Journal of Siberian Federal University. Mathematics & Physics 2009, 2(2), 137-145

УДК 517.55

Iterates of the Bochner-Martinelli Integral Operator in a Ball

Alexander M.Kytmanov[*]
Simona G.Myslivets[†]
Institute of Mathematics,
Siberian Federal University,
av. Svobodny 79, Krasnoyarsk, 660041,
Russia

Received 10.01.2009, received in revised form 20.03.2009, accepted 29.04.2009

In the present paper we prove the convergence of iterates of the integral Bochner-Martinelli operator in a ball in various spaces: the infinitely-smooth functions, the analytic functions and the spaces conjugate to them, the distributions and the analytic functionals. We give a description of a spectrum of this operator in these spaces as well as the space $\mathcal{L}^p$.

Keywords: Bochner-Martinelli integral operator, iterates.

Iterates of the Bochner-Martinelli integral operator were considered by A.V.Romanov and his result was applied to research the solvability of $\overline{\partial}$-equation. For a ball B in $\mathbb{C}^n$ and for the space $\mathcal{L}^2(\partial B)$ they have been considered in [1], while for any domain D and spaces $\mathcal{W}_2^1(D)$ they have been considered in [2]. Since that, there was no further advancement in generalization of these results to other classes of spaces. Moreover, E.L.Shtraube has shown (se example in [3]) that in any domain it is impossible to expect the convergence of these iterates in spaces $\mathcal{W}_2^s$ for any s.

In the present paper one proves the convergence of iterates of the Bochner-Martinelli integral operator in a ball in various spaces: infinitly-smooth functions, analytic functions and the spaces conjugate to them, distributions and analytic functionals. We give a the description of the spectrum of this operator in these spaces and in the space $\mathcal{L}^p$.

1. Let $B = \{z \in \mathbb{C}^n : \ |z| < 1\}$ be the unit ball in $\mathbb{C}^n$, $n > 1$ and let $S = \partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary. Denote by $\mathcal{W}_2^s(B)$ the Sobolev space, $s \in \mathbb{N}$. We remind that this space consists of functions $f \in \mathcal{L}_2(B)$ such that the derivatives $\partial^\alpha f$ of the order s belong to the space $\mathcal{L}_2(B)$, where

$$\partial^\alpha f = \frac{\partial^{\|\alpha\|} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\alpha_{n+1}} \ldots \partial \bar{z}_n^{\alpha_{2n}}}$$

and $\alpha = (\alpha_1, \ldots, \alpha_{2n})$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_{2n}$. If

$$(f, g)_{\mathcal{L}_2(B)} = \int\limits_B f \cdot \bar{g} \, dv$$

and

$$\|f\|_{\mathcal{L}_2(B)} = \sqrt{(f, f)_{\mathcal{L}_2(B)}}$$

are the scalar product and the norm in $\mathcal{L}_2(B)$, where dv is the element of volume in $\mathbb{C}^n$, then the scalar product and the norm in space $\mathcal{W}_2^s(B)$ are given by the formulas

$$(f, g)_{\mathcal{W}_2^s(B)} = \sum_{\|\alpha\| \leqslant s} (\partial^\alpha f, \partial^\alpha g)_{\mathcal{L}_2(B)}$$

[*]e-mail adress: kytmanov@lan.krasu.ru
[†]e-mail adress: simona@lan.krasu.ru

and

$$\|f\|_{\mathcal{W}_2^s(B)} = \sqrt{(f,f)_{\mathcal{W}_2^s(B)}}.$$

Consider the space $\mathcal{W}_2^{s+\lambda}(S)$ for $0<\lambda<1$. It consists of functions $f\in\mathcal{W}_2^s(S)$ such that the integral

$$\int\limits_S\int\limits_S\sum_{\|\alpha\|=s}\frac{|\partial^\alpha f(z)-\partial^\alpha f(\zeta)|^2}{|\zeta-z|^{2n+2\lambda-1}}\,d\sigma_\zeta d\sigma_z,$$

converges, where $d\sigma$ is an element of the surface S.

