha}) = 0.
\end{aligned}
\tag{19}
$$

Therefore, in variables α, β for new "velocities" $u_1(\alpha,\beta)$, $v_1(\alpha,\beta)$ the first condition (15) is met if, for example,

$$
\Delta\alpha = \Delta\beta = 0, \quad \nabla\alpha\cdot\nabla\beta = 0, \quad |\nabla\alpha|^2 = |\nabla\beta|^2 = 1. \tag{20}
$$

The last two equations in (20) mean that the functions α, β (or $-\beta$, α) satisfy the Cauchy –Riemann system. Therefore, in this case, the equivalence transformation $(\xi,\eta)\to(\alpha,\beta)$ is conformal and preserves the area, and, as follows from (17), the harmonic function $\alpha(\xi,\eta)$ satisfies the Eikonal equation

$$
\alpha_\xi^2 + \alpha_\eta^2 = 1. \tag{21}
$$

It is easy to see that transformations that satisfy formulas (20) and (21), are reduced to shifts and rotation to an angle γ in the plane ξ,η:

$$
\alpha = \xi\cos\gamma + \eta\sin\gamma + d_1, \quad \beta = -\xi\sin\gamma + \eta\cos\gamma + d_2, \tag{22}
$$

γ, d_1, d_2 — are constant.

Remark 1 *If to the function φ a linear combination of variables ξ, η is added*

$$
\varphi = \varphi_1 + a_1\xi + a_2\eta + d(t), \tag{23}
$$

where a_1, a_2 – are arbitrary constants, $d(t)$ – is an arbitrary function, and functions $u(\xi,\eta)$ and $v(\xi,\eta)$ are replaced with

$$
u_1(\xi,\eta) = a_1 + u(\xi,\eta), \qquad v_1(\xi,\eta) = a_2 + v(\xi,\eta), \tag{24}
$$

then the structure of the system of equations (12)–(15) *will not change. So, the replacement* (23), (24) *is the equivalence transformation for system* (12)–(15).

Lemma 1 *Equivalence transformations are given by the following formulas* (22), (23), (24).

For system (12)–(14) an operator in the form of

$$
Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial \xi} + \xi^3\frac{\partial}{\partial \eta} + \eta^1\frac{\partial}{\partial x} + \eta^2\frac{\partial}{\partial y} + \eta^3\frac{\partial}{\partial \varphi} \tag{25}
$$

is found with unknown coordinates ξ^i, η^i, depending on all variables t, ξ, η, x, y, φ, $i = 1,2,3$.

To further study the system of equations (12)–(14) it is necessary to continue the operator (25) onto the first derivatives

$$\begin{aligned}\underset{1}{Y} =&\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial \xi}+\xi^3\frac{\partial}{\partial \eta}+\eta^1\frac{\partial}{\partial x}+\eta^2\frac{\partial}{\partial y}+\eta^3\frac{\partial}{\partial \varphi}+\\ &+\zeta_1^1\frac{\partial}{\partial x_t}+\zeta_2^1\frac{\partial}{\partial x_\xi}+\zeta_3^1\frac{\partial}{\partial x_\eta}+\zeta_1^2\frac{\partial}{\partial y_t}+\zeta_2^2\frac{\partial}{\partial y_\xi}+\zeta_3^2\frac{\partial}{\partial y_\eta}+\\ &+\zeta_1^3\frac{\partial}{\partial \varphi_t}+\zeta_2^3\frac{\partial}{\partial \varphi_\xi}+\zeta_3^3\frac{\partial}{\partial \varphi_\eta},\end{aligned} \tag{26}$$

here ζ_i^α, $\alpha=1,2,3$, $i=1,2,3$.

Let us use the criterion of invariance of the manifold [3], defined by equations (12)–(14). The transition to a manifold is determined by the fact that from equations (12)–(13) the elements x_t, y_t are expressed through the remaining variables, and from equation (14) we express x_ξ as follows

$$x_\xi=\frac{1}{y_\eta}(1+x_\eta y_\xi).$$

In particular, after the operator (26) acted on equation (14)

$$\underset{1}{Y}\,(x_\xi y_\eta-x_\eta y_\xi-1)=0$$

or in expanded form

$$\zeta_2^1 y_\eta+x_\xi\zeta_3^2-\zeta_3^1 y_\xi-x_\eta\zeta_2^2=0,$$

where

$$\begin{aligned}\zeta_2^1=&\frac{\partial\eta^1}{\partial\xi}+x_\xi\frac{\partial\eta^1}{\partial x}+y_\xi\frac{\partial\eta^1}{\partial y}+\varphi_\xi\frac{\partial\eta^1}{\partial\varphi}-x_t\left(\frac{\partial\xi^1}{\partial\xi}+x_\xi\frac{\partial\xi^1}{\partial x}+y_\xi\frac{\partial\xi^1}{\partial y}+\varphi_\xi\frac{\partial\xi^1}{\partial\varphi}\right)-\\ &-x_\xi\left(\frac{\partial\xi^2}{\partial\xi}+x_\xi\frac{\partial\xi^2}{\partial x}+y_\xi\frac{\partial\xi^2}{\partial y}+\varphi_\xi\frac{\partial\xi^2}{\partial\varphi}\right)-x_\eta\left(\frac{\partial\xi^3}{\partial\xi}+x_\xi\frac{\partial\xi^3}{\partial x}+y_\xi\frac{\partial\xi^3}{\partial y}+\varphi_\xi\frac{\partial\xi^3}{\partial\varphi}\right),\end{aligned}$$

$$\begin{aligned}\zeta_2^2=&\frac{\partial\eta^2}{\partial\xi}+x_\xi\frac{\partial\eta^2}{\partial x}+y_\xi\frac{\partial\eta^2}{\partial y}+\varphi_\xi\frac{\partial\eta^2}{\partial\varphi}-y_t\left(\frac{\partial\xi^1}{\partial\xi}+x_\xi\frac{\partial\xi^1}{\partial x}+y_\xi\frac{\partial\xi^1}{\partial y}+\varphi_\xi\frac{\partial\xi^1}{\partial\varphi}\right)-\\ &-y_\xi\left(\frac{\partial\xi^2}{\partial\xi}+x_\xi\frac{\partial\xi^2}{\partial x}+y_\xi\frac{\partial\xi^2}{\partial y}+\varphi_\xi\frac{\partial\xi^2}{\partial\varphi}\right)-y_\eta\left(\frac{\partial\xi^3}{\partial\xi}+x_\xi\frac{\partial\xi^3}{\partial x}+y_\xi\frac{\partial\xi^3}{\partial y}+\varphi_\xi\frac{\partial\xi^3}{\partial\varphi}\right),\end{aligned}$$

$$\begin{aligned}\zeta_3^1=&\frac{\partial\eta^1}{\partial\eta}+x_\eta\frac{\partial\eta^1}{\partial x}+y_\eta\frac{\partial\eta^1}{\partial y}+\varphi_\eta\frac{\partial\eta^1}{\partial\varphi}-x_t\left(\frac{\partial\xi^1}{\partial\eta}+x_\eta\frac{\partial\xi^1}{\partial x}+y_\eta\frac{\partial\xi^1}{\partial y}+\varphi_\eta\frac{\partial\xi^1}{\partial\varphi}\right)-\\ &-x_\xi\left(\frac{\partial\xi^2}{\partial\eta}+x_\eta\frac{\partial\xi^2}{\partial x}+y_\eta\frac{\partial\xi^2}{\partial y}+\varphi_\eta\frac{\partial\xi^2}{\partial\varphi}\right)-x_\eta\left(\frac{\partial\xi^3}{\partial\eta}+x_\eta\frac{\partial\xi^3}{\partial x}+y_\eta\frac{\partial\xi^3}{\partial y}+\varphi_\eta\frac{\partial\xi^3}{\partial\varphi}\right),\end{aligned}$$

$$\begin{aligned}\zeta_3^2=&\frac{\partial\eta^2}{\partial\eta}+x_\eta\frac{\partial\eta^2}{\partial x}+y_\eta\frac{\partial\eta^2}{\partial y}+\varphi_\eta\frac{\partial\eta^2}{\partial\varphi}-y_t\left(\frac{\partial\xi^1}{\partial\eta}+x_\eta\frac{\partial\xi^1}{\partial x}+y_\eta\frac{\partial\xi^1}{\partial y}+\varphi_\eta\frac{\partial\xi^1}{\partial\varphi}\right)-\\ &-y_\xi\left(\frac{\partial\xi^2}{\partial\eta}+x_\eta\frac{\partial\xi^2}{\partial x}+y_\eta\frac{\partial\xi^2}{\partial y}+\varphi_\eta\frac{\partial\xi^2}{\partial\varphi}\right)-y_\eta\left(\frac{\partial\xi^3}{\partial\eta}+x_\eta\frac{\partial\xi^3}{\partial x}+y_\eta\frac{\partial\xi^3}{\partial y}+\varphi_\eta\frac{\partial\xi^3}{\partial\varphi}\right),\end{aligned}$$

after the transition to the manifold the following equations are obtained

$$\eta^1_x + \eta^2_y - \xi^2_\xi - \xi^3_\eta = 0, \quad \eta^2_\eta - \xi^2_x = 0, \quad \eta^1_\eta + \xi^2_y = 0. \tag{27}$$

In addition, parts of the derived coordinates of the operator (25) for some variables are equal to zero:

$$\begin{gathered} \frac{\partial\xi^1}{\partial\xi} = \frac{\partial\xi^1}{\partial\eta} = \frac{\partial\xi^1}{\partial x} = \frac{\partial\xi^1}{\partial y} = \frac{\partial\xi^1}{\partial\varphi} = 0, \\ \frac{\partial\xi^2}{\partial\varphi} = \frac{\partial\xi^2}{\partial x} = 0, \quad \frac{\partial\xi^3}{\partial\varphi} = \frac{\partial\xi^3}{\partial x} = 0, \quad \frac{\partial\eta^1}{\partial\varphi} = \frac{\partial\eta^2}{\partial\varphi} = 0. \end{gathered} \tag{28}$$

Taking into account the obtained equalities (27), (28) the coordinates of operator (26) are simplified. Acting via the continued operator (26) on the equations of motion (12), (13)

$$\underset{1}{Y}\left(x_t - y_\eta(\varphi_\xi + u) + y_\xi(\varphi_\eta + v)\right) = 0, \tag{29}$$

$$\underset{1}{Y}\left(y_t + x_\eta(\varphi_\xi + u) - x_\xi(\varphi_\eta + v)\right) = 0, \tag{30}$$

and excluding x_t, y_t, x_ξ, after long calculations in addition to (28) the following equalities are obtained

$$\begin{gathered} \frac{\partial\xi^2}{\partial t} = \frac{\partial\xi^2}{\partial y} = 0, \quad \frac{\partial\xi^3}{\partial t} = \frac{\partial\xi^3}{\partial y} = 0, \\ \frac{\partial\eta^1}{\partial\xi} = \frac{\partial\eta^1}{\partial\eta} = 0, \quad \frac{\partial\eta^2}{\partial\xi} = \frac{\partial\eta^2}{\partial\eta} 0. \end{gathered} \tag{31}$$

Also from equations (29) and (30) follow the relations

$$\begin{gathered} (\xi^3_\eta - \xi^1_t + \eta^1_x - \eta^2_y)u - \xi^3_\xi v - (u_\xi\xi^2 + u_\eta\xi^3) - \eta^3_\xi = 0, \\ (\xi^3_\xi - \xi^1_t + \eta^1_x - \eta^2_y)v - \xi^2_\eta u - (v_\xi\xi^2 + v_\eta\xi^3) - \eta^3_\eta = 0, \\ \xi^2_\xi + \xi^3_\eta - \xi^1_t - \eta^\varphi = 0, \quad \eta^1_y + \eta^2_x = 0, \quad \eta^1_t - \eta^3_x = 0, \\ \eta^2_t - \eta^3_y = 0, \quad \eta^1_x = \eta^2_y, \quad \xi^2_\xi + \xi^3_\eta = \eta^1_x + \eta^2_y. \end{gathered} \tag{32}$$

So, from (28), (31) the dependence of the coordinates of the operator (25) is obtained:

$$\begin{gathered} \xi^1(t), \quad \xi^2(\xi,\eta), \quad \xi^3(\xi,\eta), \\ \eta^1(x,y,t), \quad \eta^2(x,y,t), \quad \eta^3(t,\xi,\eta,x,y,\varphi). \end{gathered} \tag{33}$$