We will use the following properties of these spaces (see [4]):

1) The restriction of the function $f\in\mathcal{W}_2^s(B)$, $s\geqslant 1$, to S belongs to the space $\mathcal{W}_2^{s-\frac12}(S)$, and the operator of restriction is continuous.

2) If we denote by $\mathcal{G}_2^s(B)$ the subspace of harmonic functions in the space $\mathcal{W}_2^s(B)$, then the operator of restriction from $\mathcal{G}_2^s(B)$ to $\mathcal{W}_2^{s-\frac12}$ is a linear topological isomorphism. Observe that the folowing decomposition holds:

$$\mathcal{W}_2^s(B)=\mathcal{G}_2^s(B)\oplus\mathcal{N}_2^s(B),$$

where the space $\mathcal{N}_2^s(B)$ consists of the functions in $\mathcal{W}_2^s(B)$ that are equal to zero on S.

3) Embedding theorems imply that there exists the compact continuous embedding $\mathcal{W}_2^s(S)\subset\mathcal{C}^k(S)$ where $s>n+k-\dfrac12$.

For $f\in\mathcal{W}_2^s(B)$, $s\in\mathbb{N}$, we consider the Bochner-Martinelli formula for smooth functions

$$f(z)=\int\limits_S f(\zeta)U(\zeta,z)-\int\limits_B\bar\partial f\wedge U(\zeta,z),\qquad z\in B, \tag{1}$$

where

$$U(\zeta,z)=\frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^n(-1)^{k-1}\frac{(\bar\zeta_k-\bar z_k)}{|\zeta-z|^{2n}}\,d\bar\zeta[k]\wedge d\zeta$$

is the Bochner-Martinelli kernel and $d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_n$, $d\bar\zeta[k]=d\bar\zeta_1\wedge\ldots\wedge d\bar\zeta_{k-1}\wedge d\bar\zeta_{k+1}\wedge\ldots\wedge d\bar\zeta_n$, $\bar\partial f=\sum\limits_{k=1}^n\dfrac{\partial f}{\partial\bar\zeta_k}d\bar\zeta$. We denote by $\mathfrak{M}f$ the first integral (in (1)) and by $\mathfrak{T}f$ the second integral. Then $\mathfrak{I}=\mathfrak{M}+\mathfrak{T}$, where $\mathfrak{I}$ is the identical operator in $\mathcal{W}_2^s(B)$. Since the operator $\bar\partial$ is aj bounded operator from $\mathcal{W}_2^s(B)$ to $\mathcal{W}_2^{s-1}(B)$, the operator $\mathfrak{T}$ is the bounded operator from $\mathcal{W}_2^s(B)$ to $\mathcal{W}_2^s(B)$ (see for example, [4]). Thus $\mathfrak{M}$ is also bounded in $\mathcal{W}_2^s(B)$.

Let's denote by $\mathcal{P}_{m,l}$ the space of homogeneous harmonic polynomials of degree m in z and of degree l in $\bar z$. It is well-known that the space $\bigcup\limits_{m,l}\mathcal{P}_{m,l}$ is dense in $\mathcal{L}_2(S)$ and all spaces $\mathcal{P}_{m,l}$ have finite dimensions. Since polynomials from different spaces $\mathcal{P}_{m,l}$ are orthogonal (in $\mathcal{L}_2(S)$ and in $\mathcal{L}_2(B)$) it follows then each function $f\in\mathcal{G}_2^s(B)$ admits a representation of the form

$$f(z)=\sum_{m,l}P_{m,l}(z), \tag{2}$$

where $P_{m,l}\in\mathcal{P}_{m,l}$. The series (2) converges uniformly on any compact in B with respect to the norm in the space $\mathcal{W}_2^s(B)$ to a function f (convergence in $\mathcal{W}_2^s(B)$ follows from usual properties of a complete orthogonal systems in a Hilbert space).

By Lemma 5.2 from [3] the following equality holds:

$$\mathfrak{M}P_{m,l}=\frac{n+m-1}{n+m+l-1}P_{m,l}. \tag{3}$$

The representation (3) and the properties of the Bochner-Martinelli integral imply the following assertion.

Proposition 1. *The operator $\mathfrak{M} : \mathcal{G}_2^s(B) \longrightarrow \mathcal{G}_2^s(B)$ is a bounded self-adjoint operator with $\|\mathfrak{M}\|_{\mathcal{W}_2^s(B)} = 1$. All rational numbers in the interval $(0,1]$ are eigenvalues of the operator $\mathfrak{M}$ of infinite multiplicity. The spectrum $\mathfrak{M}$ coincides with the interval $[0,1]$.*

Proof. Consider the decomposition of the form (2) for the function $f \in \mathcal{G}_2^s(B)$. Using (3), we obtain the decomposition

$$\mathfrak{M}f = \sum_{m,l} \frac{n+m-1}{n+m+l-1} P_{m,l}.$$

It follows that

$$\|\mathfrak{M}f\|^2_{\mathcal{L}_2(B)} = \sum_{m,l} \Big(\frac{n+m-1}{n+m+l-1}\Big)^2 \|P_{m,l}\|^2_{\mathcal{L}_2(B)} \leqslant \|f\|_{\mathcal{L}^2(B)}.$$

Since

$$\mathfrak{M}\partial^\alpha f = \sum_{m,l} \frac{n+m-\alpha_1-\ldots-\alpha_n-1}{n+m+l-\alpha_1-\ldots\alpha_{2n}-1}\,\partial^\alpha P_{m,l},$$

it is obvious that

$$\|\mathfrak{M}\partial^\alpha f\|^2_{\mathcal{L}_2(B)} \leqslant \|\partial^\alpha f\|^2_{\mathcal{L}_2(B)}.$$

From here and from the norm definition in $\mathcal{W}_2^s(B)$ we have that $\|\mathfrak{M}\|_{\mathcal{W}_2^s(B)} = 1$.

The self-adjointness of the operator $\mathfrak{M}$ follows from the formula (3) and the decomposition (2). The formula (3) also implies that all rational numbers from an interval $(0,1]$ (and only such numbers) are the eigenvalues of the Bochner-Martinelli operator of infinite multiplicity. □

Proposition 2. *Let $\pi^s_{\mathcal{O}}$ be the projection operator from $\mathcal{G}_2^s(B)$ onto the subspace of holomorphic functions $\mathcal{O}_2^s(B) \subset \mathcal{G}_2^s(B)$. Then $\mathfrak{M}^k \xrightarrow[k\to\infty]{} \pi^s_{\mathcal{O}}$ in the strong operator topology of space $\mathcal{G}_2^s(B)$ for any $s \in \mathbb{N}$.*

Proof the proposition follows from the Banach-Steinhaus theorem, the Proposition 1 and the equality (3). □

Let's introduce the Szegö operator $\mathfrak{K}$ in a ball B by the integral

$$\mathfrak{K}\varphi(z) = \frac{(n-1)!}{(2\pi i)^n} \int_S \varphi(\zeta) \frac{d\sigma_\zeta}{(1-\langle z, \bar\zeta\rangle)^n},$$

where $\langle z, \bar\zeta\rangle = z_1\bar\zeta_1 + \ldots + z_n\bar\zeta_n$, and $d\sigma_\zeta$ is an element of a surface S. The operator $\mathfrak{K}$ coincides with the projection operator $\pi^s_{\mathcal{O}}$ from the space $\mathcal{G}_2^s(B)$ onto the subspace $\mathcal{O}_2^s(B)$ for all s. It is known that for any function $\varphi \in \mathcal{C}^\infty(S)$ the operator $\mathfrak{K}\varphi \in \mathcal{O}(B) \cap \mathcal{C}^\infty(\overline{B})$ (see, for example, [5]).

Let's denote by $\mathcal{E}(S)$ the space of functions in $\mathcal{C}^\infty(S)$ with a topology of the uniform convergence on S with all derivatives.