Substituting these relationships into (27) and (32) to a system of defining equations

$$\begin{gathered} \xi^2_\xi + \xi^3_\eta = \eta^1_x + \eta^2_y, \\ (\xi^3_\eta - \xi^1_t)u - \xi^3_\xi v - (\xi^2 u_\xi + \xi^3 u_\eta) - \eta^3_\xi = 0, \\ (\xi^2_\eta - \xi^1_t)v + \xi^3_\eta u - (\xi^2 v_\xi + \xi^3 v_\eta) - \eta^3_\eta = 0, \\ \eta^1_y + \eta^2_x = 0, \quad \eta^1_x = \eta^2_y, \quad \xi^2_\xi + \xi^3_\eta - \xi^1_t - \eta^3_\varphi = 0, \\ \xi^1_t(v_\xi - u_\eta) + \xi^2(v_\xi - u_\eta) + \xi^3(v_\xi - u_\eta) = 0. \end{gathered} \tag{34}$$

Let us find the solution of the determining equations (34) taking into account the dependences (33) of the coordinates of the operator (25). First, it must be $\xi^1 = C_3 t + C_4$, where C_3, C_4 – are constants, secondly,

$$\begin{aligned}
&\xi^2_\xi + \xi^3_\eta = 2C_1, \quad \xi^2 = \xi^2(\xi,\eta), \quad \xi^3 = \xi^3(\xi,\eta),\\
&\eta^1 = C_1 x + C_2 y + n(t), \quad \eta^2 = C_1 y - C_2 x + m(t),\\
&\eta^3 = (2C_1 - \xi^1_t)\varphi + n_t x + m_t y + h(\xi,\eta) + d(t),\\
&u(\xi^3_\eta - \xi^1_t) - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v - h_\xi = 0,\\
&v(\xi^2_\xi - \xi^1_t) - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u - h_\eta = 0.
\end{aligned} \tag{35}$$

here $n(t)$, $m(t)$, $d(t)$ – are arbitrary functions of the class C^∞; C_1, C_2 — and are permanent.

In addition, the compatibility of the last two equations (35) implies the relation

$$\xi^2\omega_\xi + \xi^3\omega_\eta + C_3\omega = 0, \tag{36}$$

where $\omega = v_\xi - u_\eta = \omega(\xi,\eta)$ — the initial vorticity of the fluid. Equation (36) is decisive; when it is executed, the function $h(\xi,\eta)$ is reconstructed from (35) using the curvilinear integral

$$\begin{aligned}
h = \int &\left[u(\xi^3_\eta - C_3) - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v\right] d\xi +\\
&+ \left[v(\xi^2_\xi - C_3) - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u\right] d\eta,
\end{aligned} \tag{37}$$

independent of the path of integration.

Let us find the kernel of the operators. To do this, let us assume that $\omega(\xi,\eta)$ – is an arbitrary function. Equation (36) can be satisfied when $C_3 = 0$, $\xi^2 = 0$, $\xi^3 = 0$, at this $\xi^1 = C_4$. From the last two defining equations (35) it follows that

$$h_\xi = 0, \qquad h_\eta = 0,$$

hence $h = h_0$ – is a constant, which, according to Remark (1) can be considered to be zero. From equation $\xi^2_\xi + \xi^3_\eta = 2C_1$ then $C_1 = 0$.

So the coordinates of the operator are

$$\begin{gathered}
\xi^1 = C_4, \quad \xi^2 = 0, \quad \xi^3 = 0,\\
\eta^1 = C_2 y + n(t), \quad \eta^2 = -C_2 x + m(t),\\
\eta^3 = n_t x + m_t y + d(t).
\end{gathered}$$

The kernel is determined by the infinitesimal operator

$$\begin{aligned}
Y^{\text{KERNEL}} =& C_4\frac{\partial}{\partial t} + (C_2 y + n(t))\frac{\partial}{\partial x} + (-C_2 x + m(t))\frac{\partial}{\partial y} +\\
&+ (n_t(t)x + m_t(t)y + d(t)\frac{\partial}{\partial \varphi},
\end{aligned}$$

whence it follows that the basic Lie algebra (the algebra admitted by system (12)–(14) for an arbitrary function ω, or u, v) is formed by the operators

$$\begin{gathered}
L_0 \,:\; Y_1 = \partial_t, \quad Y_2 = y\partial_x - x\partial_y, \quad Y_n = n(t)\partial_x + x n_t(t)\partial_\varphi,\\
Y_m = m(t)\partial_y + y m_t(t)\partial_\varphi, \quad Y_d = d(t)\partial_\varphi.
\end{gathered} \tag{38}$$

For further convenience, let us make the replacement

$$\xi^2 = C_1\xi + \bar{\xi}^2(\xi,\eta), \quad \xi^3 = C_1\eta + \bar{\xi}^3(\xi,\eta), \tag{39}$$

then the last three equations in (35) and (36) will be rewritten as:

$$\begin{aligned} &\bar{\xi}^2_\xi + \bar{\xi}^3_\eta = 0, \\ &h_\xi = u(C_1 - C_3 + \bar{\xi}^3_\eta) - (C_1\xi + \bar{\xi}^2)u_\xi - (C_1\eta + \bar{\xi}^3)u_\eta - \bar{\xi}^3_\xi v, \\ &h_\eta = v(C_1 - C_3 + \bar{\xi}^2_\xi) - (C_1\xi + \bar{\xi}^2)v_\xi - (C_1\eta + \bar{\xi}^3)v_\eta - \bar{\xi}^2_\eta u, \end{aligned} \tag{40}$$

$$(C_1\xi + \overline{\xi^2})\omega_\xi + (C_1\eta + \overline{\xi^3})\omega_\eta + C_3\omega = 0. \tag{36$'$}$$

Clearly, this equation (36′) is the classifying one.