Theorem 1. *For a function $\varphi \in \mathcal{E}(S)$ the sequence $\mathfrak{M}^k\varphi$ converges to $\mathfrak{K}\varphi$ in $\mathcal{E}(S)$ as $k \to \infty$.*

Proof. It suffices to show that $\mathfrak{M}^k\varphi$ tends to $\mathfrak{K}\varphi$ in the spaces $\mathcal{C}^l$ (S) for any $l \in \mathbb{N}$. Properties 2), 3) of the spaces $\mathcal{W}_2^s(B)$ and the Proposition 2 imply the theorem. □

Denote by $\mathcal{E}'(S)$ the space adjoint to $\mathcal{E}(S)$ (relatively of a measure $d\sigma$), i.e., the space of distributions on S.

It is known [3], that the restriction of the Bochner-Martinelli kernel on S has the form $M(\zeta, z)d\sigma_\zeta$, where

$$M(\zeta, z) = \frac{(n-1)!}{2\pi^n}\sum_{k=1}^{n}\zeta_k\frac{\bar{\zeta}_k\bar{z}_k}{|\zeta - z|^{2n}}.$$

Let's denote by

$$\mathfrak{M}T(z) = T_\zeta(M(\zeta, z))$$

the Bochner-Martinelli transform for the distribution $T \in \mathcal{E}'(S)$.

The function $\mathfrak{M}T$ is harmonic in the ball B and has a finite order of growth near S. Therefore $\mathfrak{M}T$ determines some distribution on S. We will denote it by $[\mathfrak{M}T]_0$.

Let's denote by $\mathcal{G}_F(B)$ the space of harmonic functions of a finite order of growth nearly S, and let's denote by $\mathcal{O}_F(B)$ the space of holomorphic functions from $\mathcal{G}_F(B)$. It is known, that the space $\mathcal{G}_F(B)$ is isomorphic to space $\mathcal{E}'(S)$ ([6]).

Theorem 2. *For a distribution $T \in \mathcal{E}'(S)$ the sequence $[\mathfrak{M}T]_0^k$ converges in $\mathcal{E}'(S)$ as $k \to \infty$ to a distribution in $\mathcal{O}_F(B)$.*

Proof. It is known that the harmonic continuation of a distribution $T \in \mathcal{E}'(S)$ is given by the Poisson transform ([6])

$$\mathfrak{P}T(z) = T_\zeta(P(\zeta, z)),$$

where the Poisson kernel is equal to

$$P(\zeta, z) = \frac{(n-1)!}{2\pi^n}\frac{(1-|z|^2)}{|\zeta - z|^{2n}}.$$

For any function $\varphi \in \mathcal{E}(S)$ the value $T\varphi$ is equal to

$$T(\varphi) = \lim_{r\to 1-0}\int_S \mathfrak{P}T(r\zeta)\varphi(\zeta)\,d\sigma_\zeta. \tag{4}$$

We show that the operator $\mathfrak{M}$ satisfies the condition

$$[\mathfrak{M}T]_0(\varphi) = T(\mathfrak{M}\varphi).$$

Since any function from $\mathcal{G}_F(B)$ expands into a series in homogeneous harmonic polynomials $P_{m,l}$ converging absolutely in B, it follows that

$$\mathfrak{P}T(z) = \sum_{m,l} P_{m,l}(z).$$

Using (3) we conclude that

$$\mathfrak{M}T(z) = \sum_{m,l}\frac{n+m-1}{n+m+l-1}P_{m,l}(z). \tag{5}$$

Indeed, $\mathfrak{M}T(z) \in \mathcal{G}_F(B)$ and for any $0 < r < 1$ the series

$$\mathfrak{P}T(rz) = \sum_{m,l} P_{m,l}(rz)$$

converges absolutely and uniformly in $\overline{B}$. Therefore

$$\mathfrak{M}\mathfrak{P}T(rz) = \sum_{m,l}\frac{n+m-1}{n+m+l-1}P_{m,l}(rz).$$

This implies (5).

Since the function φ admits a similar decomposition, it follows that

$$\varphi(z) = \sum_{m,l} Q_{m,l}(z),$$

where $Q_{m,l}(z)$ are homogeneous harmonic polynomials in B. Therefore

$$\mathfrak{M}\varphi(z) = \sum_{m,l} \frac{n+m-1}{n+m+l-1} Q_{m,l}(z)$$

and hence for $0 < r < 1$ we have

$$\mathfrak{M}\mathfrak{P}T(\varphi) = \sum_{m,l} \frac{n+m-1}{n+m+l-1} \int_S P_{m,l}(r\zeta)Q_{m,l}(r\zeta)\, d\sigma_\zeta = \mathfrak{P}T(\mathfrak{M}\varphi).$$

Applying the formula (4) we conclude that

$$[\mathfrak{M}T]_0(\varphi) = T(\mathfrak{M}\varphi).$$

Therefore,

$$[\mathfrak{M}T]_0^k(\varphi) = T(\mathfrak{M}^k\varphi). \tag{6}$$

Since by the Theorem 1 $\mathfrak{M}^k\varphi \xrightarrow[k\to\infty]{} \mathfrak{K}\varphi$ in $\mathcal{E}(S)$, for any $\varphi \in \mathcal{E}(S)$ there exists the limit $\lim\limits_{k\to\infty} [\mathfrak{M}T]_0^k(\varphi)$. This limit determines a distribution $L \in \mathcal{E}'(S)$.

We show that $\mathfrak{P}L \in \mathcal{O}_F(B)$. It suffices to prove that L is a CR-distribution. For this it is necessary to show that $\bar\partial_\tau L = 0$ where $\bar\partial_\tau L$ is the value L on functions $\bar\partial_\tau\varphi$ satisfying the condition

$$\bar\partial_\tau\varphi\, d\sigma = \bar\partial\varphi \wedge d\bar\zeta[l,m] \wedge d\zeta\big|_S.$$

Using the formula (6) and the Theorem 1 we obtain

$$\bar\partial_\tau L(\varphi) = \lim_{k\to\infty} \bar\partial_\tau [\mathfrak{M}^k T]_0(\varphi) =$$
$$= -\lim_{k\to\infty} [\mathfrak{M}^k T]_0(\bar\partial_\tau\varphi) = -\lim_{k\to\infty} T(\mathfrak{M}^k(\bar\partial_\tau\varphi)) = -\lim_{k\to\infty} T(\mathfrak{K}(\bar\partial_\tau\varphi).$$

Since the function $\bar\partial_\tau\varphi$ is orthogonal to the holomorphic functions in $\mathcal{L}_2(S)$, it follows that $\mathfrak{K}(\bar\partial_\tau\varphi) = 0$. By the Hartogs-Bochner theorem (see, for example, th. 7.1 from [3]) we have that $\mathfrak{P}L \in \mathcal{O}_F(B)$. □

2. Consider the space $\mathcal{A}(S)$ of real-valued analytic functions on S as on a real analytic manifold in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Then any function $\varphi \in \mathcal{A}(S)$ has holomorphic continuation to some complex neighbourhood $U \subset \mathbb{C}^{2n}$.

The linear functional T on $\mathcal{A}(S)$ is analytic if for each complex neighbourhood U of the sphere S there exists a constant $c(U)$ such that

$$|T(\varphi)| \leqslant c(U) \sup_U |\varphi| \tag{7}$$

for any entire function φ on $\mathbb{C}^{2n}$ [7]. In this definition we can take the holomorphic functions on $\overline{U}$ instead of entire functions. We denote by $\mathcal{A}'(S)$ the set of analytic functionals on S.

The property (7) can be expressed in terms of the function φ and its derivatives on S. Namely we consider the following expression

$$\ell(\varphi) = \overline{\lim_{\|\alpha\|\to\infty}} \sqrt[\|\alpha\|]{\frac{\sup_S |\partial^\alpha\varphi|}{\alpha!}},$$

where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_{2n}!$. Then the function $\varphi \in \mathcal{E}(S)$ is belongs to $\mathcal{A}(S)$ if and only if $\ell(\varphi) < +\infty$ and $\dfrac{1}{\ell(\varphi)}$ is the radius r of a polydisc $U(\zeta^0, r)$ with the center at the point $\zeta^0 \in S$ to which φ holomorphically continues. The union of all such polydiscs gives a neighbourhood U to which φ can be holomorphically continued.