If $\omega = \omega_0 = \text{const}$, then $C_3 = 0$. Then the curvilinear integral (37) will be rewritten as:

$$h = \int \left[u\xi^3_\eta - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v\right] d\xi + \left[v\xi^2_\xi - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u\right] d\eta.$$

Let us write out the last two equalities from (35) and the connection between ξ^2_ξ and ξ^3_η :

$$\begin{aligned} &h_\xi = u(\xi^3_\eta - C_3) - \xi^2 u_\xi - \xi^3 u_\eta - \xi^3_\xi v, \\ &h_\eta = v(\xi^2_\xi - C_3) - \xi^2 v_\xi - \xi^3 v_\eta - \xi^2_\eta u, \\ &\xi^2_\xi + \xi^3_\eta = 2C_1. \end{aligned}$$

Let us introduce the function $\psi(\xi,\eta)$, such that $\overline{\xi^2} = \psi_\eta$, $\overline{\xi^3} = -\psi_\xi$, then (39) will take the form of

$$\xi^2 = C_1\xi + \psi_\eta, \quad \xi^3 = C_1\eta - \psi_\xi,$$

and from (40) it's evident that

$$\begin{aligned} &h_\xi = u(-\psi_{\xi\eta}) - \psi_\eta u_\xi + \psi_\xi u_\eta + \psi_{\xi\xi} v, \\ &h_\eta = v\psi_{\eta\xi} - \psi_\eta v_\xi + \psi_\xi v_\eta - \psi_{\eta\eta} u. \end{aligned}$$

Let's rewrite the operator (25), considering that $\omega = \omega_0 = \text{const}$, $\mathrm{C}_3 = 0$, $\xi^1 = \mathrm{C}_4$:

$$\begin{aligned} Y =& C_4\frac{\partial}{\partial t} + (C_1\xi + \psi_\eta)\frac{\partial}{\partial \xi} + (C_1\eta - \psi_\xi)\frac{\partial}{\partial \eta} + (C_1 x + C_2 y + n(t))\frac{\partial}{\partial x} + \\ &+ (C_1 y - C_2 x + m(t))\frac{\partial}{\partial y} + (2C_1\varphi + n_t x + m_t y + h(\xi,\eta) + d(t))\frac{\partial}{\partial \varphi}. \end{aligned}$$

Then the L_0 algebra is extended by the operators:

$$\begin{aligned} &Y_\psi = \psi_\eta\partial_\xi - \psi_\xi\partial_\eta + h\partial_\varphi, \\ &Y_3 = \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + (2\varphi)\partial_\varphi \end{aligned} \tag{41}$$

with an arbitrary smooth function $\psi(\xi,\eta)$.

We make the change of variables $u = \overline{u}(\xi,\eta) - \omega_0\eta$ (shift over u) and substitute it into (15): $\overline{u}_\xi + v_\eta = 0$, $v_\xi - \overline{u}_\eta = 0$. The resulting system is a Cauchy-Riemann system for determining $\overline{u}$, v, which means u, v.

In the case $\omega(\xi,\eta) \neq \text{const}$ we write down the classifying equation (36) and the equation with dependence on ξ^2, ξ^3 from (35):

$$\xi^2\omega_\xi + \xi^3\omega_\eta + C_3\omega = 0, \quad \xi^2_\xi + \xi^3_\eta = 2C_1.$$

Let $F(\xi,\eta)$ – be an arbitrary function and $\xi^2 = F_\eta + C_1\xi,\ \xi^3 = -F_\xi + C_1\eta$. Then it is obvious that the equality from (35), holds, and the classifying equation takes the form:

$$(F_\eta + C_1\xi)\omega_\xi + (-F_\xi + C_1\eta)\omega_\eta = -C_3\omega.$$

Let us consider the possible options

1. $C_1 = C_3 = 0$, then the classifying equation: $F_\eta\omega_\xi - F_\xi\omega_\eta = 0$, means $F = F(\omega)$. Given the choice of $\xi^2,\ \xi^3,\ \xi^2 = F_\eta(\omega),\ \xi^3 = -F_\xi(\omega)$. In this case $Y_F = F_\eta(\omega)\partial_\xi - F_\xi(\omega)\partial_\eta$.

2. $F = \text{const}$. Then $C_1\xi\omega_\xi - C_1\eta\omega_\eta = -C_3\omega$, $\xi^2 = C_1\xi,\ \xi^3 = C_1\eta$, denote $C_3/C_1 = \delta$. The resulting differential equation is solved via the characteristic method.

$$\frac{d\xi}{\xi} = \frac{d\eta}{\eta} = \frac{d\omega}{-\delta\omega}$$

$$\frac{\xi}{\eta} = J_1; \quad \frac{d\eta}{\eta} = -\frac{d\omega}{\delta\omega}, \quad \text{then} \quad \omega\eta^{-\delta} = J_2.$$

$$\omega(\xi,\eta) = \eta^{-\delta} f(\xi/\eta). \tag{42}$$

In that case

$$Y = \delta t\partial_t + \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + (2-\delta)\varphi\partial_\varphi.$$

Thus for any given initial vorticity$\omega(\xi,\eta) \neq \text{const}$ the basis of a Lie algebra that extends L_0 is also represented by two operators:

$$\begin{aligned} Y_F =& \frac{\partial F(\omega)}{\partial\eta}\partial_\xi - \frac{\partial F(\omega)}{\partial\xi}\partial_\eta, \\ Y_4 =& \delta t\partial_t + \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + (2-\delta)\varphi\partial_\varphi. \end{aligned} \tag{43}$$

From the structure of the vorticity representation (42), let's assume that the components of the initial velocity field have the following representations

$$u = \eta^{1-\delta}u_1(\zeta), \quad v = \eta^{1-\delta}v_1(\zeta), \quad \zeta = \xi/\eta, \tag{44}$$

wherein functions u_1, v_1 are related by

$$u_{1\zeta} + (1-\delta)v_1 - \zeta v_{1\zeta} = 0, \tag{45}$$

which is a consequence of the mass conservation equation: $u_\xi + v_\eta = 0$. In this case the function $f(\zeta)$, which determines the vorticity in formula (42) by the known $u_1(\zeta)$, $v_1(\zeta)$, taking into account (15) is given by the equation

$$f(\zeta) = -(1-\delta)u_1 + \zeta u_{1\zeta} + v_{1\zeta}. \tag{46}$$

Using formulas (44) it is easy to show that $h_\xi = h_\eta = 0$, and, without limitation generality, it can be set that $h = 0$.