The sequence of functions $\{\varphi_k\}$ converges to φ in $\mathcal{A}(S)$ as $k \to \infty$ if the functions φ_k holomorphically continue to some complex neighbourhood U and in this neighbourhood uniformly converge to φ. Therefore, if $T \in \mathcal{A}'(S)$ then $T(\varphi_k) \xrightarrow[k\to\infty]{} T(\varphi)$.

We can identify the space $\mathcal{A}(S)$ with the space $\mathcal{G}(\overline{B})$ of functions harmonic in $\overline{B}$. Each function $\varphi \in \mathcal{A}(S)$ can be harmonically continued to B as a function Φ. This function Φ is a function of class $\mathcal{C}^\infty$ up to a boundary S [8] and since φ is real analytic on S it follows that $\Phi \in \mathcal{G}(\overline{B})$. The maximum principle shows that the topologies in $\mathcal{A}(S)$ and in $\mathcal{G}(\overline{B})$ coincide.

For a function $\varphi \in \mathcal{A}(S)$ we consider $\mathfrak{M}\varphi$. Since the coefficients of the Bochner-Martinelli kernel are the derivatives of the potential of a simple layer, by Lemma 24.1 in [3] we have that $\ell\mathfrak{M}\varphi \leqslant \ell\varphi$. Hence $\ell\mathfrak{M}^k\varphi \leqslant \ell\varphi$. I.e., the iterates $\mathfrak{M}^k\varphi$ holomorphically continue to the same neighbourhood $U \subset \mathbb{C}^{2n}$ of the closure of the ball $\overline{B}$ as the function φ.

Theorem 3. *For a function $\varphi \in \mathcal{A}(S)$ the sequence $\{\mathfrak{M}^k\varphi\}$ converges to $\mathfrak{K}\varphi$ at $k \to \infty$ in some neighbourhood U of the closure $\overline{B}$.*

Proof. We consider a neighbourhood U to which all functions $\mathfrak{M}^k\varphi$ have a holomorphic continuation. We continue φ as a harmonic function in $\mathcal{G}(\overline{B})$ and expand it into a series in homogeneous harmonic polynomials $P_{m,l}$, i.e.,

$$\varphi(z, \bar{z}) = \sum_{m,l} P_{m,l}(z, \bar{z}). \tag{8}$$

This series converges uniformly and absolutely in some ball $B_r \supset B$, $r > 1$ (§4, ch. 11 [9]). By the Kiselman theorem a series of the form $\sum\limits_{m,l} P_{m,l}(z, w)$ absolutely and uniformly converges to the holomorphic continuation $\tilde{\varphi}$ of the function φ in some neighbourhood $U \subset \mathbb{C}^{2n}$ of a ball B_r. We can chose this neighbourhood to be the ball Li (the symmetric domain of the fourth type) [10]. Since

$$\mathfrak{M}^k\varphi = \sum_{m,l} \Big(\frac{n+m-1}{n+m+l-1}\Big)^k P_{m,l},$$

it follows that the holomorphic continuation $\widetilde{\mathfrak{M}^k\varphi}$ in U satisfies the estimate

$$|\widetilde{\mathfrak{M}^k\varphi}| \leqslant \sum_{m,l} |\tilde{P}_{m,l}(z, w)|$$

on any compact from U.

Thus the sequence $\{\widetilde{\mathfrak{M}^k\varphi}\}$ is uniformly bounded on any compact lying in U. Hence the sequence $\{\widetilde{\mathfrak{M}^k\varphi}\}$ converges uniformly in some neighbourhood compactly lying in U. The limit of this sequence is the function

$$\mathfrak{K}\varphi(z) = \sum_m P_{m,0}(z).$$

In particular, a function $\mathfrak{K}\varphi \in \mathcal{O}(\overline{B})$. □

Theorem 4. *For the analytic functional $T \in \mathcal{A}'(S)$ the sequence $[\mathfrak{M}^k T]_0$ converges weakly in $\mathcal{A}'(S)$ to the analytic functional $[\mathfrak{K}T]_0$ at $k \to \infty$.*

The *proof* repeats the proof of Theorem 2 with $\mathcal{E}(S)$ replaced by $\mathcal{A}(S)$ and $\mathcal{E}'(S)$ by $\mathcal{A}'(S)$.

3. Let function $\varphi \in \mathcal{L}_p(S)$, $p > 1$. The harmonic continuation of the function φ in a ball B is given by Poisson integral

$$\mathfrak{P}\varphi(z) = \int\limits_S \varphi(\zeta) P(\zeta, z)\, d\sigma_\zeta.$$

It is known that (ch. 7, §4 [11])

$$\sup_{0<r<1} \int\limits_S |\mathfrak{P}\varphi(r\zeta)|^p d\sigma < \infty$$

and the function $\mathfrak{P}\varphi(r\zeta) \underset{r\to 1}{\longrightarrow} \varphi(\zeta)$ with respect to the norm of the space $\mathcal{L}_p(S)$ and almost everywhere.

Let's consider the decomposition of the function $\mathfrak{P}\varphi$ into homogeneous harmonic polynomials of the form (2):

$$\mathfrak{P}\varphi(\zeta) = \sum_{m,l} P_{m,l}(\zeta).$$

This series converges absolutely and uniformly inside the ball B. For the function $\psi \in \mathcal{L}_q(S)$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ the decomposition $\mathfrak{P}\psi$ has the form

$$\mathfrak{P}\psi(\zeta) = \sum_{m,l} Q_{m,l}(\zeta).$$

Then the pairing of the functions φ and ψ is given by the formula

$$(\varphi, \psi) = \int\limits_S \varphi(\zeta)\bar{\psi}(\zeta) d\sigma_\zeta = \lim_{r\to 1} \int\limits_S \mathfrak{P}\varphi(r\zeta)\mathfrak{P}\bar{\psi}(r\zeta)\, d\sigma_\zeta =$$
$$= \lim_{r\to 1} \int\limits_S \sum_{m,l} P_{m,l}(r\zeta)\overline{Q}_{m,l}(r\zeta)\, d\sigma_\zeta = \lim_{r\to 1} \sum_{m,l} r^{2(m+l)} \int\limits_S P_{m,l}(\zeta)\overline{Q}_{m,l}(\zeta)\, d\sigma_\zeta =$$
$$= \lim_{r\to 1} \sum_{m,l} (P_{m,l}, Q_{m,l}) r^{2(m+l)}. \quad (9)$$

Thus the sum of the series

$$\sum_{m,l} (P_{m,l}, Q_{m,l})$$

in the Abel-Poisson sense equals (φ, ψ).

Let's denote by $gr_p(B)$ the space of functions φ which are harmonic in B and such that the condition

$$\sup_{0<r<1} \int\limits_S |\varphi(r\zeta)|^p d\sigma_\zeta < \infty$$

is satisfied. It is known that the normal boundary values of the function $\varphi \in g_p(B)$ determine the function from $\mathcal{L}_p(S)$ [12]. Moreover, the Poisson integral determines an isometry of the spaces $\mathcal{L}_p(S)$ and $g_p(B)$.

Proposition 3. *The operator $\mathfrak{M}$ is a bounded linear operator from $\mathcal{L}_p(S)$ to $\mathcal{L}_p(S)$, its spectrum coincides with the interval* $[0, 1]$ *and for any $\varphi \in \mathcal{L}_p(S)$, $\psi \in \mathcal{L}_q(S)$ the following equality holds:*

$$(\mathfrak{M}\varphi, \psi) = (\varphi, \mathfrak{M}\psi). \quad (10)$$

Proof. It is known [13] (and also [3]) that $\mathfrak{M}\varphi \in g_p(B) \simeq \mathcal{L}_p(S)$ if $\varphi \in \mathcal{L}_p(S)$. Moreover, this operator is bounded because it is the singular integral operator satisfying a cancelation condition (see Th. 3, ch. 2 [11]). These conditions are checked in [14]. Equality (10) evidently follows from equality (9).