3 Invariance of initial conditions

When solving the system of equations (12)–(14), which is not normal in the time variable, it is necessary to take into account the initial conditions

$$x = \xi, \quad y = \eta, \quad t = 0. \tag{47}$$

The invariance of data (47) with respect to the operator Y simplifies the system of defining equations (35) to the following:

$$\begin{gathered} \xi^1 = C_3 t, \quad \xi^2 = C_1\xi + C_2\eta + n(0), \quad \xi^3 = C_1\eta - C_2\xi + m(0), \\ \eta^1 = C_1 x + C_2 y + n(t), \quad \eta^2 = C_1 y - C_2 x + m(t), \\ \eta^3 = (2C_1 - C_3)\varphi + n_t x + m_t y + h(\xi,\eta) + d(t), \\ h_\xi = (C_1 - C_3)u - \xi^2 u_\xi - \xi^3 u_\eta + C_2 v, \\ h_\eta = (C_1 - C_3)v - \xi^2 v_\xi - \xi^3 v_\eta - C_2 u. \end{gathered} \tag{48}$$

The classifying equation (36) here is:

$$(C_1\xi + C_2\eta + n(0))\omega_\xi + (C_1\eta - C_2\xi + m(0))\omega_\eta + C_3\omega = 0. \tag{49}$$

The basic L_{00} algebra consists of the following operators

$$\begin{gathered} Z_n = n(t)\partial_x + n_t(t)x\partial_\varphi, \quad Z_m = m(t)\partial_y + m_t(t)y\partial_\varphi, \\ Z_d = d(t)\partial_\varphi, \quad n(0) = m(0) = 0. \end{gathered} \tag{50}$$

For group classification, let us write the equation (49) in the form of

$$(A\xi + B\eta + C)\omega_\xi + (A\eta - B\xi + D)\omega_\eta + H\omega = 0 \tag{51}$$

with any constants A, B, C, D, H. Any particular function $\omega(\xi,\eta)$ with which it is possible to expand the L_{00}core, should be a solution to equation (51). The general equivalence transformation consists of all the transformations corresponding to the kernel of Lie algebras (50) and from the transformations of (a_i, $i = 1, \dots, 4$, — constants)

$$\begin{gathered} \bar t = a_1 t, \quad \bar x = a_1(x\cos a_2 + y\sin a_2) + a_3, \\ \bar y = a_1(-x\sin a_2 + y\cos a_2) + a_4, \\ \bar\xi = a_1(\xi\cos a_2 + \eta\sin a_2) + a_3, \\ \bar\eta = a_1(-\xi\sin a_2 + \eta\cos a_2) + a_4, \\ \bar\varphi = a_1\varphi + d(t). \end{gathered} \tag{52}$$

At the same time, an arbitrary element ω changes like this:

$$\bar\omega(\bar\xi,\bar\eta) = a_5\omega(\bar\xi,\bar\eta) \neq 0, \quad a_5 = \text{const}. \tag{53}$$

It can be verified that equation (51) is invariant under equivalence transformations (52), (53), supplemented by a transformation of the constants A, B, C, D, H in the form of:

$$\begin{gathered} \bar A = A, \quad \bar B = B, \quad \bar C = \cos a_2 C - \sin a_2 D - a_3 A - a_4 B, \\ \bar D = \cos a_2 D - \sin a_2 C - a_4 A + a_3 B, \quad \bar H = H. \end{gathered} \tag{54}$$

Let us suppose that $A \neq 0$, $B \neq 0$ at the same time, then, according to (54), due to shift transformations, it is obvious that $\bar{C} = \bar{D} = 0$. In this case, the solution to equation (51) is:

$$\omega = e^{\gamma_2 \operatorname{arctg}(\xi/\eta)} f\left((\xi^2 + \eta^2)^{1/2} e^{\gamma_1 \operatorname{arctg}(\xi/\eta)}\right) \tag{55}$$

with constants $\gamma_1 \neq 0$, γ_2 and an arbitrary function f.

If $A \neq 0$, $B = 0$, then let's assume that $\bar{C} = 0$ and the classification equation has the form of

$$\xi\omega_\xi + (\eta + D)\omega_\eta + H\omega = 0 \tag{56}$$

with other D and H constants. Its solution is:

$$\omega = (\eta + \gamma_3)^{\gamma_4} f\left(\frac{\xi}{\eta + \gamma_3}\right), \tag{57}$$

where γ_3, γ_4 are constants ($\gamma_3 = D$, $\gamma_4 = -H$).

If $A = 0$, $B \neq 0$, then let's also take into consideration $\bar{C} = 0$ and the equation (51) can be re-written as:

$$\eta\omega_\xi + (-\xi + D)\omega_\eta + H\omega = 0.$$

Its solution is as follows:

$$\omega = \exp\left[\gamma_6 \arcsin\left(\frac{\gamma_5 - \xi}{\sqrt{\eta^2 + (\gamma_5 - \xi)^2}}\right)\right] f(2\gamma_5\xi - \xi^2 - \eta^2), \tag{58}$$

($\gamma_5 = D$, $\gamma_6 = H$).

Now let $A = B = 0$, then the coefficients of the equation (51) are constant and its solution is

$$\omega = e^{-\gamma_7\xi} f(\gamma_8\xi - \gamma_9\eta) \tag{59}$$

given that $C \neq 0$, $\gamma_7 = H/C$, $\gamma_8 = D$, $\gamma_9 = C$;

$$\omega = e^{-\gamma_7\eta} f(\gamma_8\xi - \gamma_9\eta) \tag{60}$$

given that $D \neq 0$, $\gamma_7 = H/D$, $\gamma_8 = D$, $\gamma_9 = C$.

In each case, the operators are now presented for which the extension of the basic L_{00} algebra occurs. For this, the derivatives ω_ξ, ω_η are calculated, which are substituted into equation (49), and then it is analyzed taking into account the equivalence transformation (54). This is done to represent the initial vorticity (55). It is consistent that

$$\omega_\xi = \frac{e^{\gamma_2 \operatorname{arctg}(\xi/\eta)}}{\rho}\left[\gamma_2\eta f(\zeta) + e^{\gamma_1 \operatorname{arctg}(\xi/\eta)}(\xi + \gamma_1\eta) f_\zeta\right],$$

$$\omega_\eta = \frac{e^{\gamma_2 \operatorname{arctg}(\xi/\eta)}}{\rho^2}\left[-\gamma_2\xi f(\zeta) + \rho e^{\gamma_1 \operatorname{arctg}(\xi/\eta)}(\eta - \gamma_1\xi)\right],$$

given that $\rho = (\xi^2 + \eta^2)^{1/2}$, $\zeta = \rho e^{\gamma_1 \operatorname{arctg}(\xi/\eta)}$. Substitution into the classifying equation (49) leads to the equality

$$\Big[(\gamma_2 C_2 + C_3)\rho^2 + \gamma_2[n(0) - m(0)]\xi\Big] f(\zeta) +$$

$$+\zeta\Big[(C_1 + C_2)\rho^2 + [n(0) - \gamma_1 m(0)]\xi + [m(0) + \gamma_1 n(0)]\eta\Big] f(\zeta) = 0.$$