Since the homogeneous polynomials $P_{m,l} \in g_p(B)$ for all m, l and $p > 1$, it follows from equality (3) that $P_{m,l}$ are the eigenfunctions and the numbers $\left(\dfrac{n+m-1}{n+m+l-1}\right)$ are the eigenvalues of the operator $\mathfrak{M}$. We show that the operator $\mathfrak{M}$ in $\mathcal{L}_p(S)$ has no other eigenvalues.

Let's assume that for some function $\varphi \in g_p(B)$

$$\mathfrak{M}\varphi = \lambda\varphi.$$

Using equality (10) we conclude that

$$(\mathfrak{M}\varphi, P_{m,l}) = (\varphi, \mathfrak{M}P_{m,l}),$$

hence,

$$\lambda(\varphi, P_{m,l}) = \frac{n+m-1}{n+m+l-1}(\varphi, P_{m,l}).$$

If $\lambda \neq \dfrac{n+m-1}{n+m+l-1}$ for all m, l then the scalar product $(\varphi, P_{m,l}) = 0$ for all m, l. Since the linear combinations of functions $P_{m,l}$ are dense in $\mathcal{L}_p(S)$, it follows that $\varphi = 0$. □

The authors were supported by RFBR, grants 08-01-00844 and 08-01-90250; by Rosobrazovanie, grant №2.1.1/4620.

References

[1] A.V.Romanov, Spectral analysis of the Bochner-Martinelli operator for the ball in $\mathbb{C}^n$ and its applications, *Functional Anal. Appl.,* **12**(1978), 232-234.

[2] A.V.Romanov, Convergence of iterates of the Bochner-Martinelli operator and the Cauchy-Riemann equation, *Soviet Math. Dokl.,* **19**(1978), no. 5, 1211-1215.

[3] A.M.Kytmanov, The Bochner-Martinelli integral and its applications, Basel - Boston - Berlin, Birkhäuser Verlag, 1995.

[4] Yu.V.Egorov, M.A.Shubin, Partial differential I. Foundations of the classical theory, *Encyclopaedia of Mathematical Sciences 30*, Berlin, Springer-Verlag, 1992.

[5] G.M.Henkin, The method of integral representations in complex analysis, *Several Complex Variables I, Encyclopaedia of Mathematical Sciences*, Springer-Verlag, **7**(1990), 19-116.

[6] E.L.Straube, Harmonic and analytic functions admitting a distribution boundary value, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **11** (1984), no. 4, 559-591.

[7] L.H örmander, The analysis of linear partial differential operators I, Berlin - Heidelberg - New York - Tokyo, Springer Verlag, 1983.

[8] M.Brelot, Elements de la theorie classique du potential, Paris, 1969.

[9] S.L.Sobolev, Introduction to the theory of cubature formulas, Philadelphia, Gordon and Breach, 1992.

[10] C.O.Kiselman, Prolongment des solutions, *Bull. Soc. Math.,* **97**(1965), no. 4, 345-355.

[11] E.M.Stein, Boundary behaviour of holomorphic functions of several complex variables, Princeton, Princeton Univ. Press, NJ, 1972.

[12] G.M.Henkin, E.M.Chirka, Boundary properties of holomorphic functions of several complex variables, Current problems in mathematics, *J. Soviet Math,* **5**(1976), no. 5.

[13] A.A.Shlapunov, N.N.Tarkhanov, On the Cauchy problem for holomorphic functions of Lebesgue class $\mathcal{L}^2$ in a domain, *Sib. Mat. Zh.,* **33**(1992), no. 5, 186-195 (in Russian).

[14] A.M.Kytmanov, S.G.Myslivets, On Asymptotic Expansion of the Conormal Symbol of the Singular Bochner-Martinelli Operator on the Surfaces with Singular Points, *J. Sib. Fed. Univ. Mathematics and Physics,* **1**(2008), no. 1, 3-12.