Since f is a function of a variable ζ, the last equation will be satisfied, when either $n(0)=m(0)=0$, or $C_2=-C_1$, $C_3=\gamma_2 C_1$, $m(0)=-n(0)$, $\gamma_1=1$. In the first case, the function f satisfies the equation

$$(C_1+C_2)\zeta f_\zeta+(\gamma_2 C_2+C_3)f=0, \tag{61}$$

in the second one —

$$\zeta f_\zeta+\gamma_2 f=0. \tag{62}$$

If in equation (61) the function f is arbitrary, then $C_2=-C_1$, $C_3=\gamma_2 C_1$ and the core of L_{00} (50) is extended by the operator

$$\begin{gathered}Z_1=\gamma_2 t\partial_t+(\xi-\eta)\partial_\xi+(\xi+\eta)\partial_\eta+(x-y)\partial_x+(x+y)\partial_y+[(2-\gamma_2)\varphi+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=(1-\gamma_2)u-(\xi-\eta)u_\xi-(\xi+\eta)u_\eta-v,\\ h_\eta=(1-\gamma_2)v-(\xi-\eta)v_\xi-(\xi+\eta)v_\eta+u.\end{gathered} \tag{63}$$

The assumption $f=1$ (the transformation of stretching (53) for ω is taken into account)is classifying. Here $C_3=-\gamma_2 C_2$ and two more are added to the operator Z_1:

$$\begin{gathered}Z_2=\xi\partial_\xi+\eta\partial_\eta+y\partial_y+(2\varphi+h(\xi,\eta)\partial_\varphi,\\ h_\xi=u-\xi u_\xi-\eta u_\eta,\quad h_\eta=v-\xi v_\xi-\eta v_\eta;\end{gathered} \tag{64}$$

$$\begin{gathered}Z_3=-\gamma_2 t\partial_t+\eta\partial_\xi-\xi\partial_\eta+y\partial_x-x\partial_y+[\gamma_2\varphi+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=\gamma_2 u-\eta u_\xi+\xi u_\eta+v,\quad h_\eta=\gamma_2 v-\eta v_\xi+\xi v_\eta-u.\end{gathered} \tag{65}$$

In general, $C_1+C_2\neq 0$, $\gamma_2 C_2+C_3\neq 0$, the function f is equivalent to $f=\zeta^{\gamma_8}$, $\gamma_8\neq 0$. At the same time, $C_3=-\gamma_8 C_1-(\gamma_1+\gamma_8)C_2$ and the following is added to the operator (63)

$$\begin{gathered}Z_4=-\gamma_8 t\partial_t+\xi\partial_\xi+\eta\partial_\eta+x\partial_x+y\partial_y+[(2+\gamma_8)\varphi+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=(1+\gamma_8)u-\xi u_\xi-\eta u_\eta,\quad h_\eta=(1+\gamma_8)v-\xi v_\xi-\eta v_\eta;\end{gathered} \tag{66}$$

$$\begin{gathered}Z_5=-(\gamma_2+\gamma_8)t\partial_t+\eta\partial_\xi-\xi\partial_\eta+y\partial_x-x\partial_y+[(\gamma_2+\gamma_8)\varphi+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=(\gamma_2+\gamma_8)u-\eta u_\xi+\xi u_\eta+v,\quad h_\eta=(\gamma_2+\gamma_8)v-\eta v_\xi+\xi v_\eta-u.\end{gathered} \tag{67}$$

For equation (62), is obtained $f=\zeta^{\gamma_2}$, where γ_2 is an arbitrary constant, $\zeta=\rho\exp(\mathrm{arctg}(\xi/\eta))$. Permitted operators have the form of Z_1 and

$$\begin{gathered}Z_6=a(0)\partial_\xi-a(0)\partial_\eta+a(t)\partial_x-a(t)\partial_y+[a_t(x+y)+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=-a(0)u_\xi+a(0)u_\eta,\quad h_\eta=-a(0)v_\xi+a(0)v_\eta\end{gathered} \tag{68}$$

with an arbitrary smooth function $a(t)$.

For the function of the form (57), the following result is obtained: if f is arbitrary, then only L_{00} is allowed; with $f=1$ (then $\gamma_4\neq 0$) the L_{00} core is expanded by the operator

$$\begin{gathered}Z_7=-\gamma_4 t\partial_t+\xi\partial_\xi+(\eta+\gamma_3)\partial_\eta+x\partial_x+(y+\gamma_3 a(t))\partial_y+[(2+\gamma_4)\varphi+\gamma_3 a_t y+h(\xi,\eta)]\partial_\varphi,\\ h_\xi=(1+\gamma_3)u-\xi u_\xi-(\eta+\gamma_3)u_\eta,\quad h_\eta=(1+\gamma_3)v-\xi v_\xi-(\eta+\gamma_3)v_\eta\end{gathered} \tag{69}$$

with function $a(t)$ such that $a(0)=1$.

If $f\neq\text{const}$, then $f=(\zeta+\gamma_6)^{\gamma_4}$ and the operator is added to L_{00}

$$Z_8=-\gamma_6\partial_\xi-\gamma_6 a(t)\partial_x+\partial_\eta+a(t)\partial_x+(-\gamma_6 a_t x+a_t y+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=-\gamma_6 u_\xi-u_\eta,\quad h_\eta=-\gamma_6 v_\xi-v_\eta,\quad a(0)=1. \tag{70}$$

For the initial vorticity defined by the formula (58), for $f=1$ there are additional operators

$$Z_9=(\xi-\gamma_5)\partial_\xi+\eta\partial_\eta+(x-\gamma_5 a(t))\partial_x+y\partial_y+(2\varphi-\gamma_5 a_t x+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=u-(\xi-\gamma_5)u_\xi-\eta u_\eta,\quad h_\eta=v-(\xi-\gamma_5)v_\xi-\eta v_\eta; \tag{71}$$

$$Z_{10}=\gamma_6 t\partial_t+\eta\partial_\xi-(\xi-\gamma_5)\partial_\eta+y\partial_x-(y-\gamma_5 a(t))\partial_y+(\gamma_5 a_t y+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=-\gamma_6 u-\eta u_\xi-(\xi-\gamma_5)u_\eta+v,\ h_\eta=-\gamma_6 v-\eta v_\xi-(\xi-\gamma_5)v_\eta-u,\ a(0)=1. \tag{72}$$

When $f\neq\text{const}$, it can be seen that $f=(\zeta-\gamma_5^2)^{\gamma_7}$, $\gamma_7\neq0$, and the L_{00} core expands via the operators

$$Z_{11}=-2\gamma_7 t\partial_t+(\xi-\gamma_5)\partial_\xi+\eta\partial_\eta+(x-\gamma_5 a(t))\partial_x+y\partial_y+[2(\gamma_7+1)\varphi-\gamma_5 a_t x+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=(2\gamma_7+1)u-(\xi-\gamma_5)u_\xi-\eta u_\eta,\quad h_\eta=(2\gamma_7+1)v-(\xi-\gamma_5)v_\xi-\eta v_\eta; \tag{73}$$

$$Z_{12}=\gamma_6 t\partial_t+\eta\partial_\xi-(\xi-\gamma_5)\partial_\eta+y\partial_x-(x-\gamma_5 a(t))\partial_y+(-\gamma_6\varphi+\gamma_5 a_t y+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=\gamma_6 u-\eta u_\xi+\xi u_\eta+v,\quad h_\eta=\gamma_6 v-\eta v_\xi+\xi v_\eta-u,\ a(0)=1. \tag{74}$$

To represent (59) of the function ω it is seen that for $f=1$ (here $\gamma_7\neq0$) one additional operator

$$Z_{13}=\gamma_7 t\partial_t+\partial_\xi+a(t)\partial_x+(-\gamma_7\varphi+a_t x)\partial_\varphi,$$
$$h_\xi=-\gamma_7 u-u_\xi,\quad h_\eta=-\gamma_7 v-v_\xi,\ a(0)=1. \tag{75}$$

When $f\neq\text{const}$ ($\gamma_8^2+\gamma_9^2\neq0$), there must be $C_2=0$ and two possibilities: 1) $\gamma_7=0$; 2) $C_1=0$. In the first case, there are two operators

$$Z_{14}=-\gamma_{10}\gamma_8 t\partial_t+\partial_\xi+a(t)\partial_x+(\gamma_{10}\gamma_8\varphi+a_t x+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=\gamma_{10}\gamma_8 u-u_\xi,\quad h_\eta=\gamma_{10}\gamma_8 v-v_\xi; \tag{76}$$

$$Z_{15}=\gamma_{10}\gamma_9 t\partial_t+\partial_\eta+a(t)\partial_y+(-\gamma_{10}\gamma_9\varphi+a_t y+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=-\gamma_{10}\gamma_9 u-u_\eta,\quad h_\eta=-\gamma_{10}\gamma_9 v-v_\eta,\ a(0)=1. \tag{77}$$

Moreover $f=e^{\gamma_{10}\zeta}$, $\gamma_{10}\neq0$, $\zeta=\gamma_8\xi-\gamma_9\eta$.

If $f=(\zeta+\gamma_{11})^{\gamma_{12}}$, $\gamma_{12}\neq0$), then at $\gamma_{11}=0$ and $\gamma_9\neq0$ the operators are present

$$Z_{16}=-\gamma_{12}t\partial_t+\xi\partial_\xi+a(t)x\partial_x+[(2+\gamma_{12}\varphi+a_t x+h(\xi,\eta)]\partial_\varphi,$$
$$h_\xi=(1+\gamma_{12})u-\xi u_\xi-\eta u_\eta,\quad h_\eta=(1+\gamma_{12})v-\xi v_\xi-\eta v_\eta; \tag{78}$$

$$Z_{17}=\partial_\xi-\gamma_9^{-1}\gamma_8\partial_\eta+a(t)\partial_x-\gamma_9^{-1}\gamma_8 a(t)\partial_y+(a_t x-\gamma_9^{-1}\gamma_8 a_t y+h(\xi,\eta))\partial_\varphi,$$
$$h_\xi=-u_\xi-\gamma_9^{-1}\gamma_8 u_\eta,\quad h_\eta=-v_\xi-\gamma_9^{-1}\gamma_8 v_\eta,\ a(0)=1. \tag{79}$$

When $\gamma_{11} \neq 0$) have the following operators are present

$$Z_{18} = -\gamma_{12}t\partial_t + \xi\partial_\xi + (\eta - \gamma_9^{-1}\gamma_{11})\partial_\eta + x\partial_x + [y - \gamma_9^{-1}\gamma_{11}a(t)]\partial_y + \\ +(\varphi\partial_\varphi - \gamma_9^{-1}\gamma_{11}a_t y + h(\xi,\eta))\partial_\varphi,$$

$$h_\xi = (1 + \gamma_{12}u - \xi u_\xi - (\eta - \gamma_9^{-1}\gamma_{11})u_\eta, \quad h_\eta = (1 + \gamma_{12}v - \xi v_\xi - (\eta - \gamma_9^{-1}\gamma_{11})v_\eta, \ a(0) = 1; \tag{80}$$

$$Z_{19} = \partial_\xi + a(t)\partial_x + \gamma_9^{-1}\gamma_8 a(t)\partial_y + \gamma_9^{-1}\gamma_8\partial_\eta + [\gamma_9^{-1}\gamma_8 a_t y + h(\xi,\eta)]\partial_\varphi, \\ h_\xi = -u_\xi - \gamma_9^{-1}\gamma_8 u_\eta, \quad h_\eta = -v_\xi - \gamma_9^{-1}\gamma_8 v_\eta, \ a(0) = 1. \tag{81}$$

Remark 2 *For potential motions, $u = \varphi_{0\xi}$, $v = \varphi_{0\eta}$, and the replacement $\varphi \to \varphi + \varphi_0$ allows us to consider in the system* (12), (13) $u = v = 0$. *From the system of defining equations* (35) *the basic Lie algebra of operators are obtained*

$$\mu(t)\partial_t - \mu_t(t)\varphi\partial_\varphi, \ y\partial_x - x\partial_y, \ n(t)\partial_x + n_t(t)\partial_\varphi, \ m(t)\partial_y + m_t(t)\partial_\varphi,$$

$$d(t)\partial_\varphi, \ \psi_\eta\partial_\xi - \psi_\xi\partial_\eta, \ \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + 2\varphi\partial_\varphi$$

with arbitrary class C^∞ functions $\mu(t)$, $n(t)$, $m(t)$, $d(t)$, $\psi(\xi,\eta)$.

4 Arbitrary Lagrangian coordinates

The Cauchy problem (4), (5) defines the Lagrange variables ξ, η, as Cartesian coordinates of liquid particles at the initial moment of time. However, instead of ξ, η, distinguishing one particle of fluid from another, it is possible to take any values of a, b, related to ξ, η corellations

$$\xi = f(a,b), \quad \eta = g(a,b), \quad J = f_a g_b - f_b g_a \neq 0. \tag{82}$$

Thus, the region $\Omega_0$is one-to-one mapped onto the region Ω_{ab} of the changes of the variables a, b. The system (12)–(14) is equivalent to the system

$$JX_t = Y_b(\Phi_a + U_1) - Y_a(\Phi_b + V_1); \tag{83}$$

$$JY_t = -X_b(\Phi_a + U_1) + X_a(\Phi_b + V_1); \tag{84}$$

$$X_aY_b - X_bY_a = J(a,b) \tag{85}$$

with the initial data

$$X\Big|_{t=0} = f(a,b), \quad Y\Big|_{t=0} = g(a,b). \tag{86}$$

New "velocities" $U_1(a,b)$, $V_1(a,b)$ are connected to $u(\xi,\eta)$, $v(\xi,\eta)$ in the following way:

$$U_1(a,b) = f_a u(f(a,b), g(a,b)) + g_a v(f(a,b), g(a,b)), \\ V_1(a,b) = f_b u(f(a,b), g(a,b)) + g_b v(f(a,b), g(a,b)). \tag{87}$$

If in the system of equations (83)–(85) the replacement is made $X(a,b,t) = x(\xi,\eta,t)$, $Y(a,b,t) = y(\xi,\eta,t)$, $\Phi(a,b,t) = \varphi(\xi,\eta,t)$, where the variables are ξ, η given by equality (82), then the functions x, y, φ are solutions of system (12)–(14) and by (85) $x = \xi$, $y = \eta$ at $t = 0$. Therefore, the Lie groups of the transformations of the system (12)–(14)

and (83)–(85) are similar, therefore the group classification of the system (83)–(85) can be omitted. Only a general view of the operator of this system is given:

$$Y_1 = \xi^1(t)\partial_t + \xi_1^2(a,b)\partial_a + \xi_1^3(a,b)\partial_b + \eta^1\partial_X + \eta^2\partial_Y + \eta^3\partial_\Phi,$$

where $\xi^1(t)$, η^1, η^2, η^3 are the same as for the operator Y of the system (12)–(14). The coordinates $\xi_1^2(a,b)$, $\xi_1^3(a,b)$ satisfy the defining equations

$$\begin{gathered}(J\xi_1^2)_a + (J\xi_1^3)_b = 2C_1 J,\\ \xi_1^2(\omega_1/J)_a + \xi_1^3(\omega_1/J)_b + C_3\omega_1/J = 0,\end{gathered} \tag{88}$$

where $\omega_1(a,b) = J(a,b)\omega(f(a,b), g(a,b)) = V_{1a} - U_{1,b}$ — is the "vorticity" in the variables a, b.

Despite the similarity of the Lie groups of systems (12)–(14) and (83)–(85), the study of the latter is sometimes more preferable from the point of view of constructing exact solutions. This is due to the arbitrariness of the transformations (82): if an exact solution is found in the variables a, b, then in variables ξ, η There is generally no analytical expression for such a solution. There is only its parametric representation and a, b are parameters. This situation often arises in the theory of the Theory of ordinary differential equations. One well-known example of Görstner waves on water is given in [4].

The representation of (82) has the form of

$$\xi = a + \frac{1}{k}\,e^{kb}\sin(ka), \quad \eta = b - \frac{1}{k}\,e^{kb}\cos(ka), \quad J = 1 - e^{2kb} \tag{89}$$

and when $k > 0$, $b \leqslant b_0 < 0$ it is univalent. It is clear that there are no explicit dependencies of a, b from ξ, η. Still, the formulas

$$\begin{gathered}X = a + \frac{1}{k}\,e^{kb}\sin[k(a+ct)], \quad Y = b - \frac{1}{k}\,e^{kb}\cos[k(a+ct)],\\ \Phi = \frac{ce^{kb}}{k}\,\{\sin[k(a+ct)] - \sin(ka)\}\end{gathered} \tag{90}$$

are the exact solution of the (83)–(85) system (k, c are constants).

Real motion vorticity in variables a, b is equal to

$$\omega = -\frac{2kce^{kb}}{1 - e^{2kb}}\,. \tag{91}$$

According to formulas (16) and (89) the representation for pressure in the same variables (here g is the acceleration of gravity) is obtained

$$p = -gb + \frac{1}{2}\,c^2e^{2kb} + \frac{1}{k}\,(g - 2kc^2)e^{kb}\cos[k(a+ct)] + \text{const}.$$

It is required that the image of the line $b = b_0$ under the mapping (89) be a free boundary. From the expression for the pressure it can be seen that for this it suffices to put $c^2 = g/k$ and choose the appropriate integration constant:

$$p = g(b_0 - b) + \frac{1}{2}\,c^2(e^{2kb} - e^{2kb_0}). \tag{92}$$

Formulas (89), (90), (92) with the constant c chosen above determine the wavemotion, stationary in the coordinate system, which moves with speed $-c$ along the x axis. The free boundary $b = b_0$ with a fixed t is a trochoid with a wavelength of $2\pi/k$ (a curve drawn by a point of a circle of radius $R = k^{-1}e^{kb}$, rolling in a straight line). The equality says that the Görstner waves are always are whirling movements. The vortex is maximal on the free boundary and decreases exponentially with depth.

When $b > 0$, the mapping (89) ceases to be univalent. Free border will be a trachoid with loops and such a decision has no physical meaning. The limiting case $b_0 = 0$ gives the free boundary equation at $t = 0$ in the form of a cycloid

$$x = a + \frac{1}{k}\sin(ka), \quad y = b - \frac{1}{k}\cos(ka).$$

If $ka = (2n+1)\pi$ (n is integer), this cycloid has cusps, i.e. here there are singular points on the free border.

In conclusion, let's point out that the Görstner waves are invariant solutions of the system (83)–(85) with respect to a two-dimensional subalgebra $\langle c^{-1}\partial_t - \partial_a - \partial_X; \partial_\Phi\rangle$ where $J = 1 - e^{2kb}$, $f = a + k^{-1}e^{kb}\sin(ka)$, $g = b - k^{-1}e^{kb}\cos(ka)$, see formula (89). "New" initial velocities are:

$$U_1 = ce^{2kb} + ce^{kb}\cos(ka), \quad V_1 = ce^{kb}\sin(ka),$$

where $c = (g/k)^{1/2}$. Let us note that $U_{1a} + V_{1b} = 0$.

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